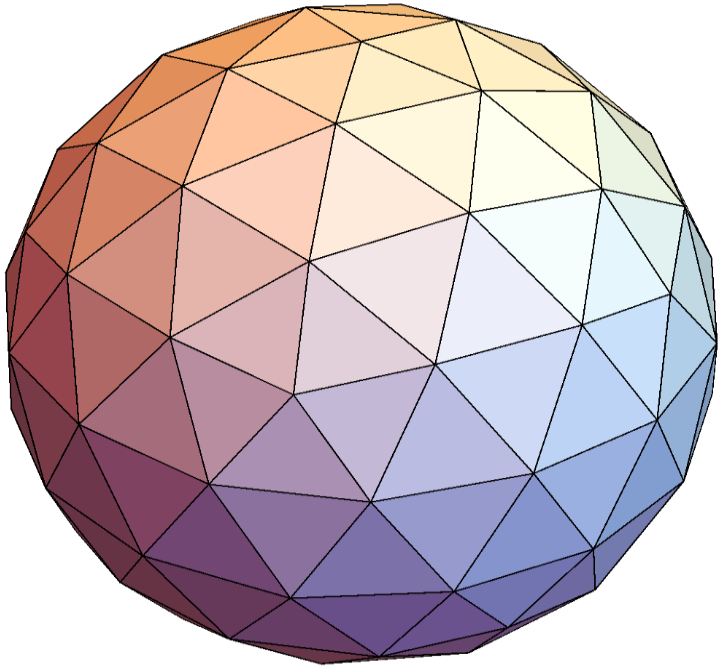
Lattice QFT in Curved Spacetimes and Applications to CFT



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CERN Theory Seminar, July 25th 2017

with Richard Brower (Boston), George Fleming (Yale), Timothy Raben (Kansas),
Chung-I Tan (Brown), and Evan Weinberg (Boston)

Refs:
1.) PRD 95 (11), 114510
2.) arXiv:1601.01367

Topics covered

- A primer: How does lattice describe continuum physics?
  - Classically and at loop level in lattice perturbation theory
- General strategy for lattice theory for curved spacetimes (classical)
  - Topology, Geometry, and Hilbert Space
- General considerations for quantum corrections
  - Counterterms required to cancel position dependent quantum loops
- Radial quantization for conformal field theory on the lattice
- ϕ^4 Theory on the Riemann sphere
- Ongoing work and future directions

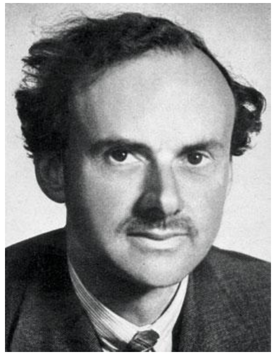
Oscar Klein

Ernst Ising

Topics omitted

- Dirac fermions on curved lattice
  - Wilson term
  - 2D Ising from free Dirac theory

PRD 95 (11), 114510



Earlier Works

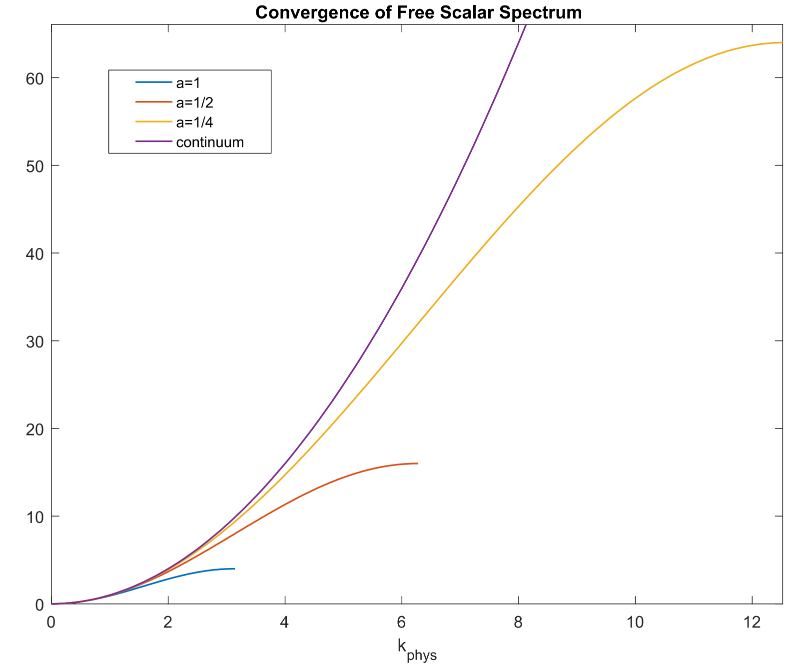
- Random Lattice Field Theory. Christ et al *Nuclear Physics B* 202.1 (1982): 89-125
- GR without Coordinates (Regge Calculus). T Regge, Nuovo Cim. 19 (1961) 558-571
- Finite Elements

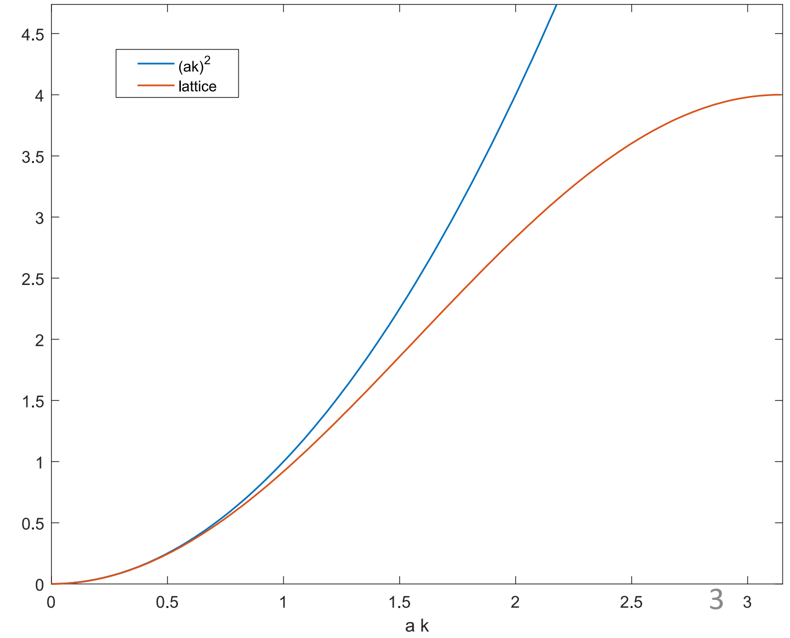
How does lattice describe the continuum (classical)

- Nonlocal operators in LGT contribute infinite tower of local operators
    - Many are not Lorentz invariant
- Classically, operators breaking Lorentz Invariance vanish as $a \to 0$

$$\Delta_\mu^+ \Delta_\mu^- \phi(x) \approx \nabla^2 \phi(x) + a^2 \sum_\mu \nabla_\mu^4 \phi(x) + O(a^4)$$

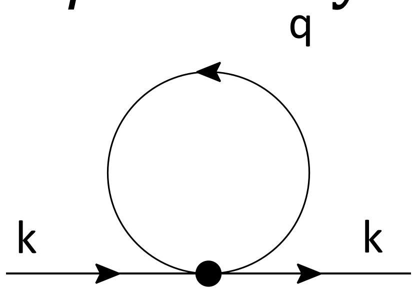
- On lattice, one takes continuum limit by taking $a \to 0$ while holding physical quantity fixed
- For free theory, each physical momentum converges to continuum result
    - Deep UV is always wrong
- One might worry about quantum effects sensitive to UV modes spoiling the continuum limit





How does lattice describe the continuum (quantum)

- Nonperturbative proofs are hard
- One can prove renormalizability and continuum limit in perturbation theory
  - "Power counting theorem" for lattice perturbation theory (T. Reisz 1988-1989)
    - $\deg(I) < 0 \rightarrow$ Integral is finite and given by naïve continuum limit as $a \rightarrow 0$
- Consider ϕ^4 theory in d=2,3 dimensions. At one loop:



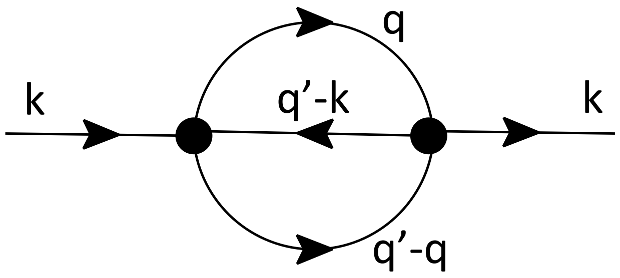
$$= I_1(k, m; a) = \frac{\lambda}{2} \int_{-\pi/a}^{\pi/a} \frac{d^d q}{(2\pi)^d} \frac{1}{\tilde{q}^2 + m^2}$$

$\deg(I_1) = d - 2$
divergent in 2 and 3 dimensions

- Divergent constant can be absorbed into counterterm δm^2
  - Diagram only has support on k=0 due to *translation invariance*
  - **This will change on a curved lattice without translation invariance!**

How does lattice describe the continuum (quantum), ctd

- At two loops:



$$= I_2(k,m;a) = \frac{\lambda^2}{3}\int_{-\pi/a}^{\pi/a} \frac{d^d q}{(2\pi)^d}\frac{d^d q'}{(2\pi)^d}\frac{1}{(\tilde{q}^2+m^2)}\frac{1}{(\widetilde{(q'-q)}^2+m^2)}\frac{1}{(\widetilde{(q'-k)}^2+m^2)}$$

$\deg(I_2) = 2d-6$ → Divergent in d=3

- Can check that divergence is independent of k and renormalized perturbation theory can be made Lorentz Invariance

$$I_2(k,m;a) = I_2(0,m;a) + D_2(k,m;a)$$

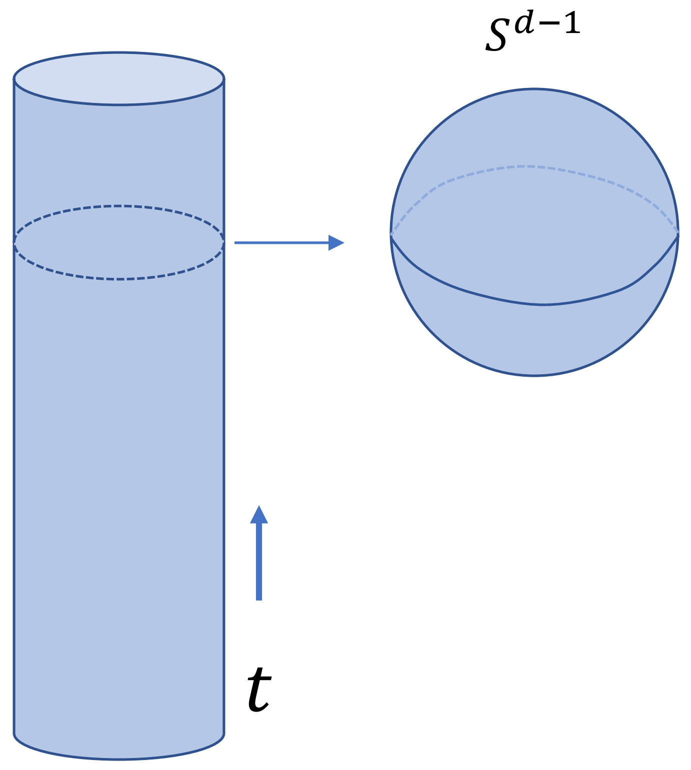
$$D_2(k,m;a) = \frac{\lambda^2}{3}\int_{-\pi/a}^{\pi/a} \frac{d^d q}{(2\pi)^d}\frac{d^d q'}{(2\pi)^d}\frac{1}{(\tilde{q}^2+m^2)}\frac{1}{(\widetilde{(q'-q)}^2+m^2)}\left[\frac{\widetilde{q'^2}-(\widetilde{q'-k})^2}{((\widetilde{q'-k})^2+m^2)(\widetilde{q'^2}+m^2)}\right]$$

$\deg(D_2) = 2d-7$ → Well defined continuum limit in d=3

Renormalizable QFTs are also renormalizable in lattice regularization
The renormalized LFT becomes Lorentz invariant in the continuum limit

Why Study Strongly Coupled QFTs in Curved Space

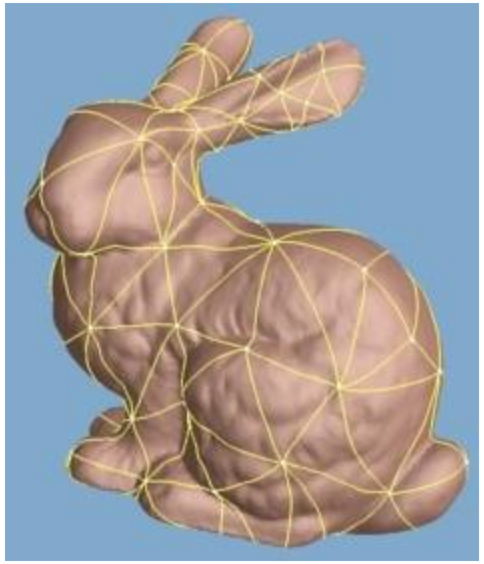
- Theoretical exercise: nonperturbative renormalization in curved space
- Many interesting QFTs in curved space and irregular lattices
  - **Conformal field theories**
    - Complementary method to the bootstrap for CFTs in $d \geq 3$ dimensions
    - Cf. El-Showk et al, Physical Review D 86 (2), 025022
  - Gauge/Gravity dualities
  - Condensed matter systems
  - Graphene
- Radial Quantization on the Lattice
  - Dilation symmetry manifest.
    - Translations must be recovered dynamically, but this is less important. Spacing of decedents will be noninteger at finite lattice spacing.
  - Exponential separations down cylinder
    - Power law correlators become exponentially decaying
    - Can isolate lowest states using usual lattice tricks



Schematic Approach to Lattice Theory for Curved Manifolds

Target Manifold, M

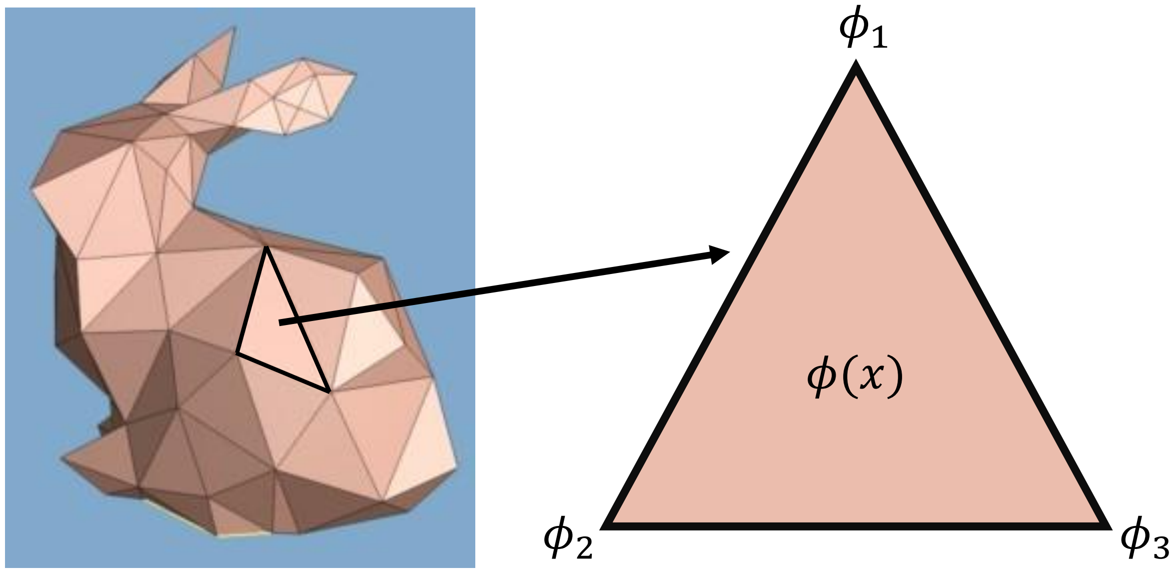
Topology $M \to M_\sigma$



- Partition space into simplices
- Simplicial Complexes

Geometry, $g^{\mu\nu} \to g_\sigma^{\mu\nu}$

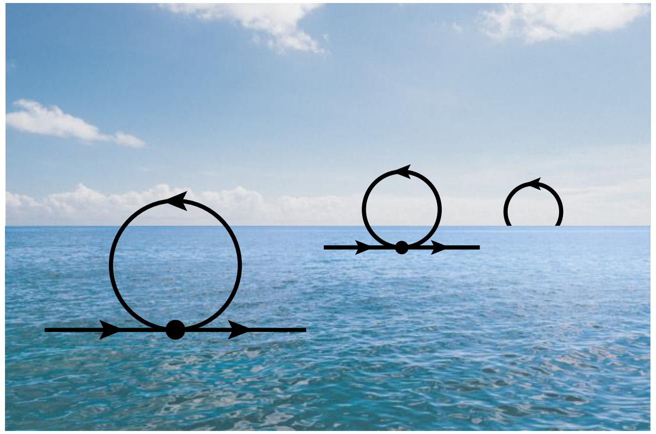
Hilbert Space



- Define metric by assigning lengths
- Regge Calculus

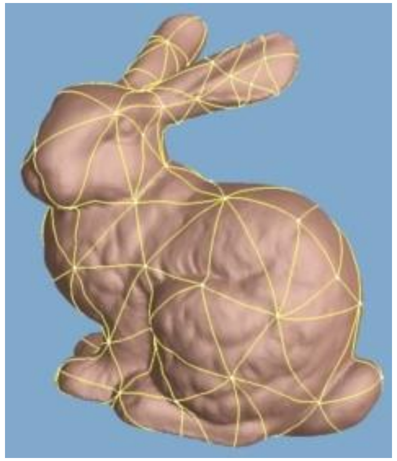
- Expand fields in finite basis
- Finite Elements

Quantum Effects and Renormalization



- Quantum loops sensitive to curvature
- “Quantum Finite Elements”

Topology and Simplicial Complexes

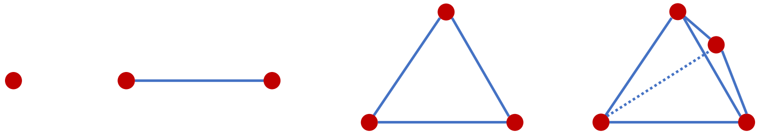


- Replace target manifold with a sequence of increasingly dense simplicial partitions of "refinement" s

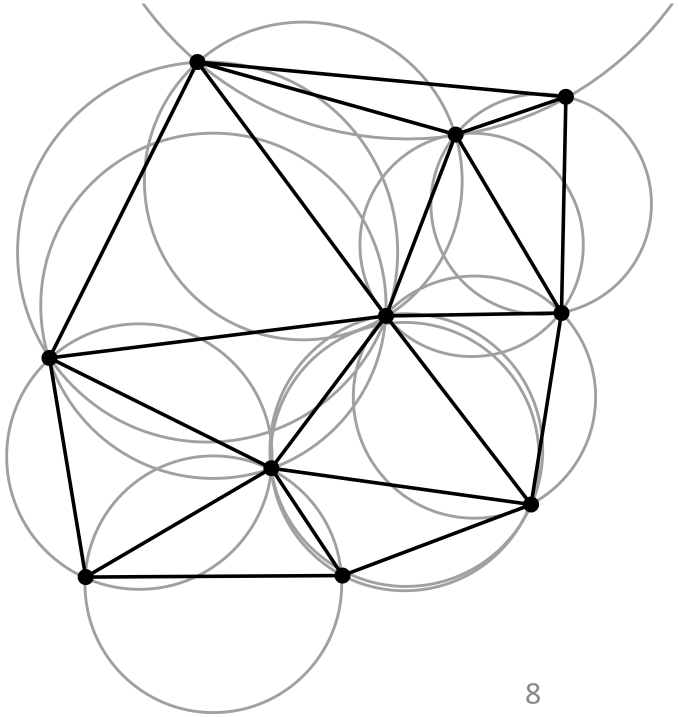
$$M \rightarrow \{M_s\}^{s \in 1,2,\ldots}$$

- At the moment, no metric. Purely topological.
  - How the simplices are glued together determines the topology of the space
  - Practically speaking, at each refinement we have a list of points and a neighbor table (amenable to intrinsic geometry)
- Simplicial complex provides an organized foundation on which to build geometrical structures (metric, vierbein, spin connection, etc)

$$\sigma_1 \rightarrow \sigma_2 \rightarrow \ldots \rightarrow \sigma_d$$



- Given a set of vertices, simplicial complex can always be constructed via the Delaney / Voronoi construction
  - Establishes links between vertices by maximizing smallest angle in simplices
  - Relies on knowing something about Geometry first, so slightly out of order

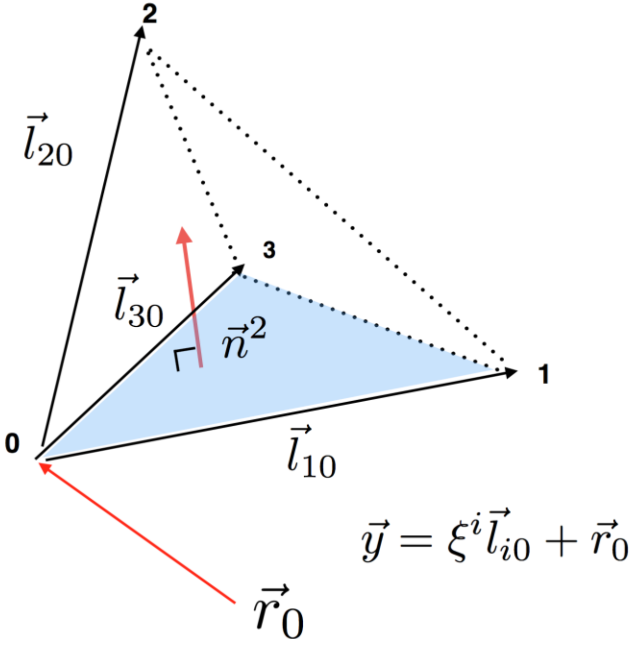


Geometry and Regge Calculus

- Define metric distance along edges by assigning lengths
  $$|\sigma_1(i,j)| \equiv l_{ij} = some\ \#$$
- Continue metric to interior of each simplex to be flat (not the only choice)
  - Then, geometry of each simplex known entirely in terms of edge lengths
- Very clean coordinate choice: Barycentric Coordinates
  - For a point $\vec{y}$ in a d-simplex

$$\vec{y} = \sum_{i=0}^{d} \xi^i \vec{r_i} \qquad 0 \le \xi^i \le 1 \qquad \sum_{i=0}^{d} \xi^i = 1 \qquad \rightarrow \qquad \vec{y} = \vec{r_0} + \sum_{i=1}^{d} \xi^i \overrightarrow{l_{i0}}$$

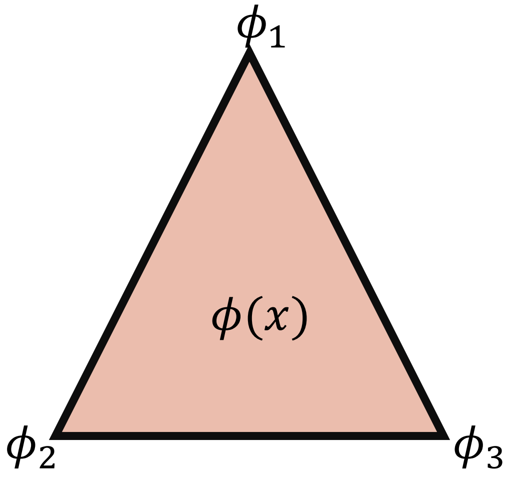
$$ds^2 = d\vec{y} \cdot d\vec{y} = g_{ij}\xi^i\xi^j \qquad g_{ij} = \overrightarrow{l_{i0}} \cdot \overrightarrow{l_{j0}} = \frac{1}{2}\left(l_{i0}^2 + l_{j0}^2 - l_{ij}^2\right)$$



- Constant flat metric everywhere inside simplex
- Can construct, e.g., Einstein Hilbert term and find EH action given entirely in terms of deficit angles
  - “GR without Coordinates”, T. Regge, 1960

$$S_s = \frac{1}{2}\sum_{\sigma_d \in M_s} \int_{\sigma_d} d^d\,\vec{y}\,\left[\vec{\nabla}\phi(y)\cdot\vec{\nabla}\phi(y) + m^2\phi(y)^2\right] = \frac{1}{2}\sum_{\sigma_d \in M_s} \int_{\sigma_d} d^d\,\xi\,\sqrt{|g_{ij}|}\,\left[g^{ij}\,\partial_i\phi(\xi)\partial_j\phi(\xi) + m^2\phi(\xi)^2\right]$$

Hilbert Space and Finite Elements



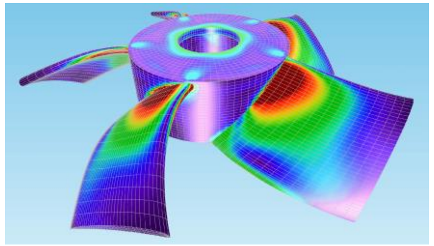
- To regulate the QFT, we truncate the Hilbert space by expanding in a finite field basis on each simplex called a **finite element basis**.

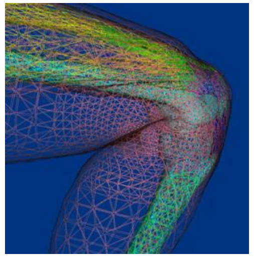
$$\phi_\sigma(\xi) = \sum_{i=0}^{d} E^i(\xi)\phi_i \qquad \sum_{i=0}^{d} E^i(\xi) = 1 \qquad E^i(\vec{r_j}) = \delta^i_j$$

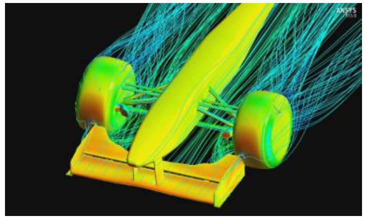
- Common tool for solving classical PDEs in engineering, E&M (cf. Jackson), fluid dynamics, …
- We use simplest case, **linear finite elements** , $E^i(\xi) = \xi^i$
- Gradients are constants everywhere in the simplex, $\partial_i \phi_\sigma(\xi) = \phi_i - \phi_0$
- Plugging expansion into action, arrive at discrete action in terms of lattice degrees of freedom located at vertices

$$S_\sigma = \frac{1}{2}\sum_{i,j=1}^{d} |\sigma_d|\, g^{ij}\,(\phi_i - \phi_0)(\phi_j - \phi_0) = \frac{1}{2}\sum_{\langle i,j\rangle} V_{ij}\,\frac{(\phi_i - \phi_j)^2}{l_{ij}^2}$$

"vertex form" "edge form"

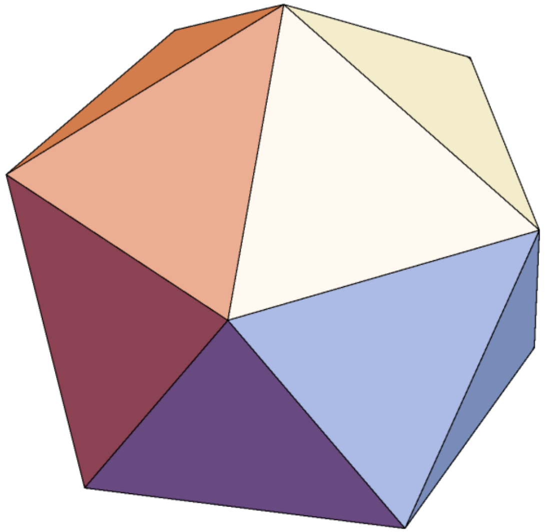




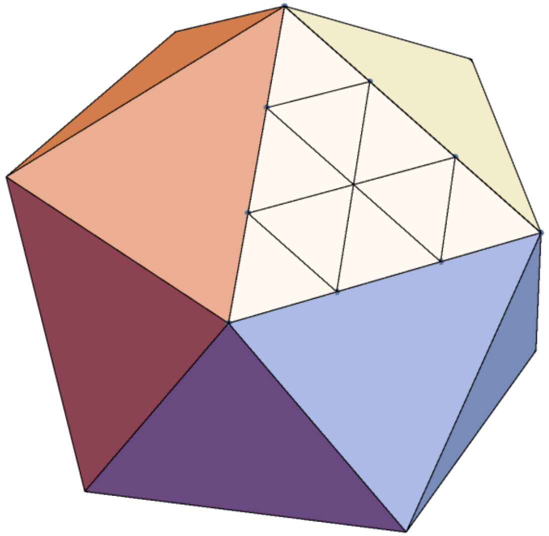


Ex. 1) Free Scalar Fields on S^2

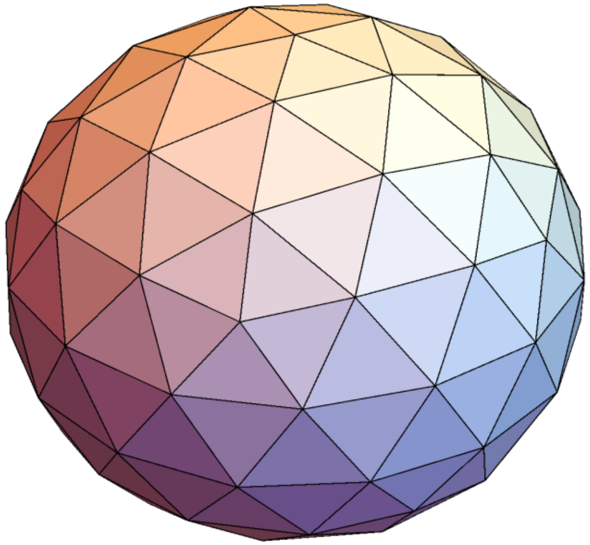
- Construction of refined simplicial lattice ($s = 3 \sim 1/a$)



Icosahedron
a.k.a s=1
Plato's best sphere



"Refine"
Divide each face into smaller
Equilateral triangles (s-1) times

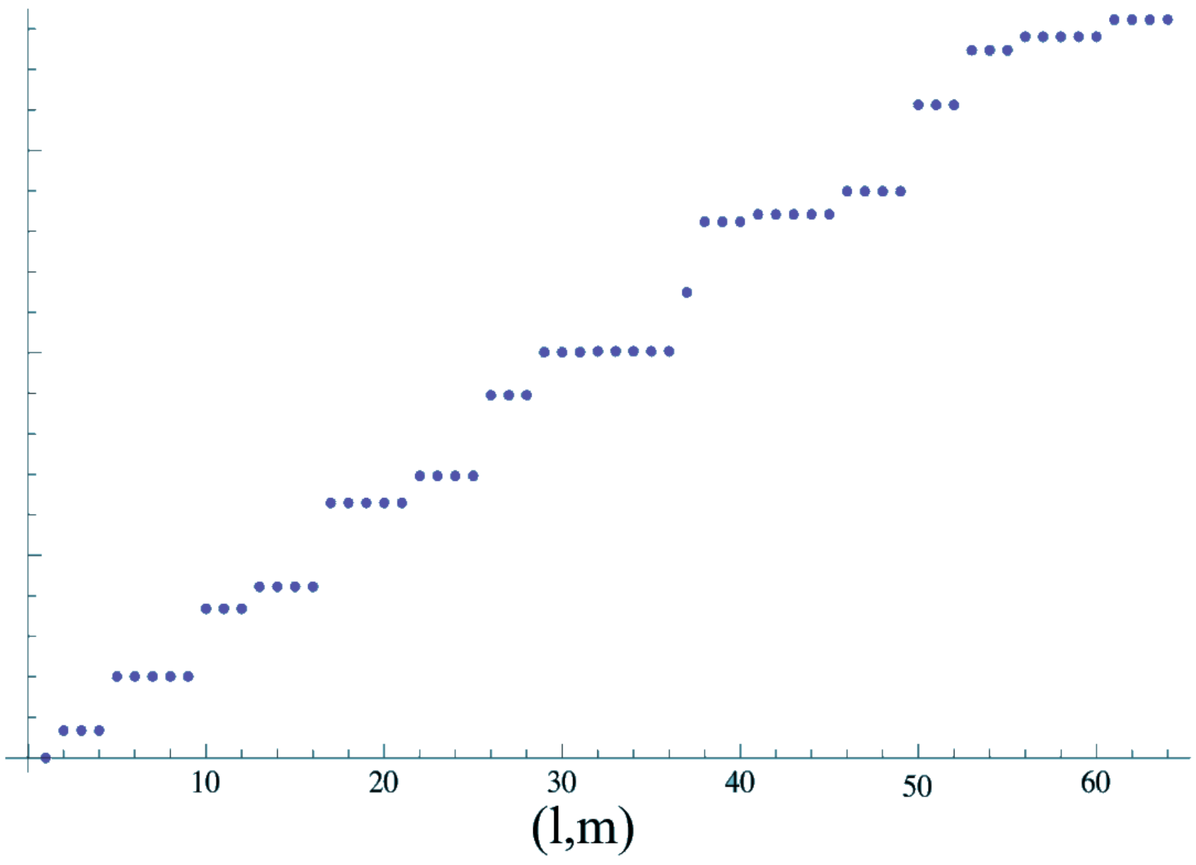


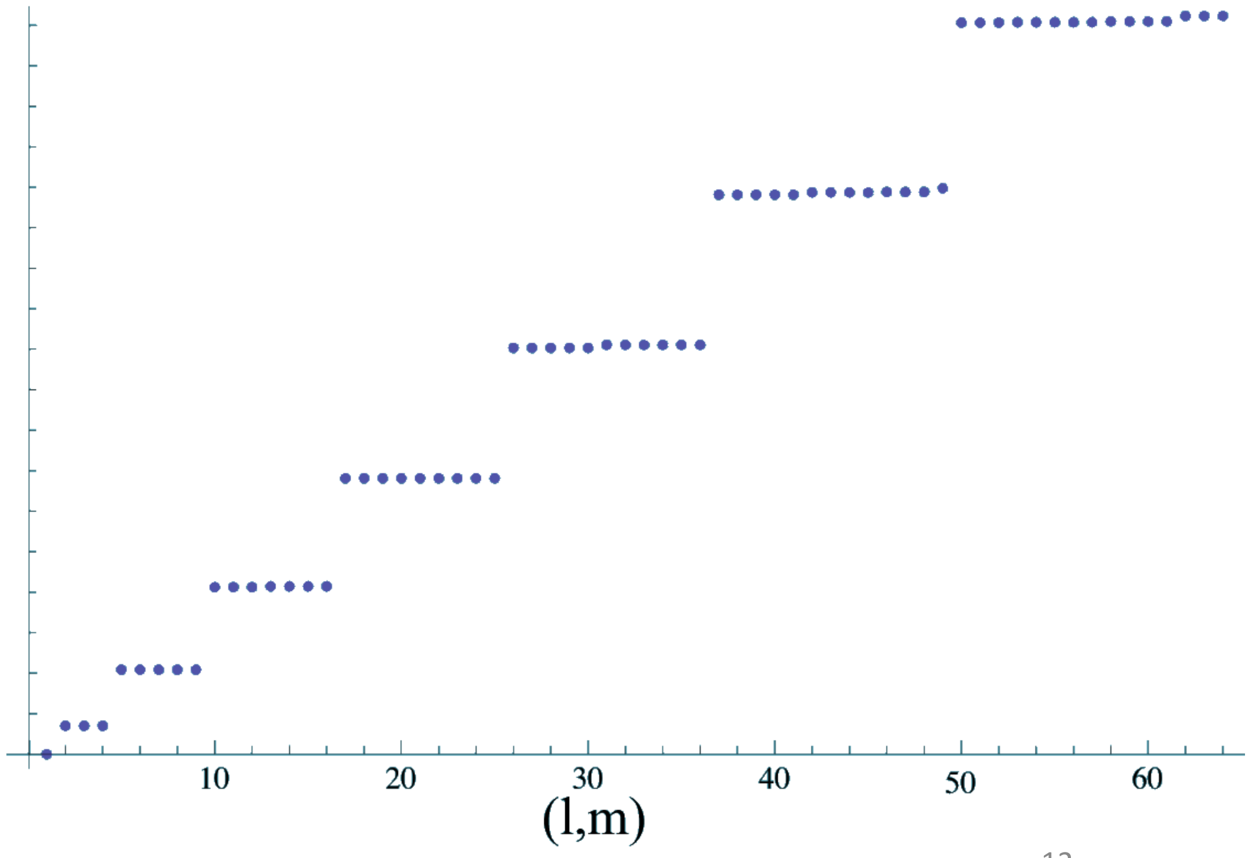
Project
All points now lie on sphere
Distances given by secant distances
In embedding space

Ex. 1) Free Scalar Fields on S^2 : Laplacian Spectrum

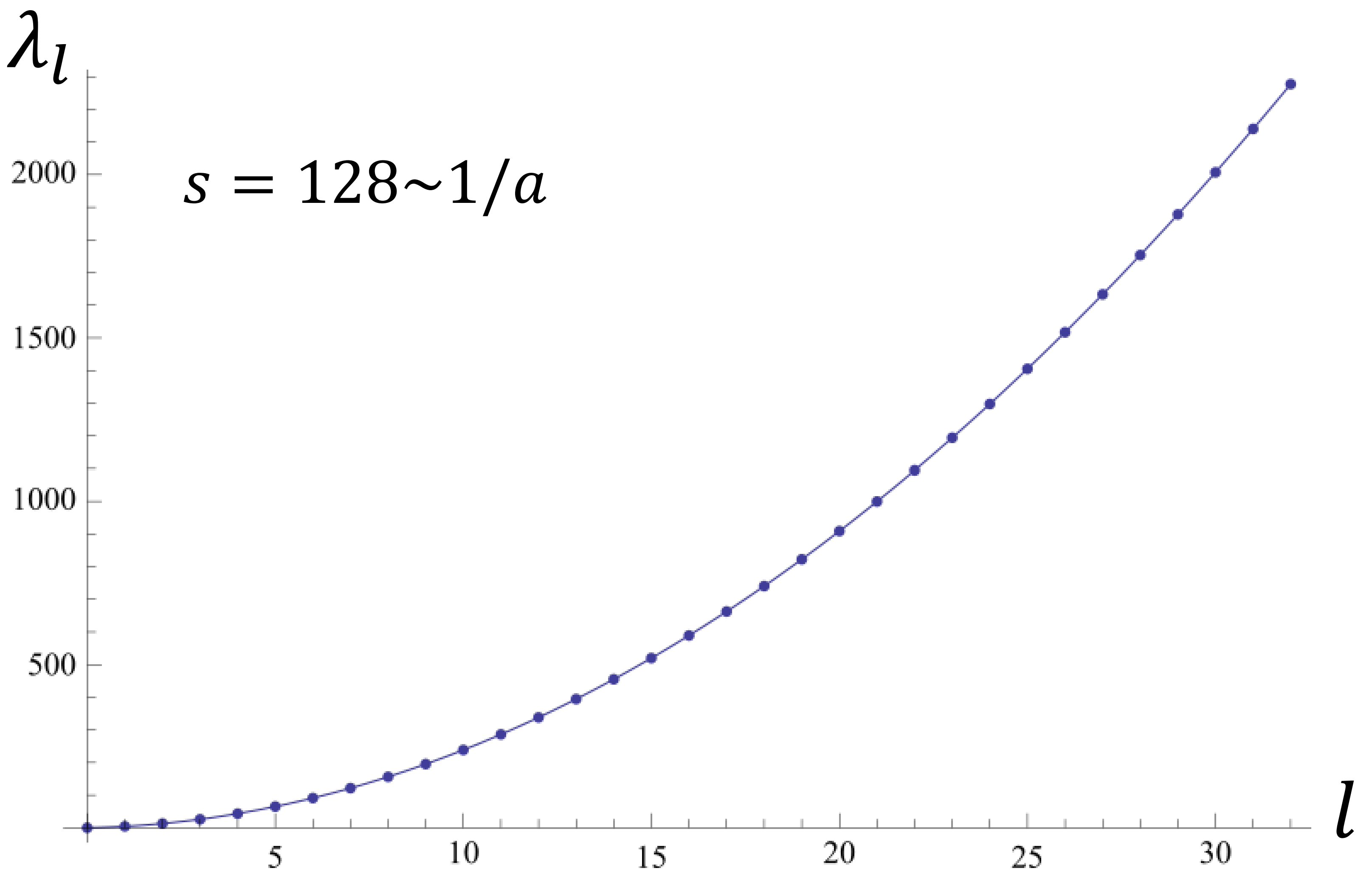
$$S_\sigma^{naive} = \frac{1}{2}\sum_{\langle i,j\rangle}(\phi_i - \phi_j)^2$$

$$S_\sigma^{FEM} = \frac{1}{2}\sum_{\langle i,j\rangle} A_{ij}\frac{(\phi_i - \phi_j)^2}{l_{ij}^2}$$





Ex. 1) Free Scalar Fields on S^2 : Laplacian Spectrum



$$\lambda(l)_{fit} = l + 1.00012 l^2 - 0.000013 l^3 - 0.000005 l^4$$

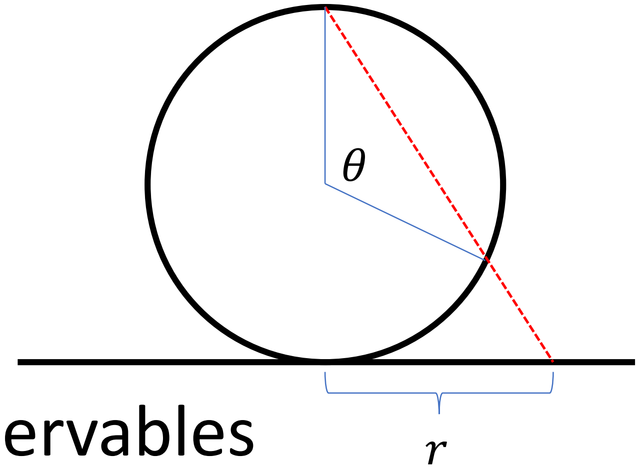
- IR spectrum becomes exact as a $\rightarrow 0$
- Each physical angular momentum is converging
- Spectrum is always bad near the UV cutoff (not shown)

- *Now we're ready to think about interacting theories!*

Ex 2) ϕ^4 Theory on S^2

- Really want to do ϕ^4 Theory on $R \times S^2$. This is a "warm up"
- S^2 is locally equivalent to R^2 up to Weyl factor (stereographic projection)

$$ds^2 = dr^2 + \mathrm{r}^2\mathrm{d}\phi^2 = \Omega^2(\theta,\phi)(d\theta^2 + \sin^2\theta\, d\phi^2)$$



- Studying 2D Ising fixed point in a very difficult way!
  - It's useful to know the answer
- First attempt: simply run monte-carlo calculation of observables using our FEM action and attempt to reach the critical point

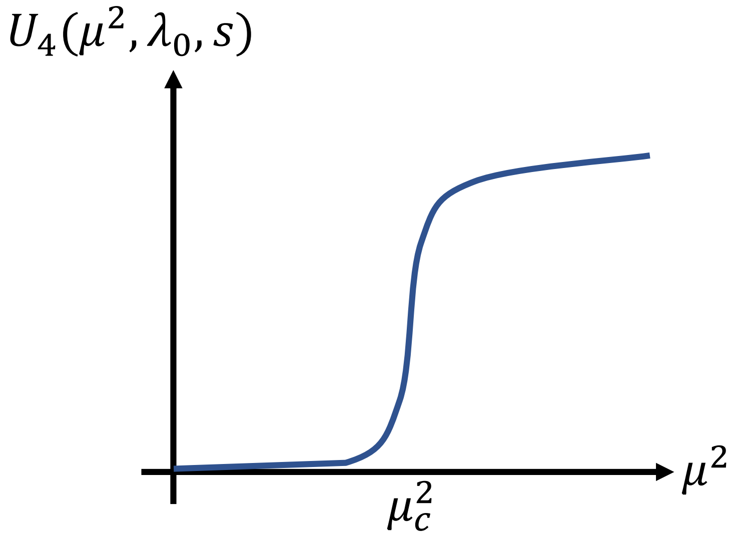
$$S = \sum_{\langle i,j \rangle} A_{ij} \frac{(\phi_i - \phi_j)^2}{l_{ij}^2} + \sum_i A_i\, \lambda \left(\phi_i^2 - \frac{\mu^2}{2\lambda}\right)^2$$

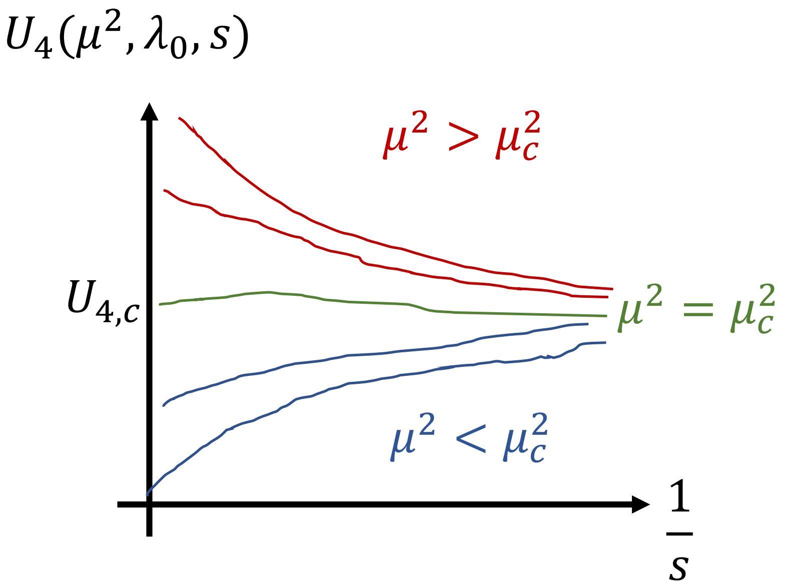
Binder Cumulant (Binder, K. 1981. Z. Physik B 43 119)

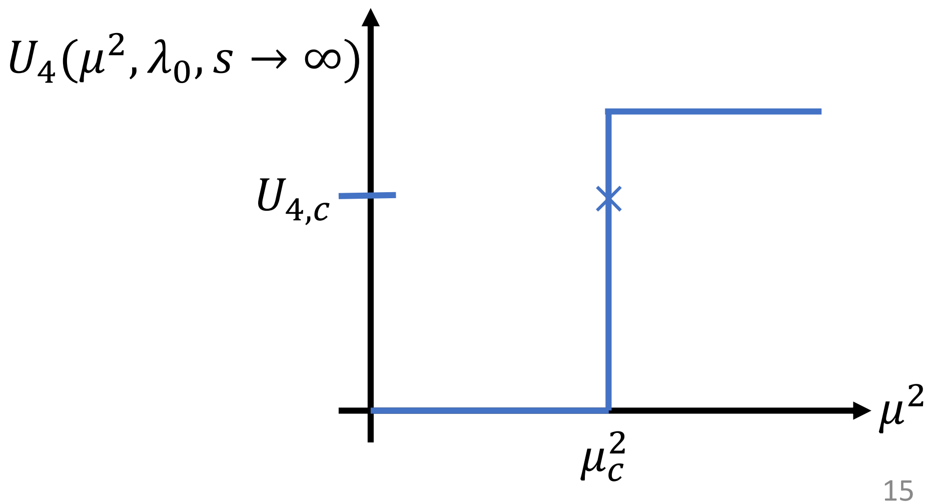
$$U_4(\mu, \lambda, s) = \frac{3}{2}\left(1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2}\right)$$

$$M = \sum_x w_x \phi_x$$

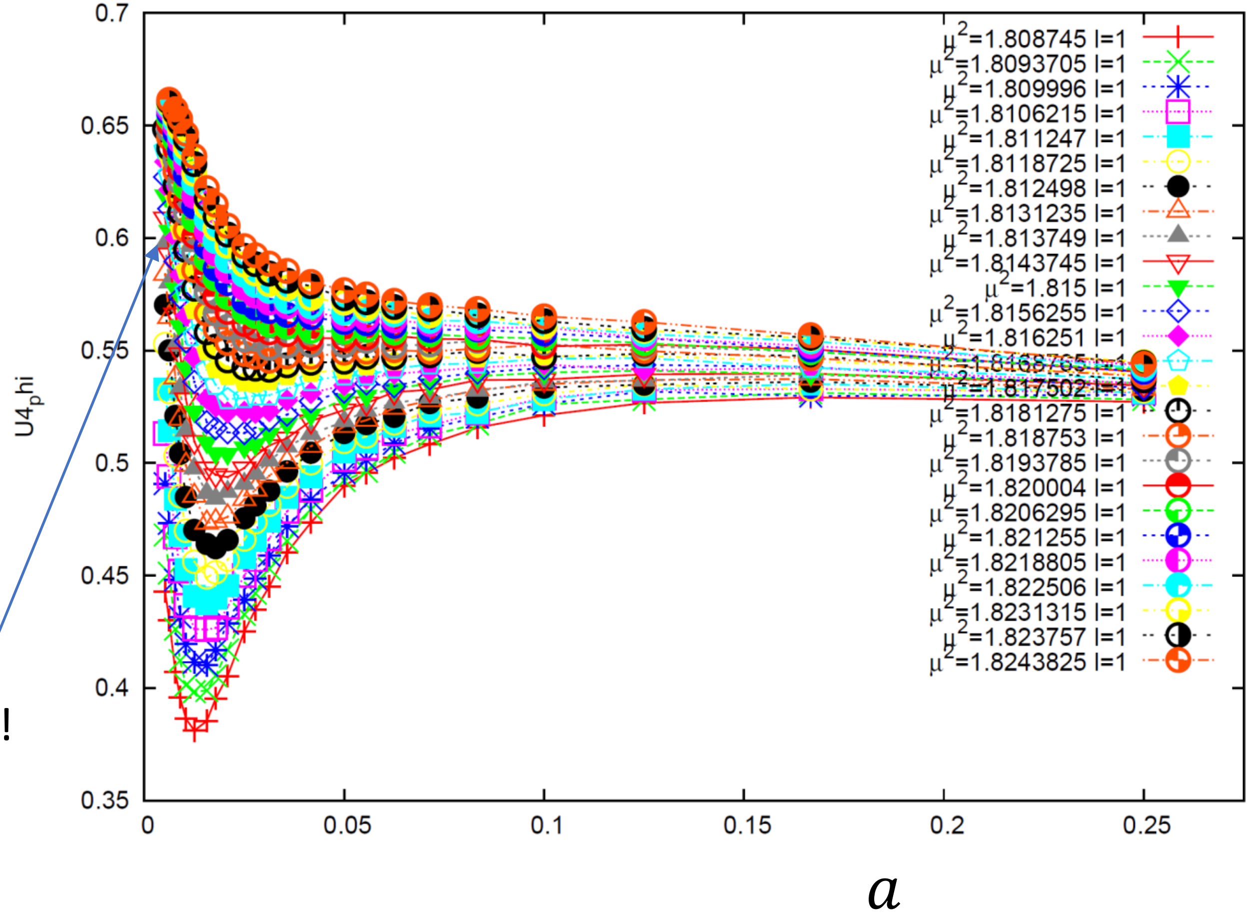
- Ordered phase, $U_4 = 1$
- Disordered phase, $U_4 = 0$







Obstruction to Criticality on S^2



Frustration!

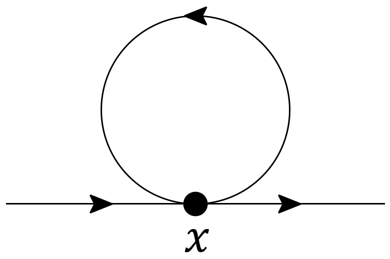
- Classical finite elements **fail** to converge to the quantum field theory
- Projection from icosahedron to sphere leads to distortion of areas even in continuum limit
  - Each vertex sees a different local UV cutoff
  - Quantum loops contribute to renormalized mass differently at different vertices
- **Low mode** distortions due to quantum loops
  - UV is always wrong on a lattice
  - But IR must be repaired!
- Essentially, we've chosen a bad scheme
  - Looks like we need to tune a volume number of couplings
  - Hopeless?

Quantum Corrections on a Curved Lattice

- General proof of renormalizability on curved lattice is hard
  - No translation symmetry, no Fourier techniques
  - No closed form for the propagator at finite lattice spacing
- Nonetheless, we propose a scheme which follows the spirit of the perturbative renormalization scheme of Reitz
- The scheme assumes the following
  1. Only divergent diagrams are sensitive to the lattice spacing in the deep UV, so only divergent diagrams remain position dependent as $a \to 0$
  2. The divergence is “universal” (the same at all positions)
- If (1) and (2) are true, then one only needs to add a **finite** position dependent counterterm to the FEM Laplacian to cancel the position dependence in the finite pieces of the UV divergent diagrams
- Then the divergence is removed as in usual lattice theory: either by explicit subtraction by a universal counterterm in perturbation theory, or nonperturbatively by tuning the universal bare mass to reach the critical surface
- We refer to this scheme as “quantum finite elements”

Quantum Corrections for ϕ^4 theory in d=2

- Only one UV divergent diagram!



$$\equiv I_1(x,m;a) = \overline{I_1}(m;a) + \left(I_1(x,m;a) - \overline{I_1}(m;a)\right)$$

$$\overline{I_1}(m;a) = \sum_x w_x I_1(x,m;a)$$

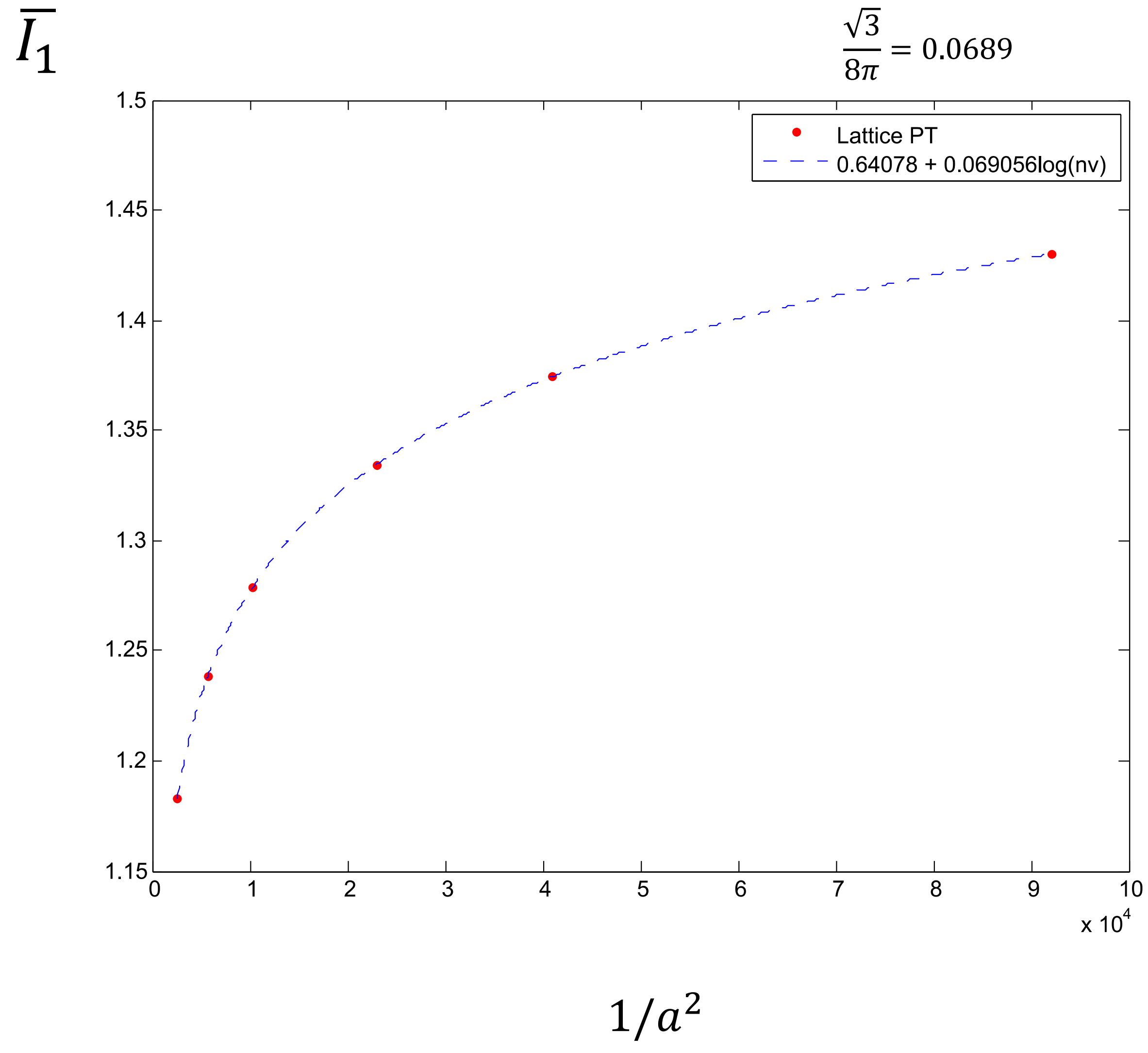
- $l = 0$ piece
- Expect divergence as a → 0

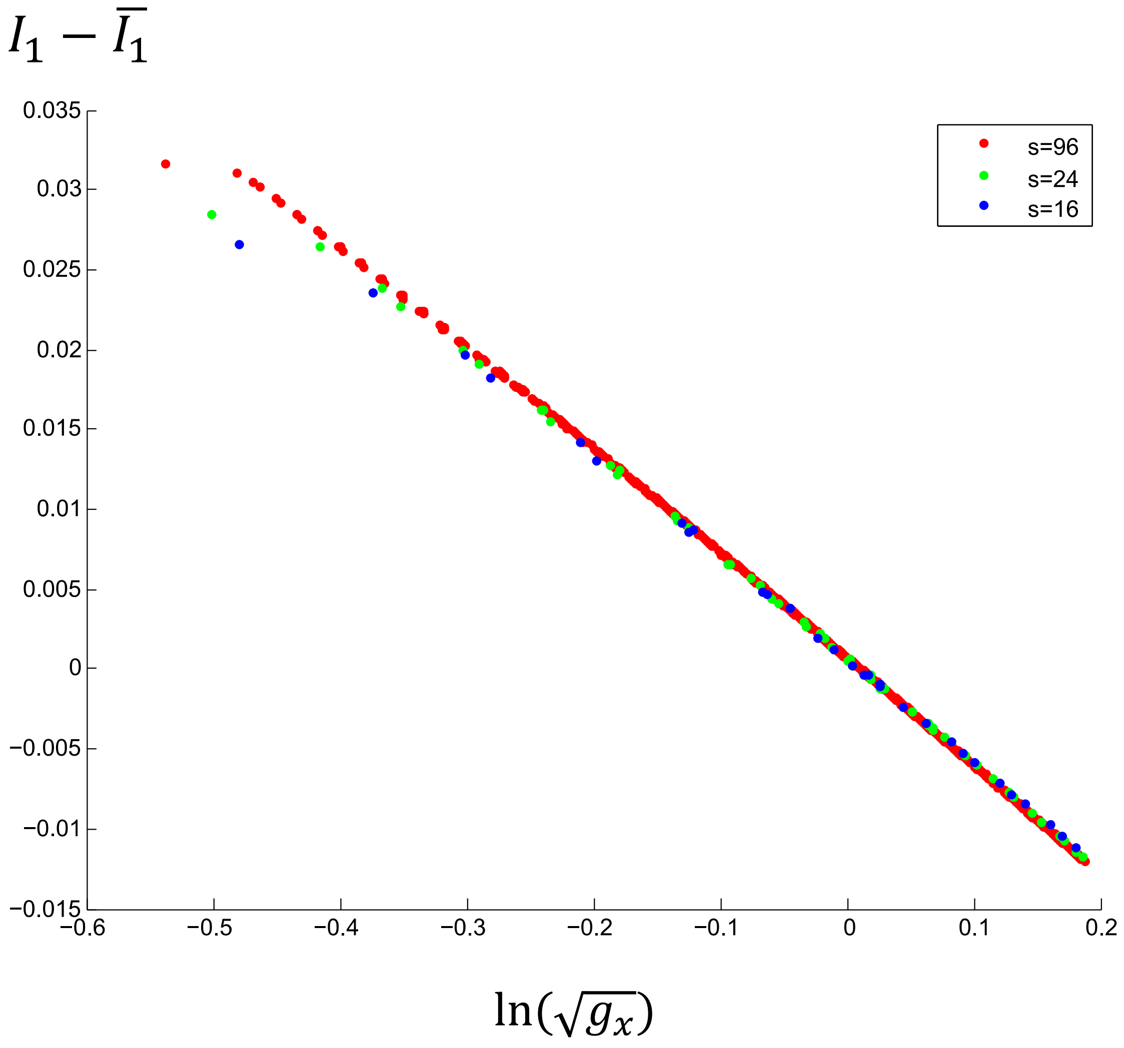
- If divergence universal, subtracted piece is finite!
- This becomes our counterterm
  - Finite function of x as a → 0

- Diagram is simply diagonal piece of inverse of FEM kinetic term

$$I_1\left(x,m;\frac{1}{s}\right) = \left[\left(\nabla_{FEM}^2 + m^2\right)^{-1}\right]_{xx}$$

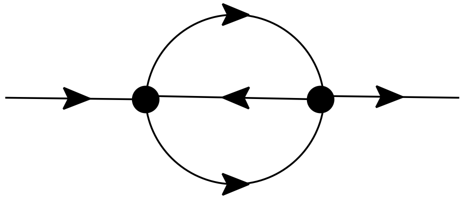
Quantum Corrections for ϕ^4 theory in d=2

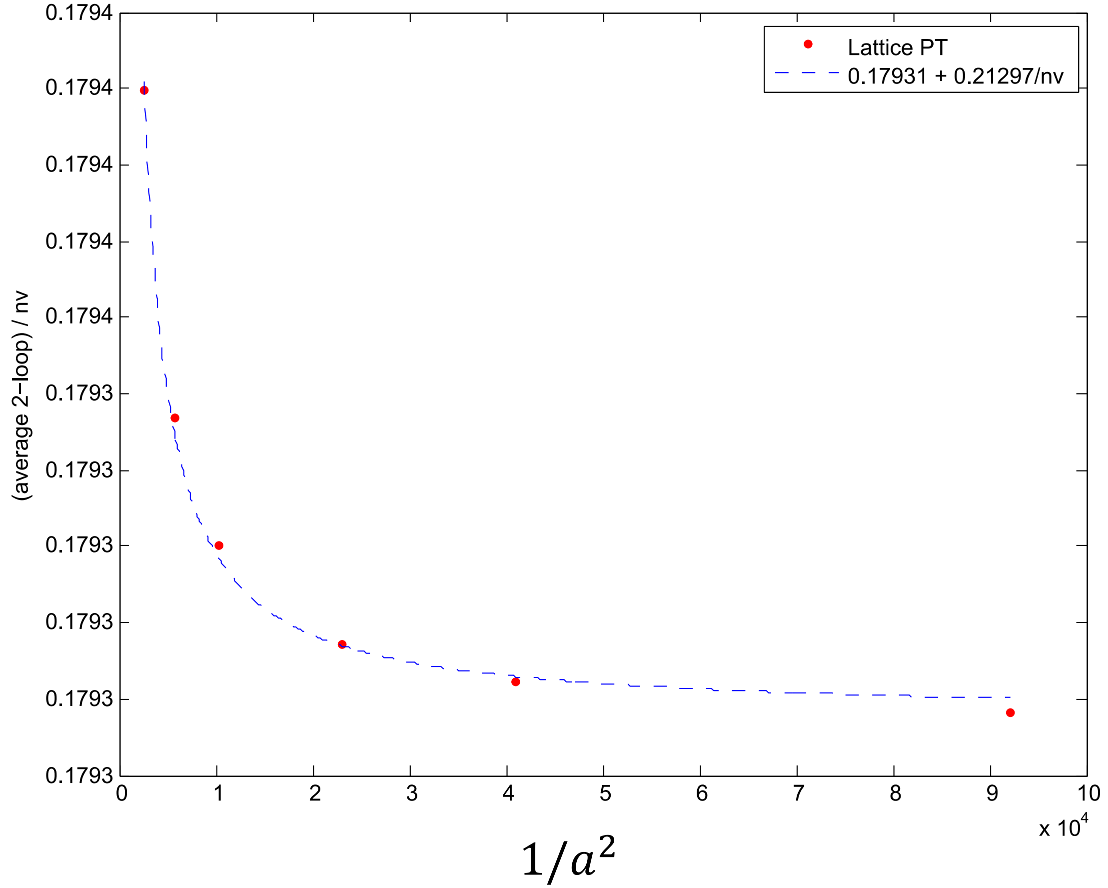


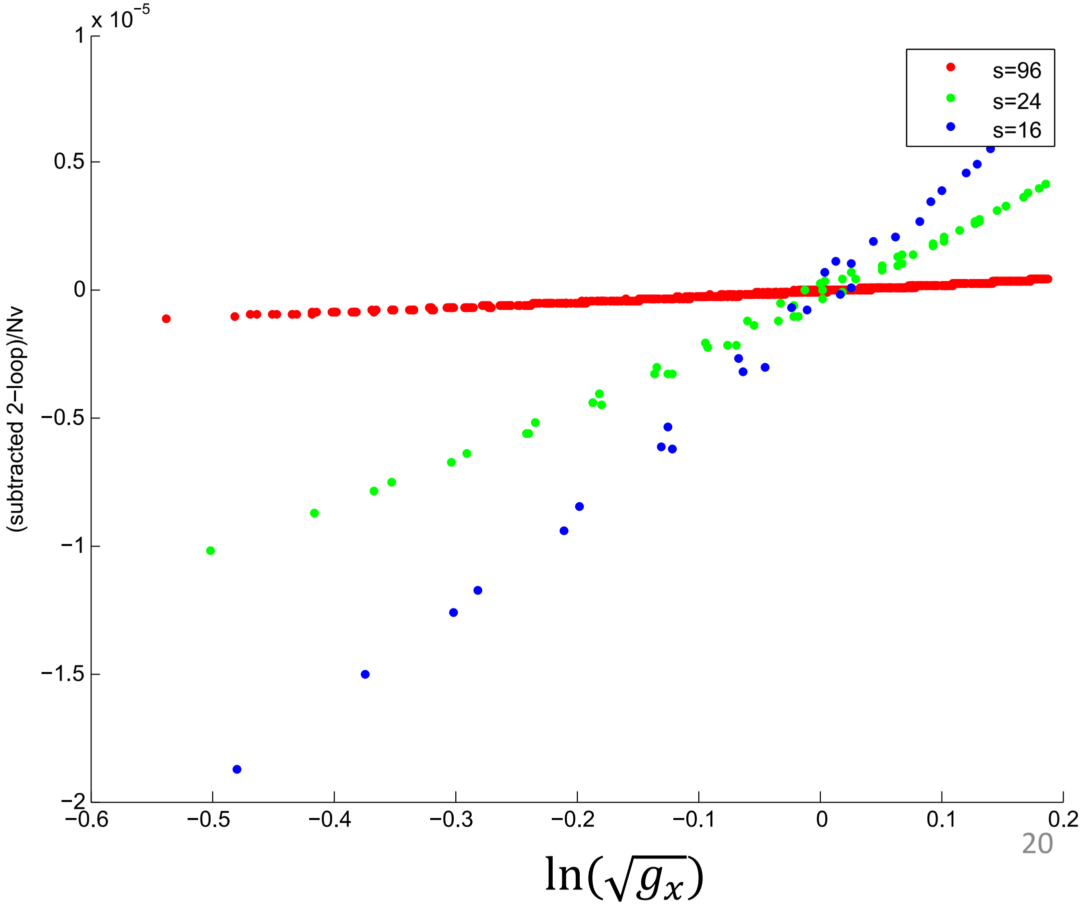


Quantum Corrections for ϕ^4 theory in d=2

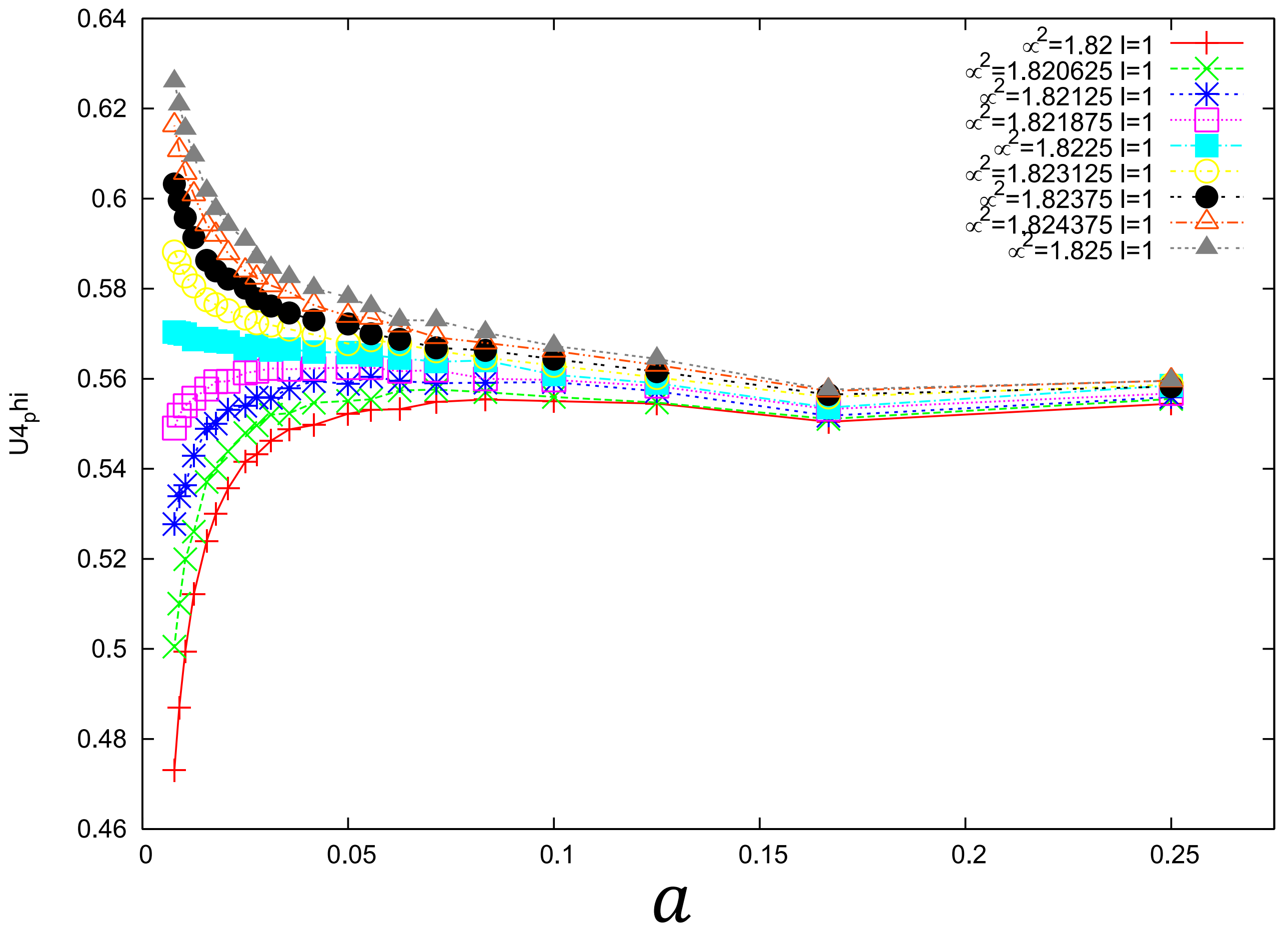
- Look at first convergent diagram, two loops

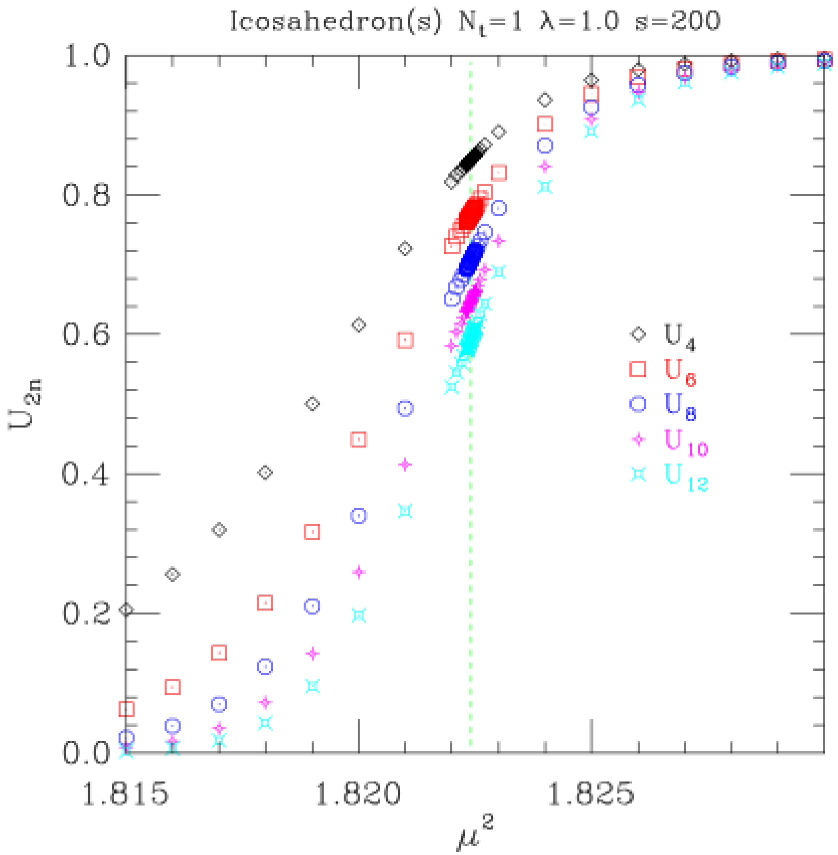






Criticality on S^2 with quantum finite elements

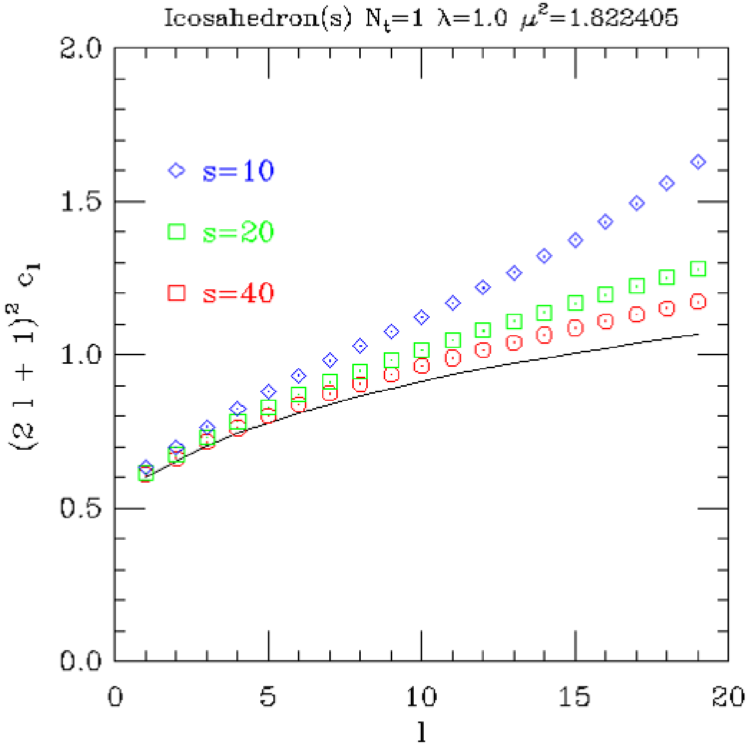


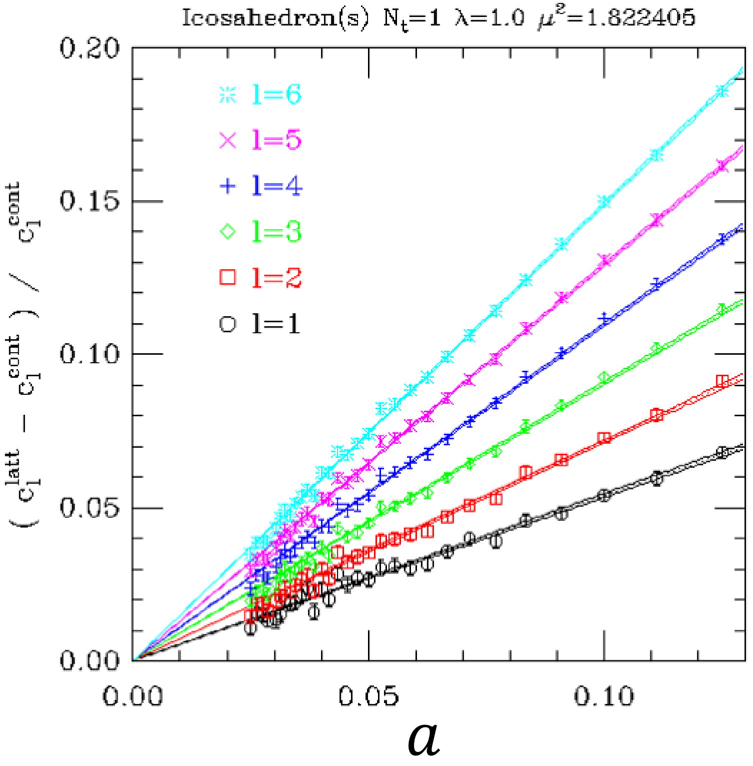


Critical 2-point function

$$\langle\phi(x)\phi(y)\rangle_{crit} = \langle\sigma(x)\sigma(y)\rangle_{ising} = \frac{C_\phi}{|x-y|^{\frac{1}{4}}} \to \frac{2\,C_\phi}{(1-\cos(\theta_{xy}))^{\frac{1}{8}}}$$

$$\langle\phi(x)\phi(y)\rangle_{crit} = \sum_l c_l P_l(\cos(\theta))$$





Critial 4-point function

$$G(z) = G(u,v) = \frac{\langle\sigma(1)\sigma(2)\sigma(3)\sigma(4)\rangle}{\langle\sigma(1)\sigma(3)\rangle\langle\sigma(2)\sigma(4)\rangle}$$

Any number of dimensions, only two real conformal invariants

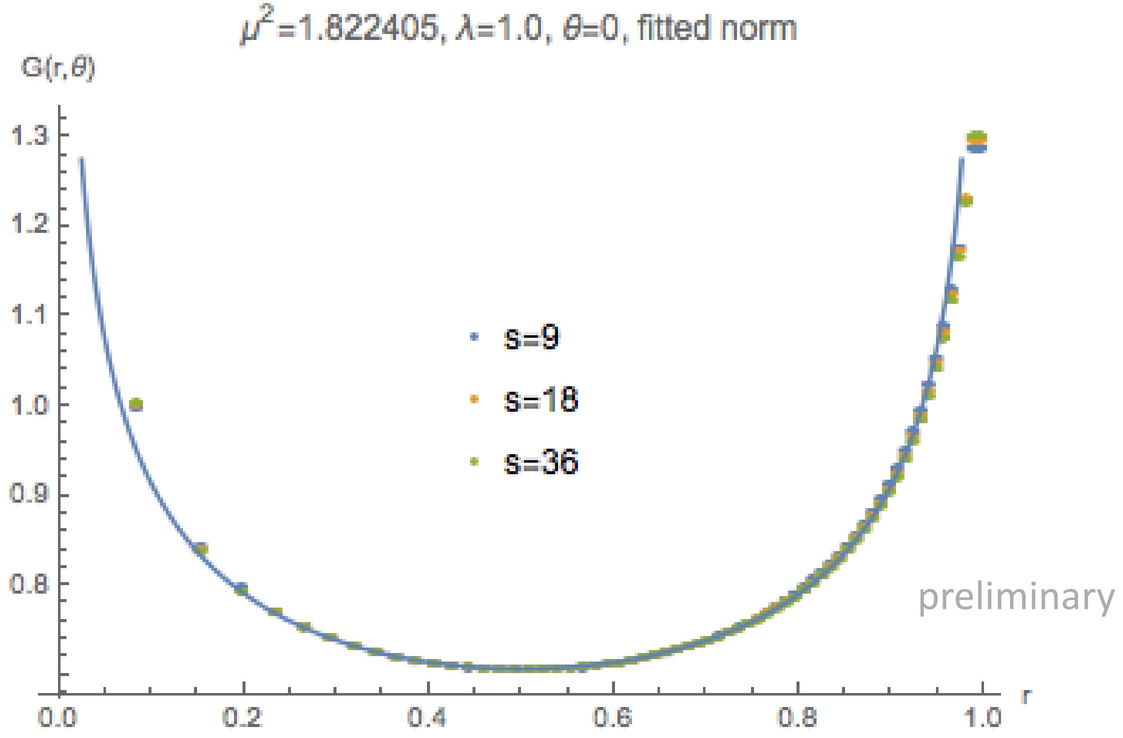
$$u = \frac{r_{12}^2 r_{34}^2}{r_{13}^2 r_{24}^2}, \qquad v = \frac{r_{14}^2 r_{23}^2}{r_{13}^2 r_{24}^2}$$

$$u = z\bar{z}, \qquad v = (1-z)(1-\bar{z})$$

$$\mathrm{G}(z) = \frac{1}{\sqrt{2}|z|^{\frac{1}{4}}|1-z|^{\frac{1}{4}}}\left[\left|1+\sqrt{1-z}\right| + \left|1-\sqrt{1-z}\right|\right]$$

Polar coordinates: $z = re^{i\theta}$

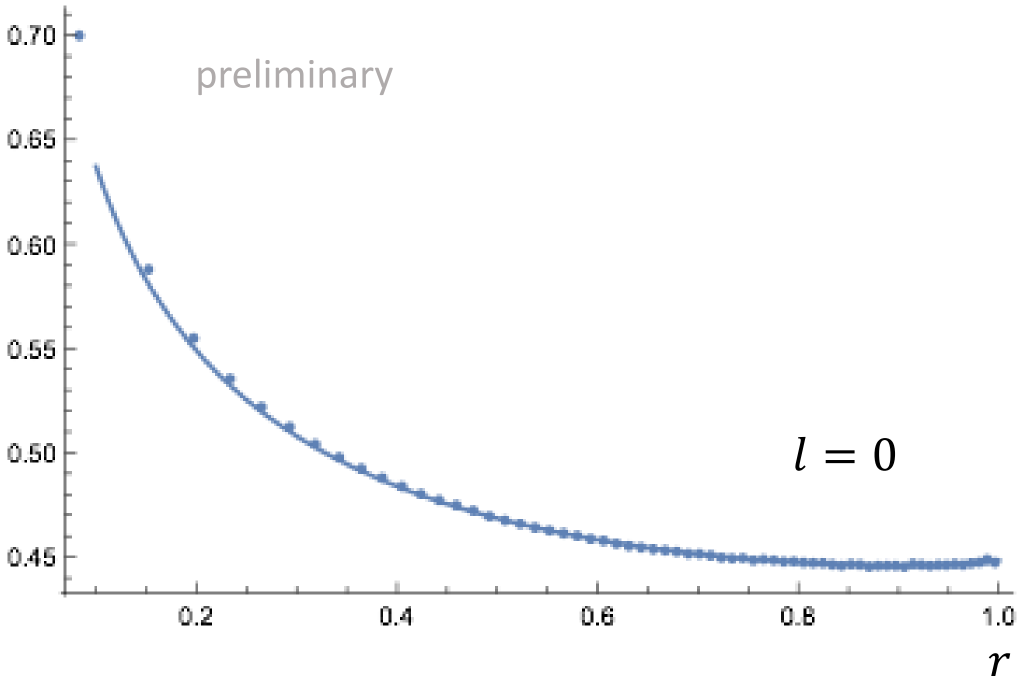
$$G(r,\theta) = \frac{1}{2}\frac{\sqrt{1+r+\sqrt{1+r^2-2r\cos(\theta)}}}{\sqrt{2}r^{\frac{1}{4}}(1+r^2-2r\cos(\theta))}$$

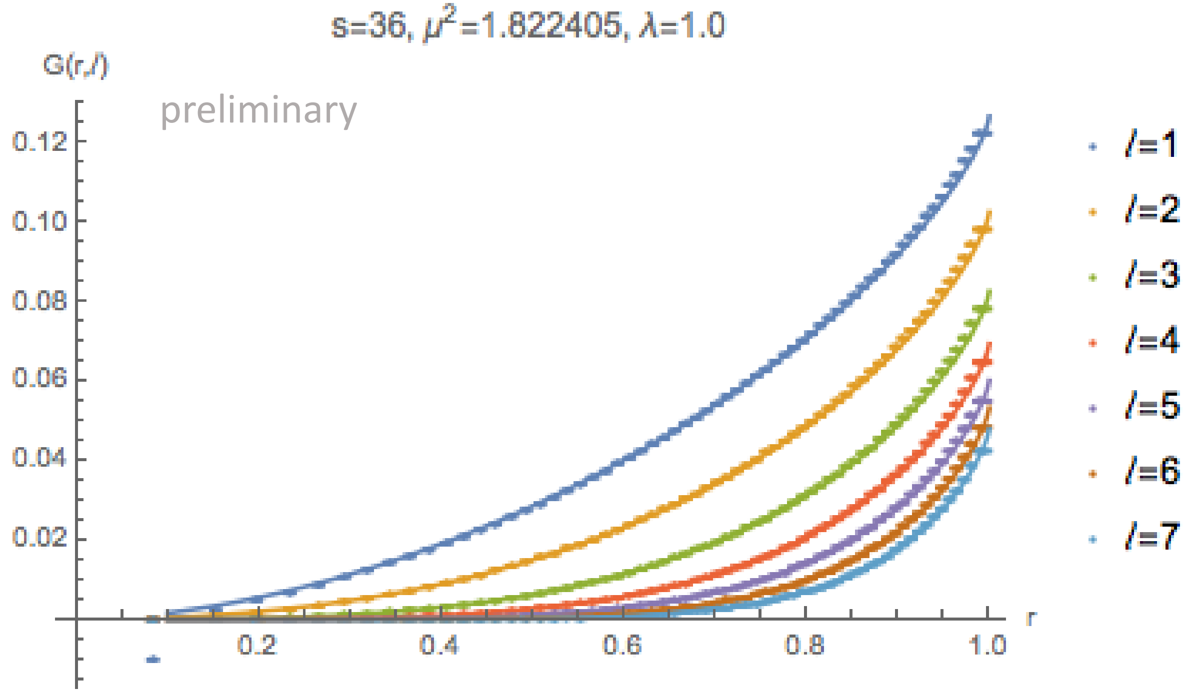


Critial 4-point function

$$G(r,\theta) = \sum_l g_l(r)\cos(l\theta)$$

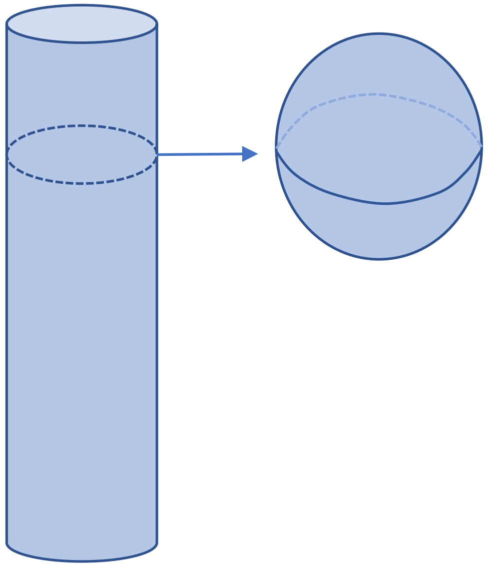
Zero free parameters! Not fits!





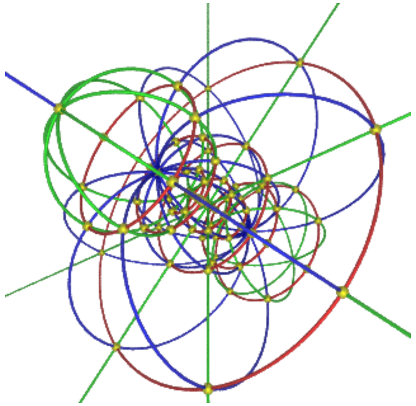
Ongoing and future work

Radial Lattice Quantization in 3d

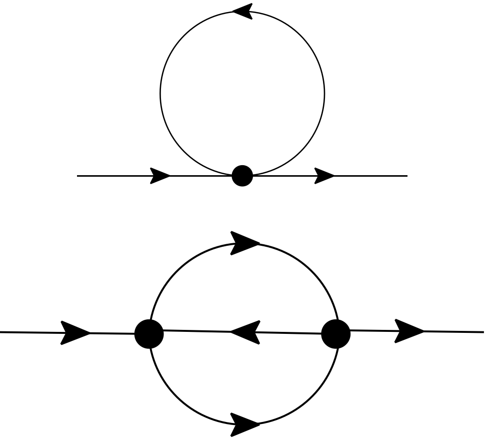


Radial quantization in 4d

3-sphere
(stereographic
Projection of geodesics)

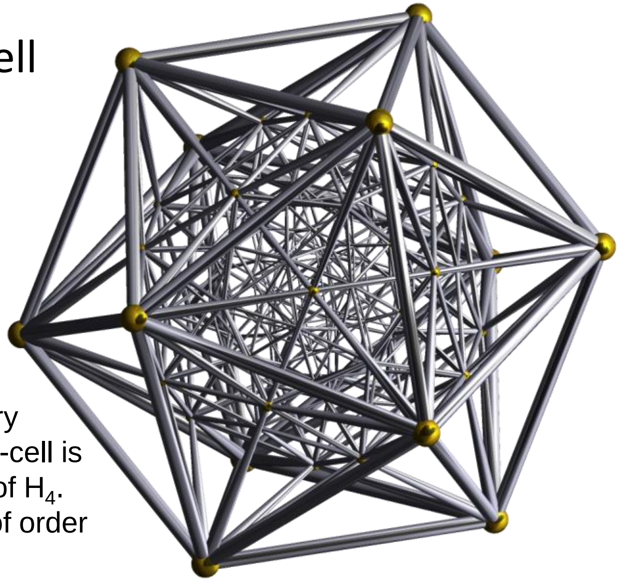


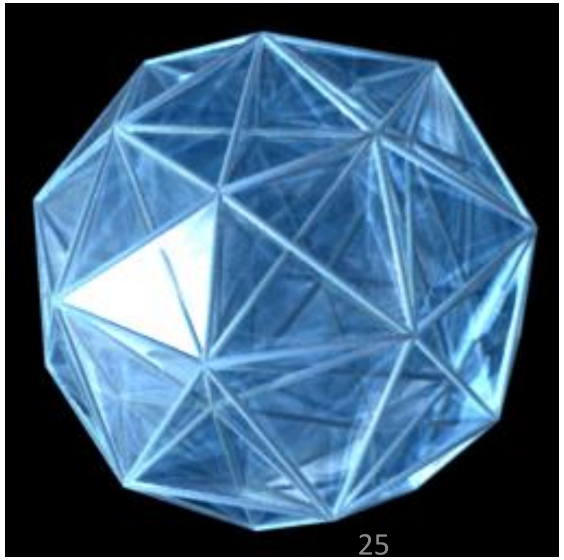
3D Ising Model



CFTs in condensed matter
e.g. Quantum criticality

600-cell





The full symmetry group of the 600-cell is the Weyl group of H_4. This is a group of order 14400