Lattice QCD and hadron structure

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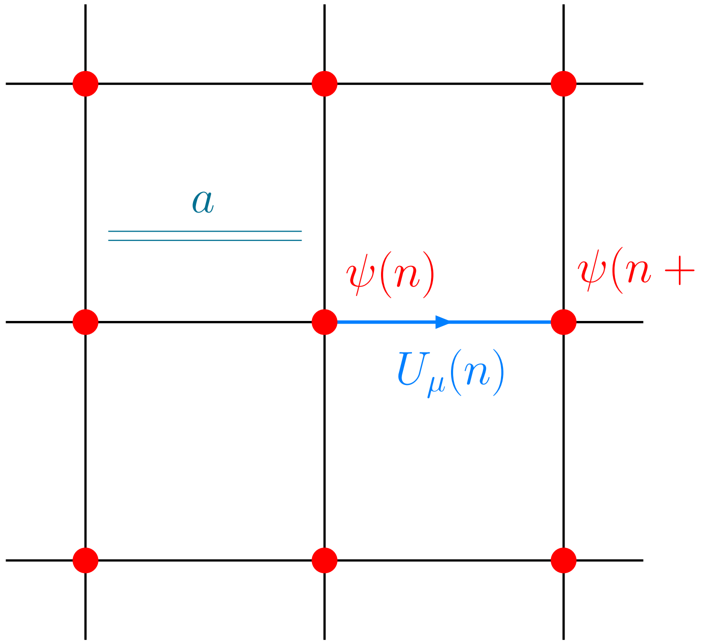
Preliminaries

- Organizers asked me to review the progress of Lattice QCD and partonic structure of nucleons
- This is a review from a person relatively away from the recent development of PDF computation
- For helping me out, I thank these experts
  - Tomomi Ishikawa
  - Takashi Kaneko
  - Huey-wen Lin
  - Shoichi Sasaki
  - Sergey Syritsyn
- This is not meant for comprehensive review of the field. But, rather picking my like and highlighting some recent developments and related matters.

items

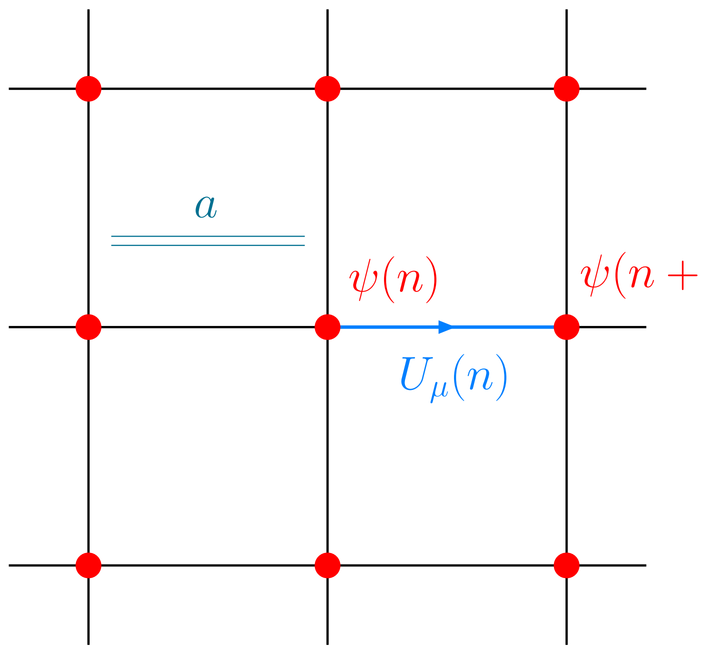
- Lattice QCD basics
- Lattice QCD low energy example for intro
- Toward PDF

QCD and Lattice QCD



QCD and Lattice QCD

- Lattice QCD = QCD defined on discretized **Euclidian** space-time
  - discreteness: lattice spacing = a ($\sim$*0.1 fm* $\sim$ *(2 GeV)*$^{-1}$)
    - eventually continuum limit: $a \rightarrow 0$ needed



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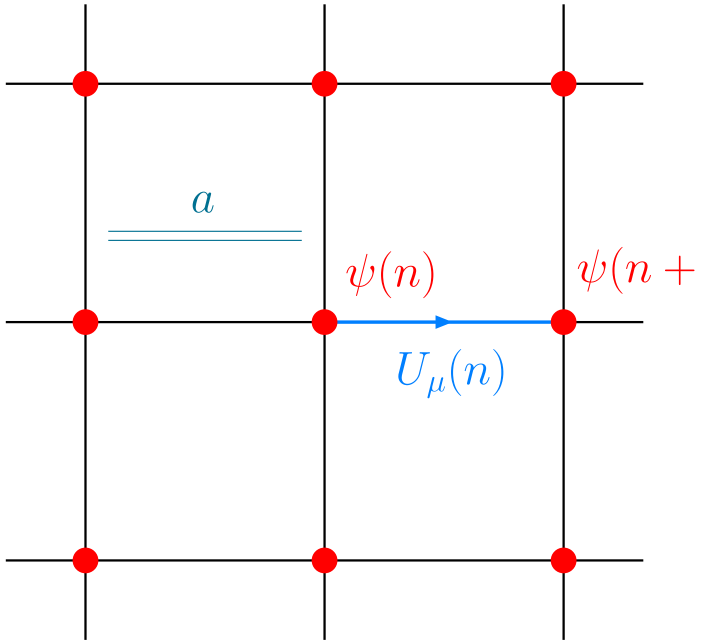

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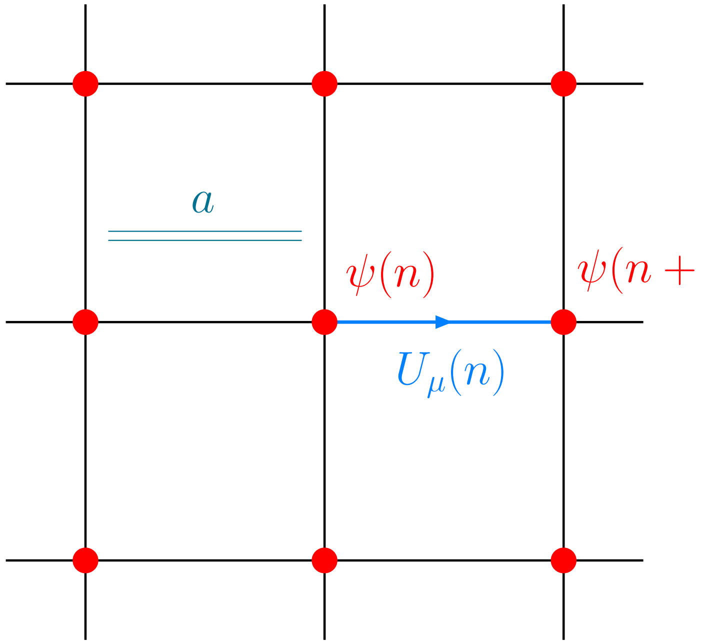
  - $Z = \Sigma \exp(-S)$ $\to$ Monte Carlo simulation



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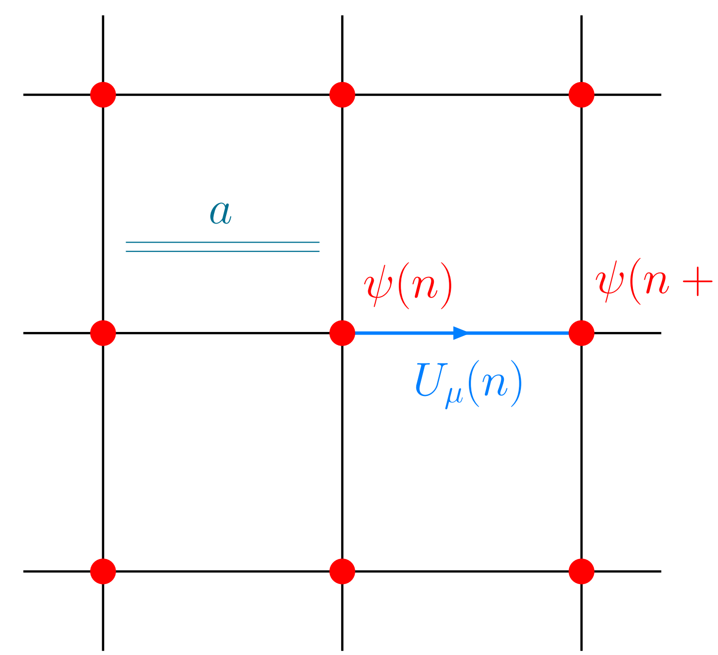

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  - chiral: partially or completely lost
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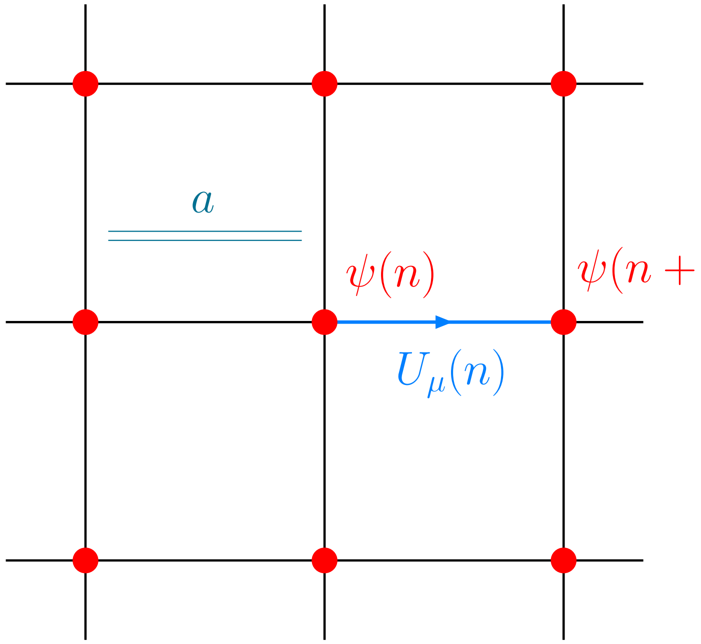

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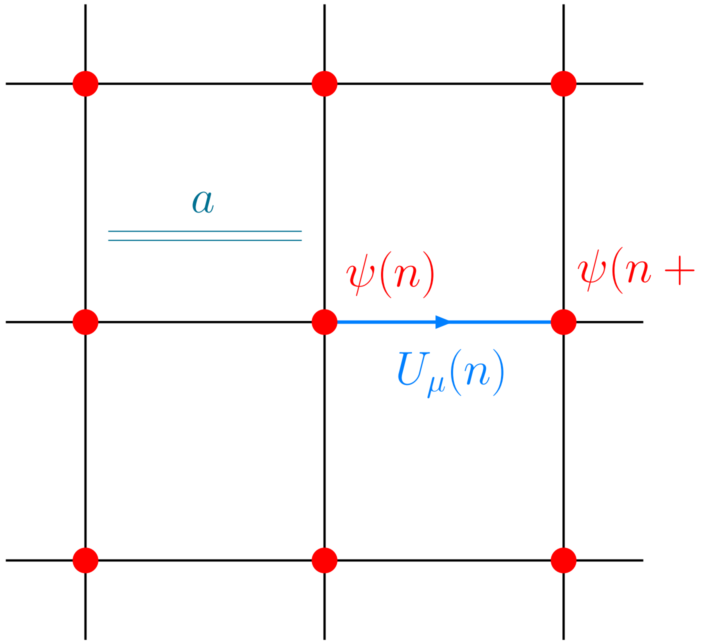
- exact symmetry
  - gauge !
  - "chiral" for special discretization
    - (close to) exact chiral symmetry crucial for some applications



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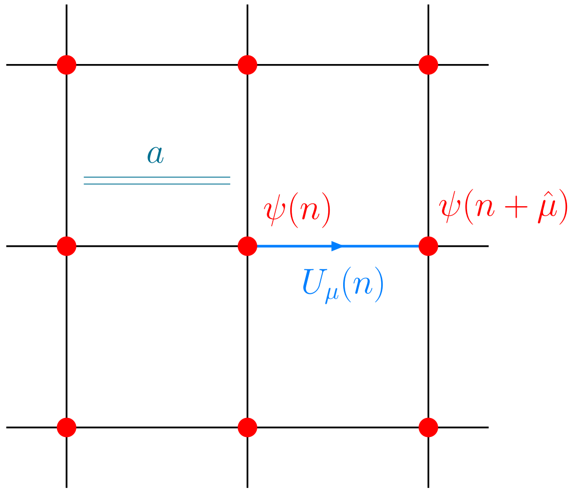

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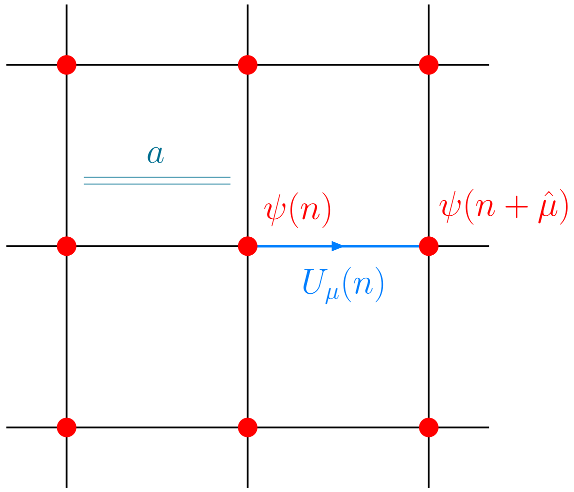

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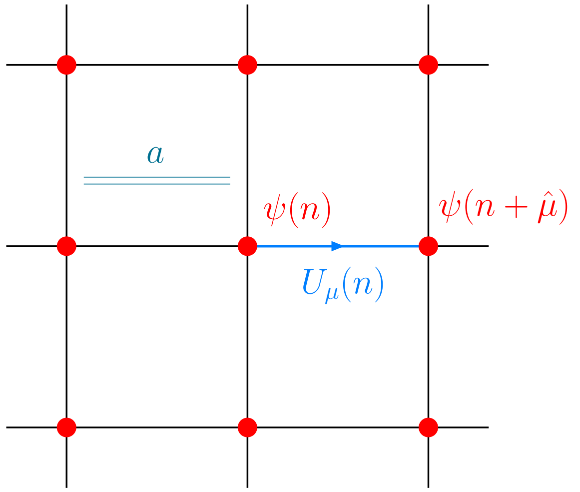

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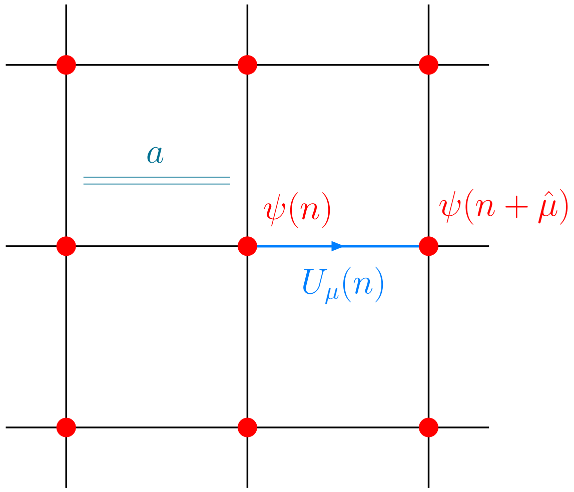
- **near the continuum limit**
  - lattice operators can be expanded in powers of a

$$\mathcal{O}|_{LQCD} = \mathcal{O}|_{QCD} + ac_1\mathcal{O}_1 + a^2c_2\mathcal{O}_2 \ldots$$

QCD and Lattice QCD



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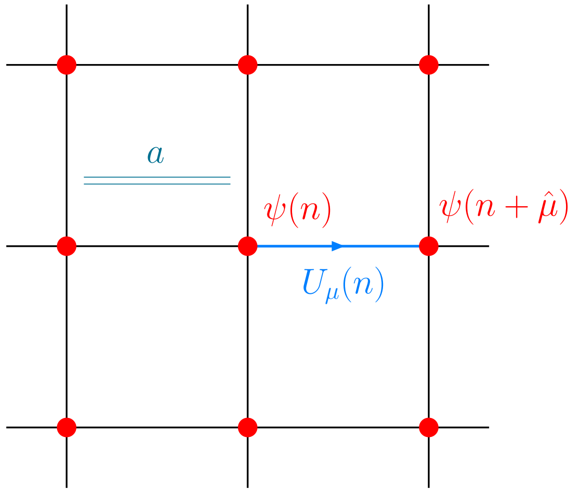
QCD and Lattice QCD



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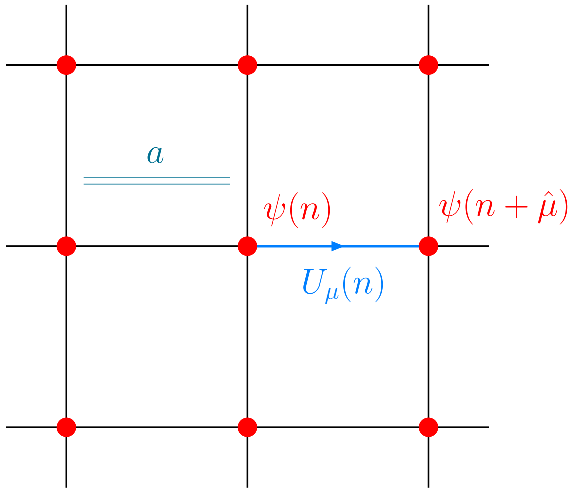
- for some operators in some lattice discretizations
  - $c_1 = 0$ automatically → effectively close to cont. lim.
  - $c_1 = 0$ by engineering = "improvements"
- most of the lattice actions used now → $c_1 = 0$ or $c_1 \simeq 0$
  - *this applies to those used for PDF computation discussed later*

Lattice QCD practicality



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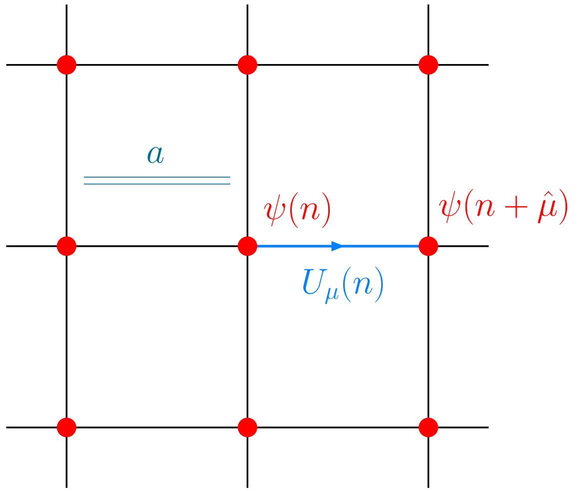

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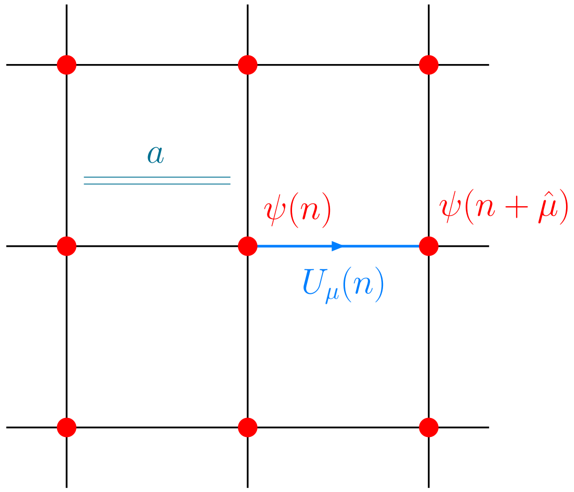

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    - excited states difficult
  - hadronic matrix elements of local operators
    - multi hadron difficult

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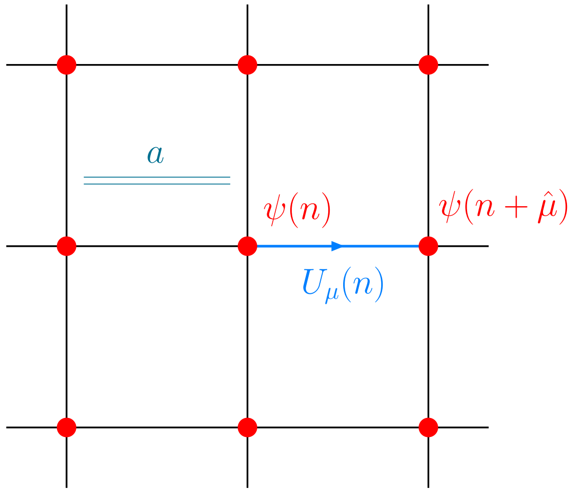

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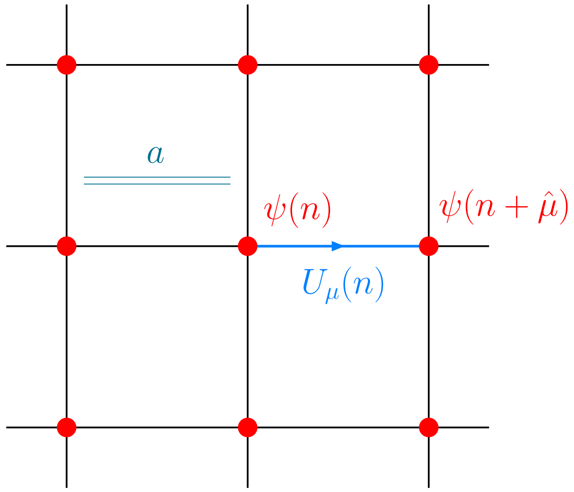
- difficult on the lattice
    - scattering phase shift (direct measurements impossible)
        - use relation with finite V energy a la Luescher
    - action with imaginary part: ex: chemical potential
        - sign problem
    - quantity defined with object on the ** light cone **

Lattice QCD practicality



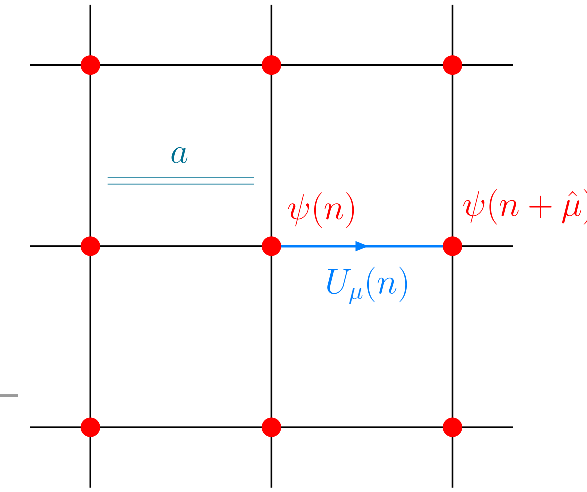
- Lattice QCD = QCD defined on discretized **Euclidian** space-time
- parameters other than a and V= L_s^3 x L_t
    - m_q : **quark masses**

- computational costs ~ $a^{-\alpha}\, V^{\beta}\, m_q^{-\gamma}$: $\alpha, \beta, \gamma > 0$
    - for long time **physical** u, d **mass** simulation was not possible !
    - this situation has been changed recently
        - due to advanced algorithms and computer power

- yet other parameters
    - p_μ injected momentum needs to be small: $p_\mu \ll 1/a$

$$\langle \mathcal{O}(p) \rangle |_{LQCD} = (1 + cp^2 a^2 + \dots) \langle \mathcal{O}(p) \rangle |_{QCD} + \dots$$

- *also applies to renormalization scale if momentum is used*
    - *matching to perturbation requires "window": $\Lambda_{QCD} \ll p_\mu \ll 1/a$*



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    - *also*
        - *matching to perturbation requires "window": $\Lambda_{QCD} \ll p_\mu \ll 1/a$*

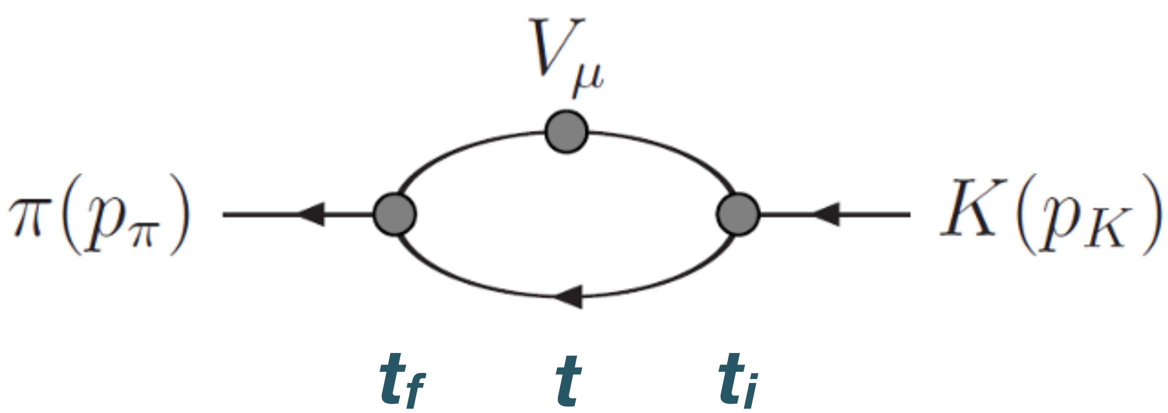
ideal procedure of computation

1. do computation with finite V, m_q, a
2. $V \to \infty$
3. $m_q \to$ physical value extrapolation
4. $a \to 0$

now 3 is becoming unnecessary

form factor calculation: one example

- Kaon semi-leptonic decay
  - $K \to \pi + l + \nu$



$$C^{3pt}_{K\pi}(t_i, t, t_f; \vec{p}_i, \vec{p}_f) = \sum_{\vec{x}_f, \vec{x}} e^{i\vec{p}_f \cdot (\vec{x}_f - \vec{x})} e^{i\vec{p}_i \cdot \vec{x}} \langle J_\pi(t_f, \vec{x}_f) V_\mu(t, \vec{x}) J^\dagger_K(t_i, \vec{0}) \rangle$$

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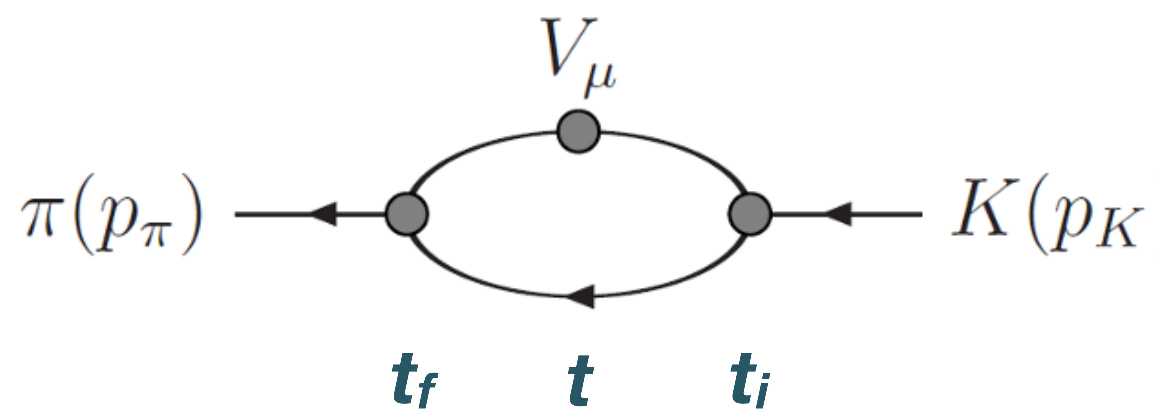

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$$1 = \sum_j |j\rangle \frac{1}{2E_j} \langle j|$$

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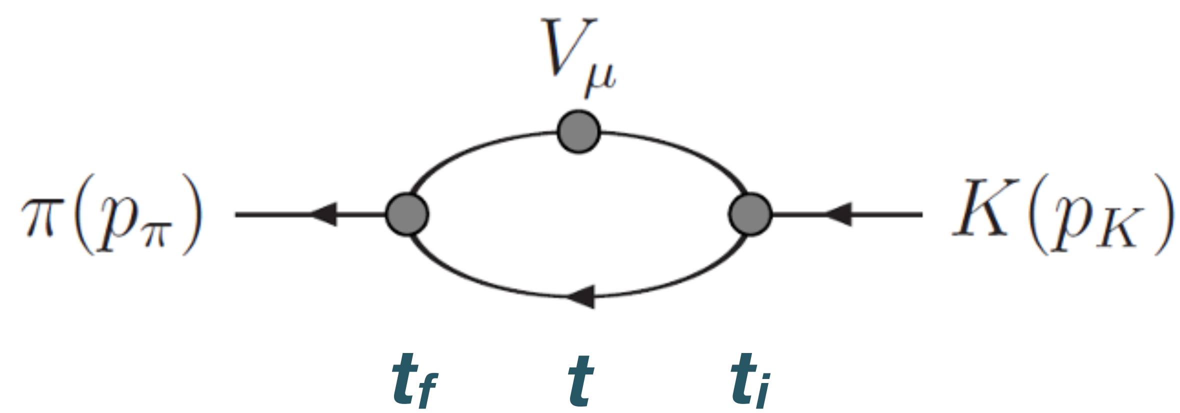

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$$= \sum_{i,f} \frac{Z_i Z_f}{4E_i E_f} \langle \pi_f(\vec{p}_f)|V_\mu(0)|K_i(\vec{p}_i)\rangle e^{-E_i(t-t_i)} e^{-E_f(t_f - t)}$$

$$Z_i = \langle K_i | J_K(0, \vec{0}) | 0 \rangle$$
$$Z_f = \langle 0 | J^\dagger_\pi(0, \vec{0}) | \pi_f \rangle$$

K_i / π_f: includes excited state

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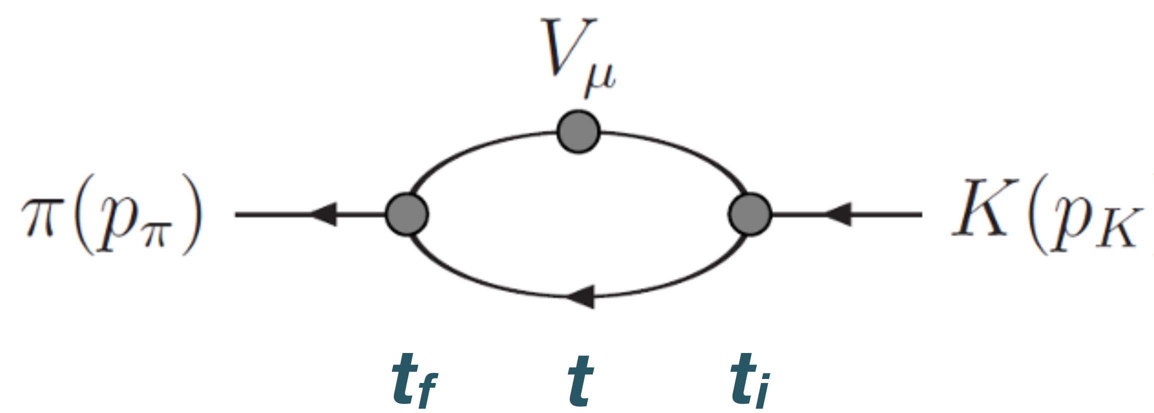

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K_i / π_f: includes excited state

exited states decay faster with Euclidian time difference

*for large (t_f-t) and (t-t_i), only **ground states** (K_i=**K** & π_f=**π**) contribute*

$$\to \frac{Z_K Z_\pi}{4E_K E_\pi} \langle \pi(\vec{p}_\pi)|V_\mu(0)|K(\vec{p}_i)\rangle e^{-E_K(t-t_i)} e^{-E_\pi(t_f-t)}$$

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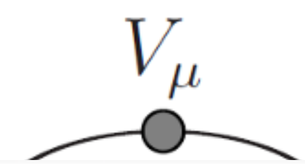

In general, multi-quark operators need to be renormalized.

There are ton's of works for renormalization perturbatively/
non-perturbatively on the lattice, that makes it possible
to reliably estimate various hadronic matrix element.

Such a legacy technique is used for quasi-PDF in a bit involved way.

(Here V_μ is usually automatically renormalized, due to lattice symmetry.)

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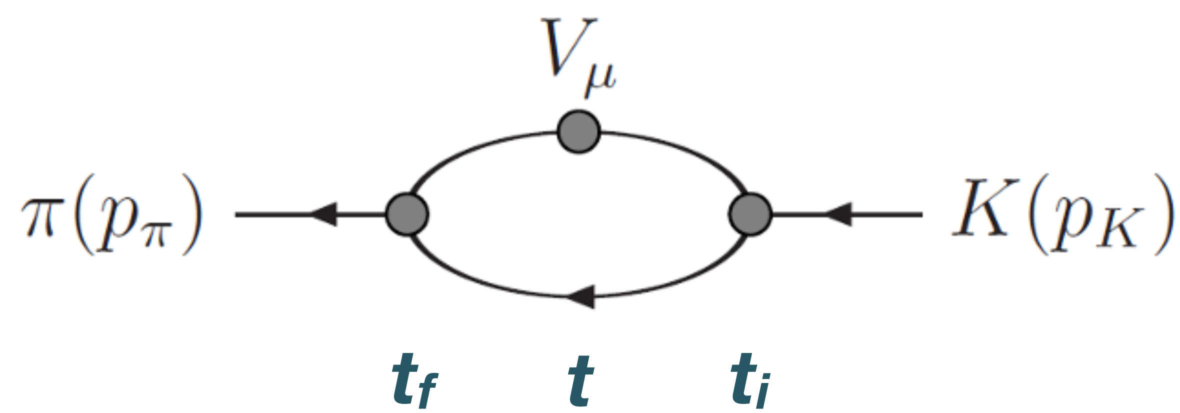


$$\langle \pi(p_f) | V_\mu(0) | K(p_i) \rangle = f_+(q^2)(p_f + p_i) + f_-(q^2)(p_f - p_i)$$

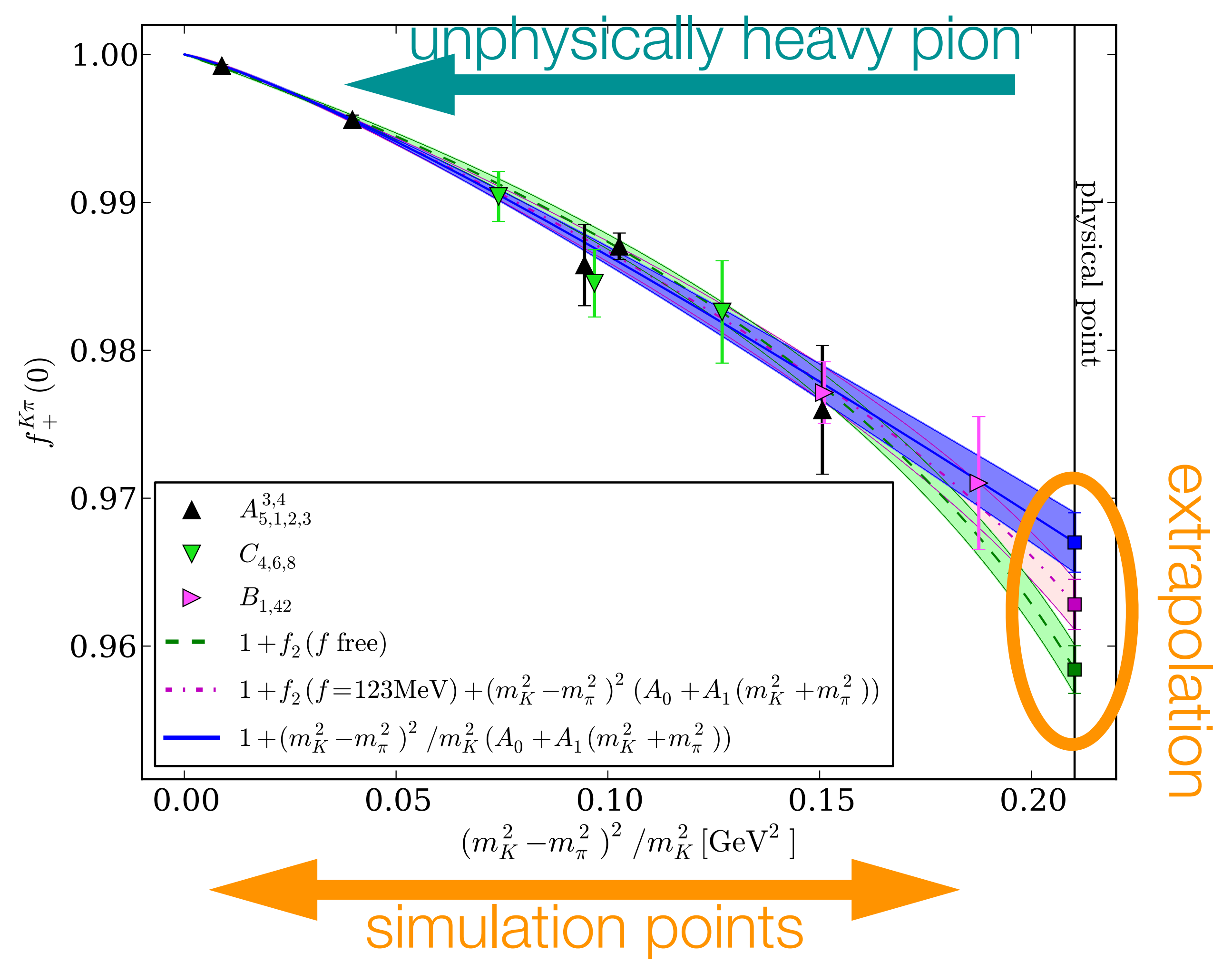
a CKM matrix element V_{us} is obtained from

$$V_{us} f_+(0) = 0.2165(4)$$

average of [$K_{L\ e3}$, $K_{L\ \mu3}$, $K_{S\ e3}$, $K_{\pm\ e3}$, and $K_{\pm\ \mu3}$] by Mouslon 2014

form factor calculation: one example
with unphysical ud mass simulation

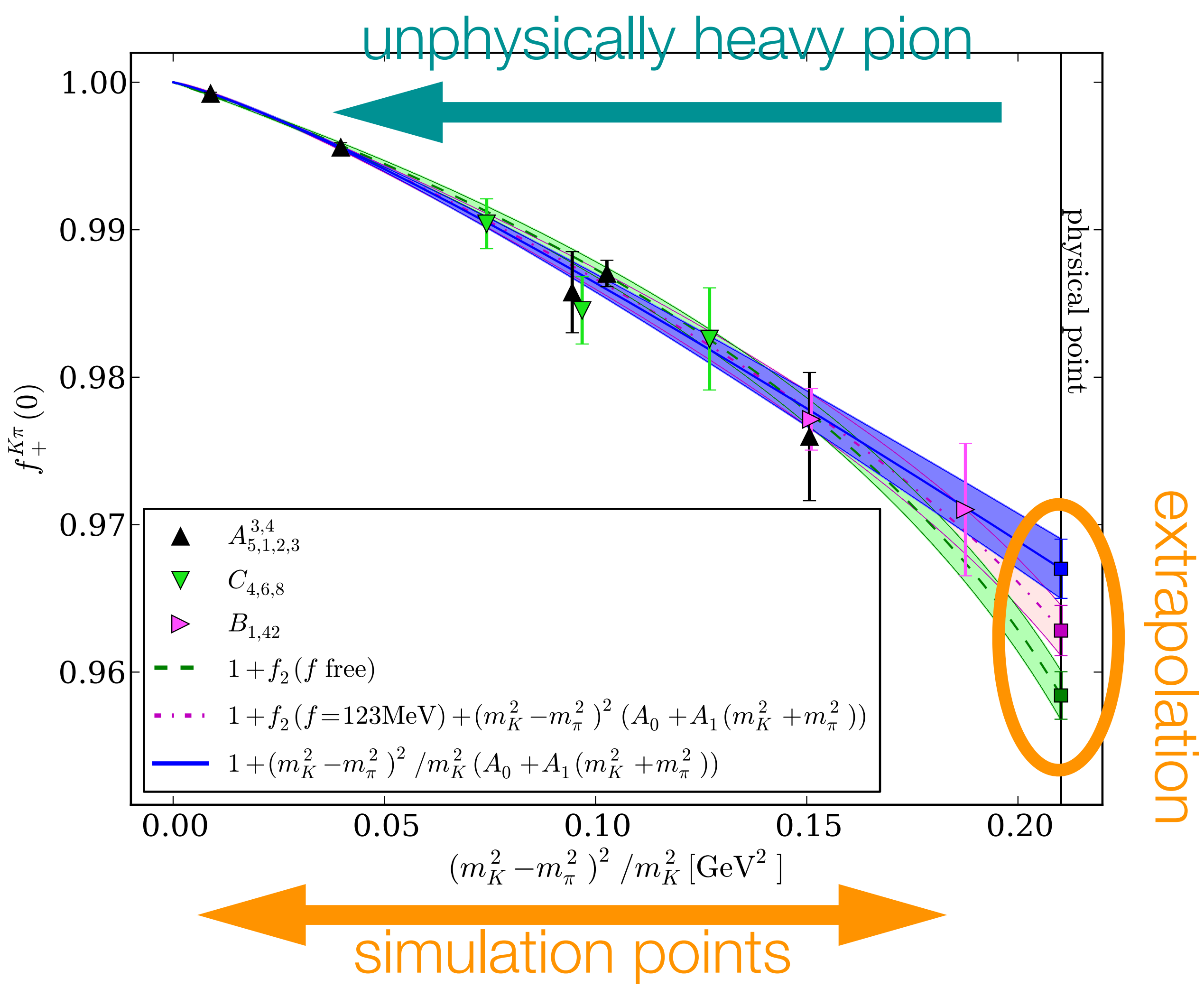
RBC/ULQCD 2013

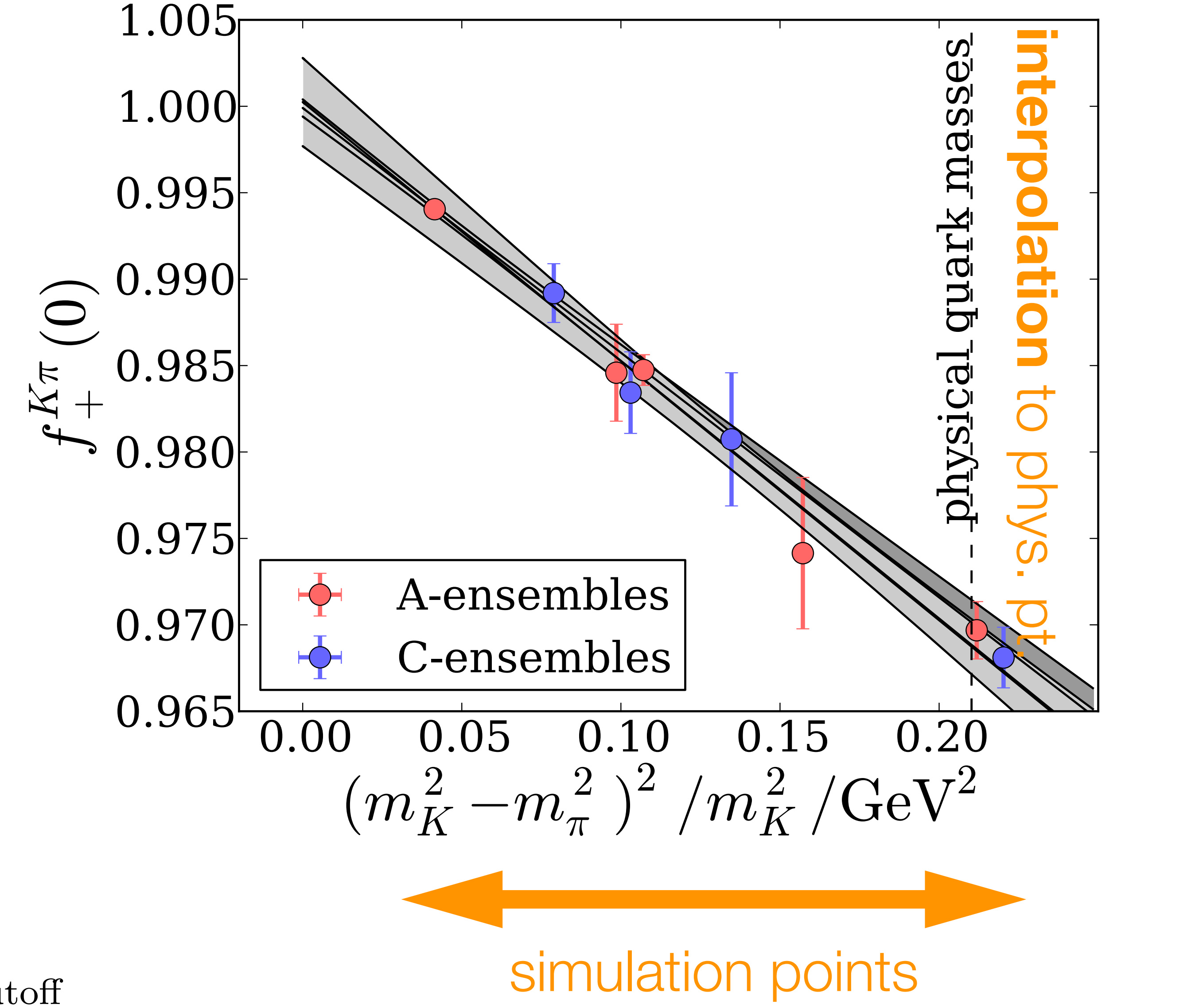


form factor calculation: one example
one big step forward: physical ud mass simulation

RBC/ULQCD 2013

RBC/ULQCD 2015





$$f_+^{K\pi}(0) = 0.9670(20)_{\text{stat}}\left(^{+\ 0}_{-42}\right)_{\text{model}}(7)_{\text{FSE}}(17)_{\text{cutoff}}$$

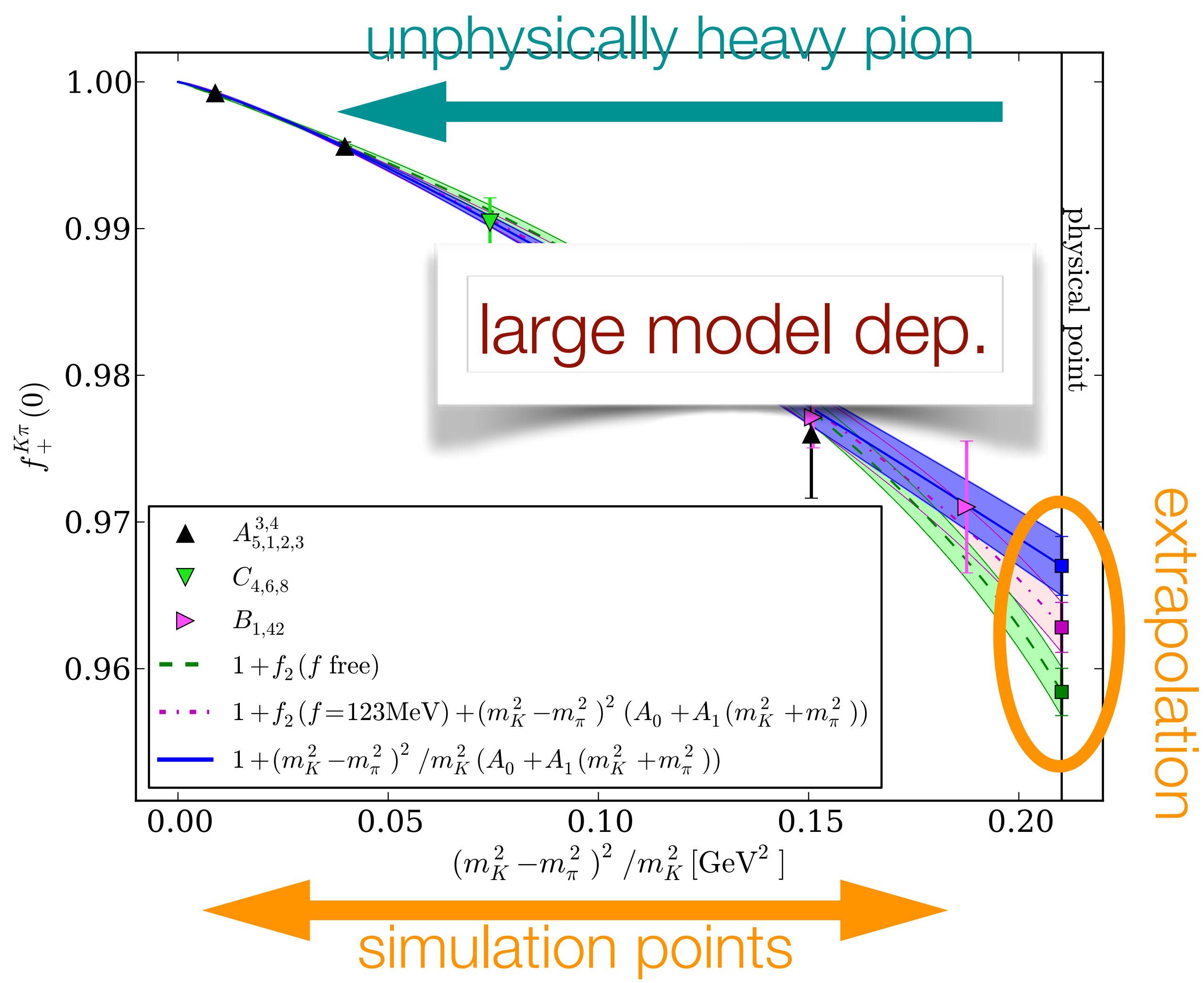
$$\qquad 0.2\% \qquad 0.4\% \qquad 0.07\% \qquad 0.2\%$$

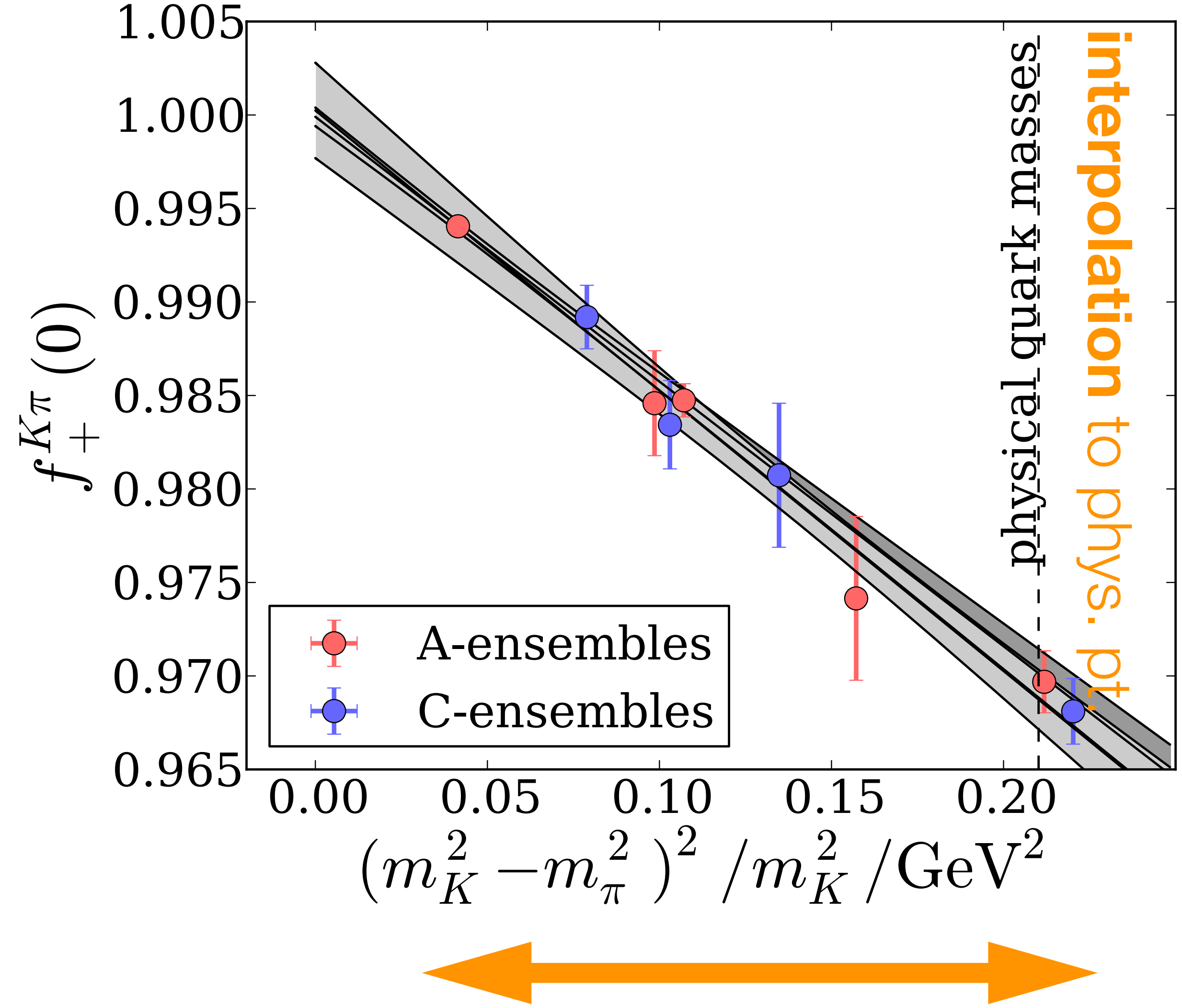
$$= 0.9670(20)\left(^{+18}_{-46}\right),$$

$$f_+^{K\pi}(0) = 0.9685(34)_{\text{stat}}(14)_{\text{finite volume}}\,.$$

form factor calculation: one example
one big step forward: physical ud mass simulation

RBC/ULQCD 2013 → RBC/ULQCD 2015





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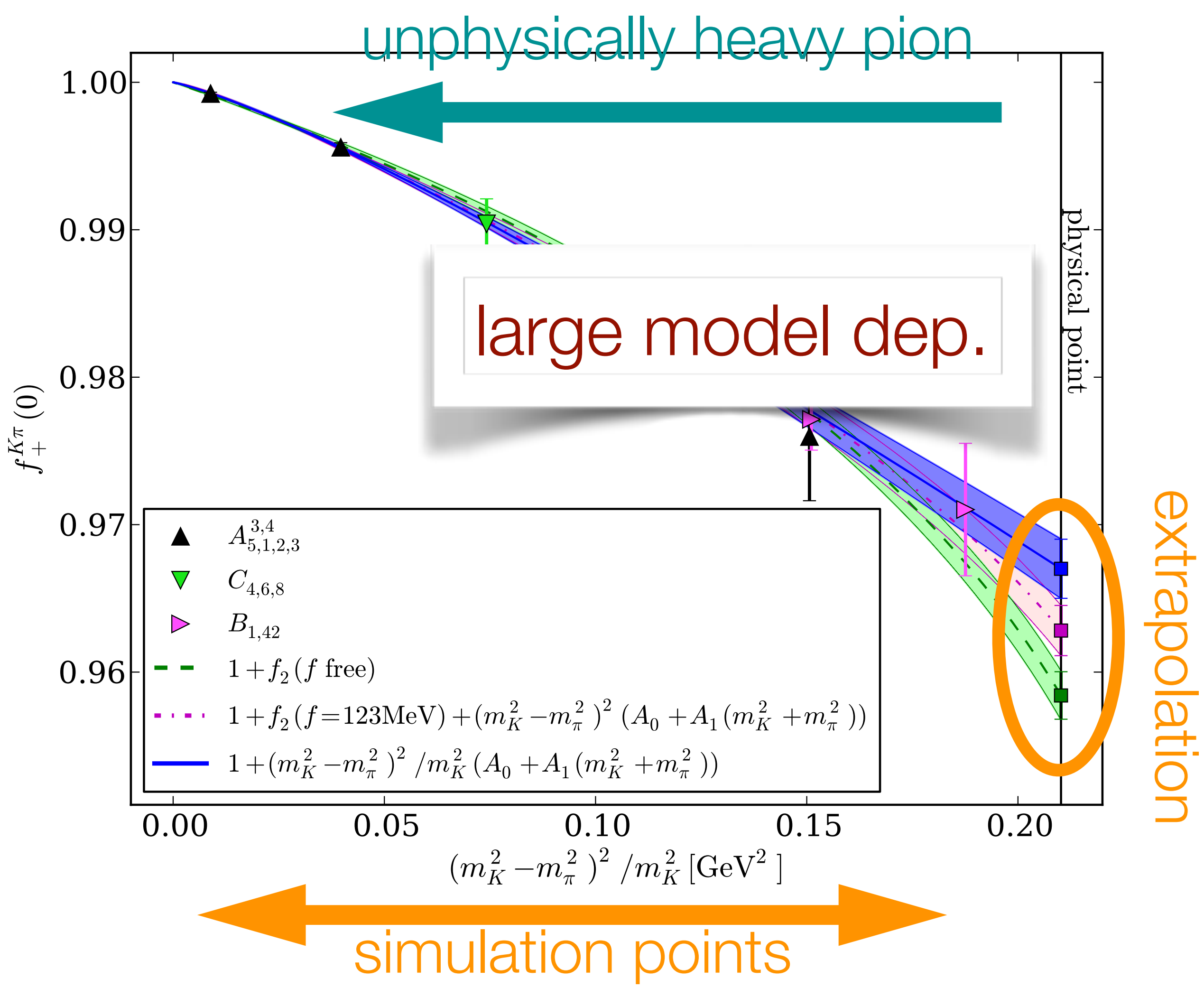
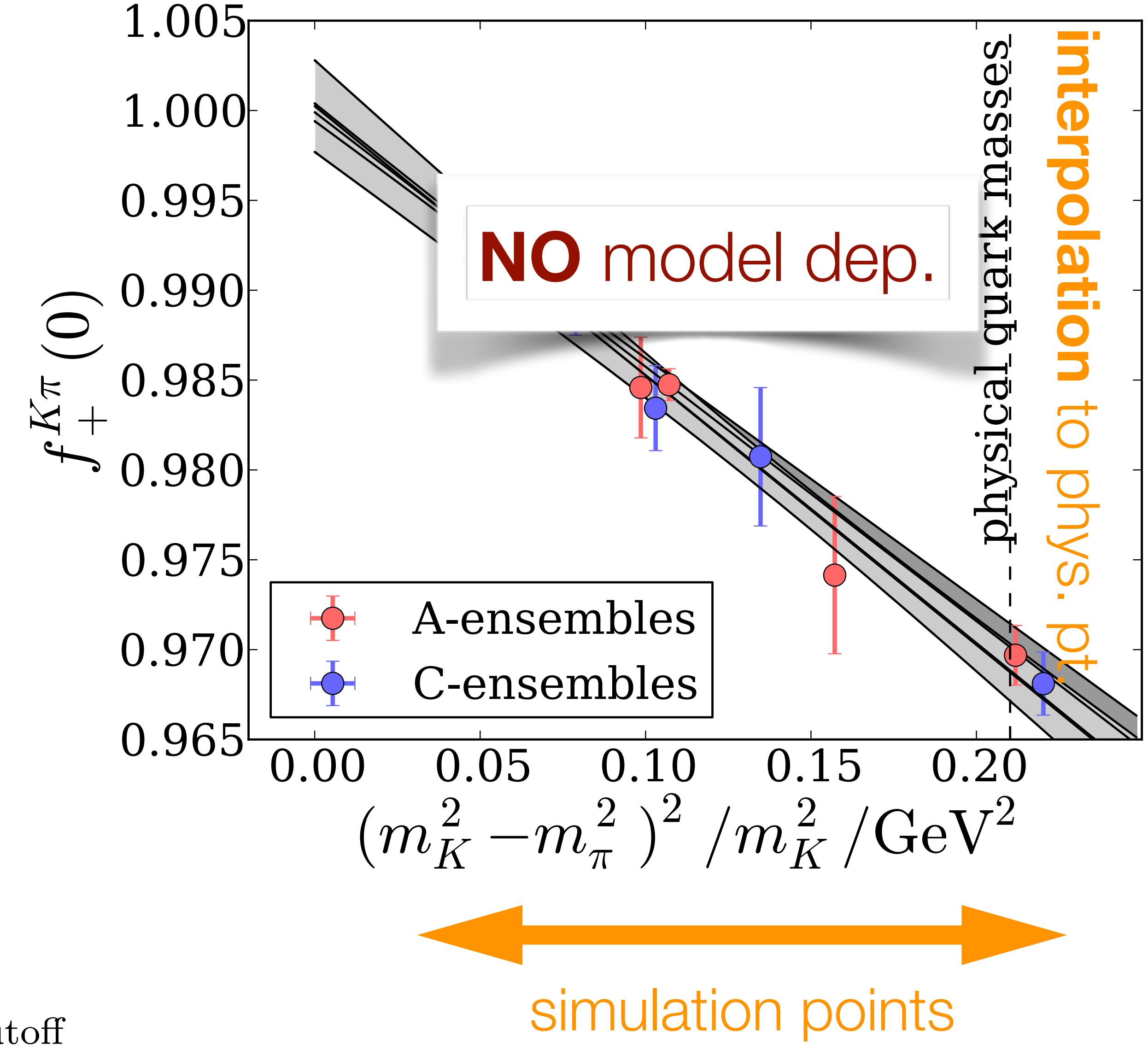
RBC/ULQCD 2015





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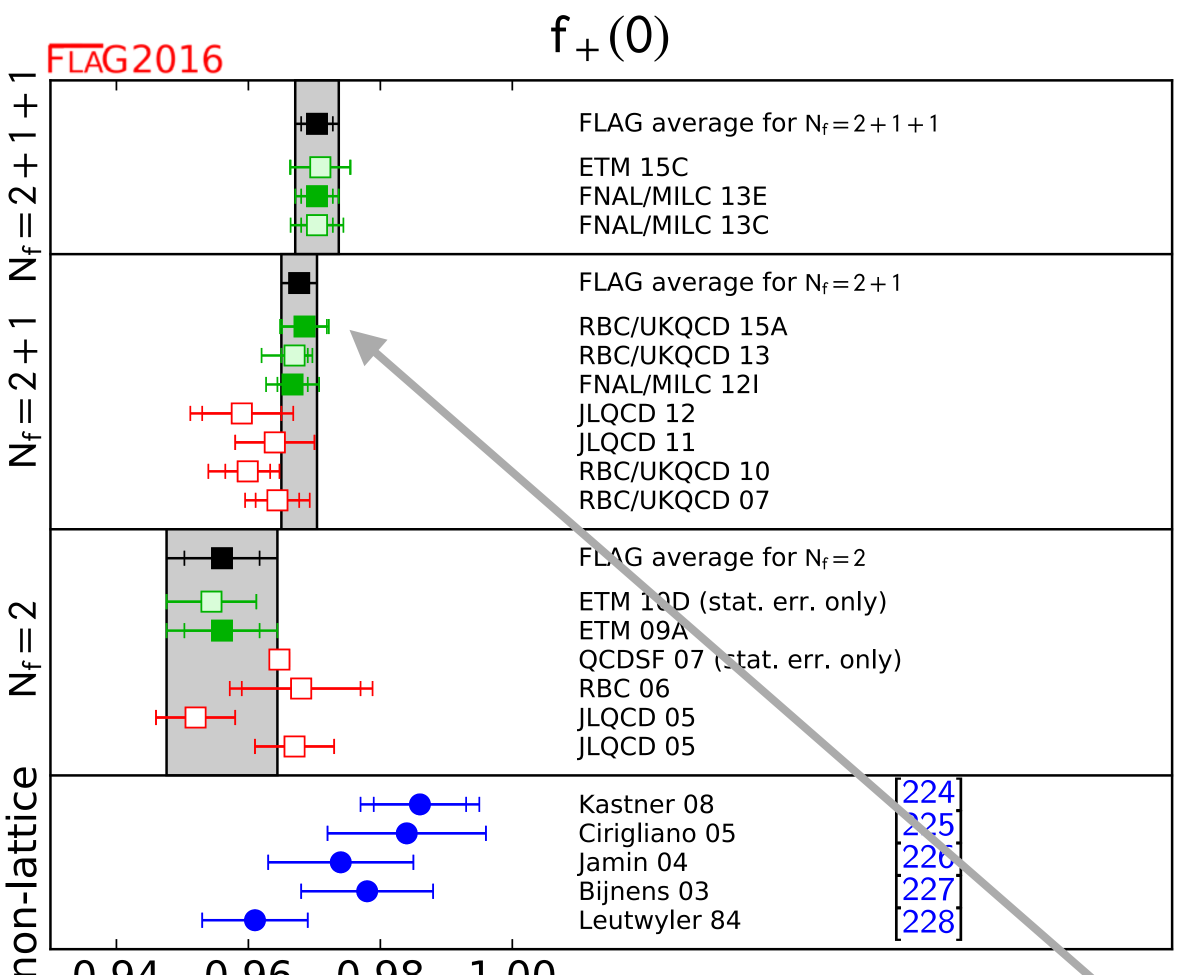
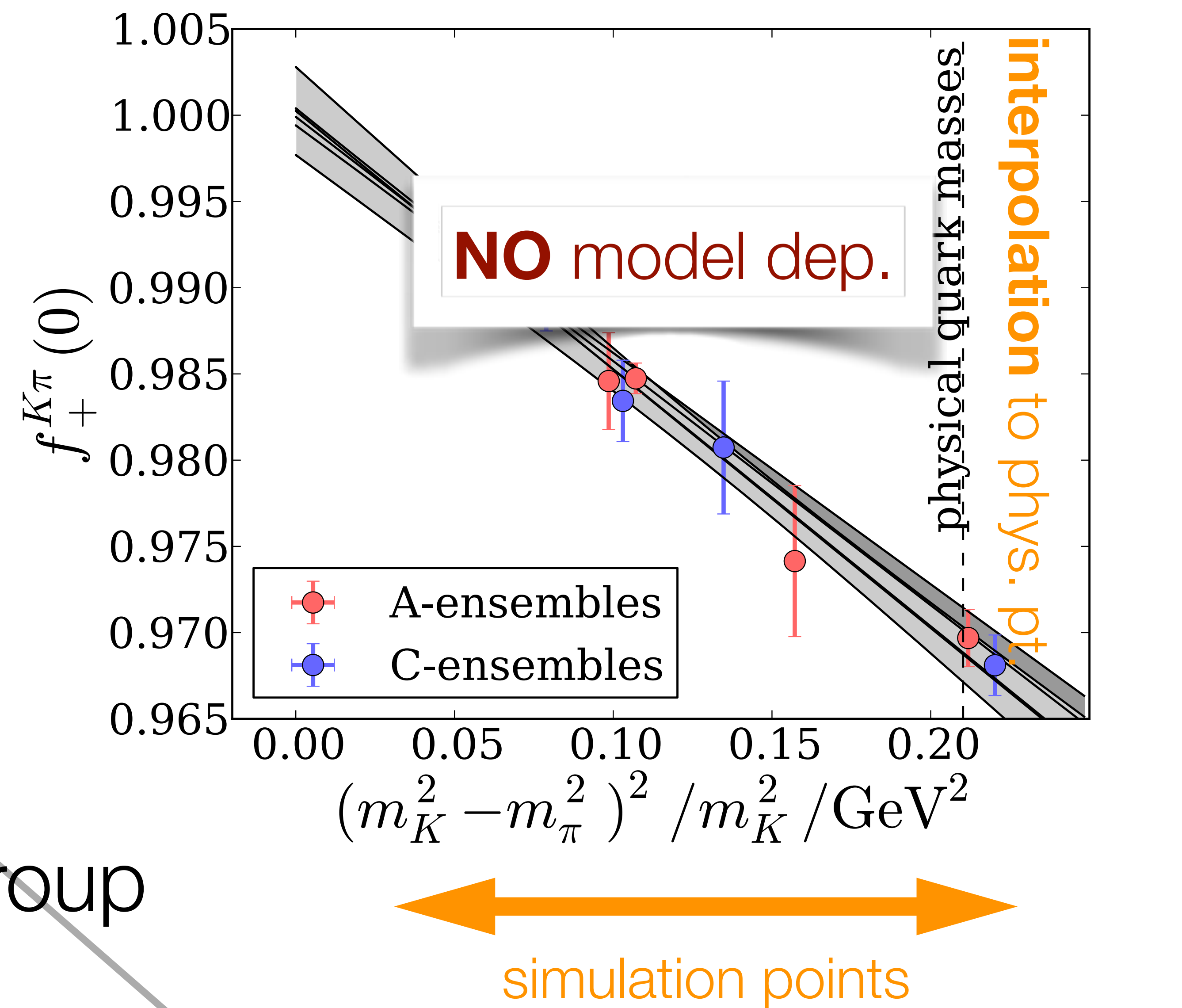
form factor calculation: one example
this is a best understood quantity → average available



RBC/ULQCD 2015



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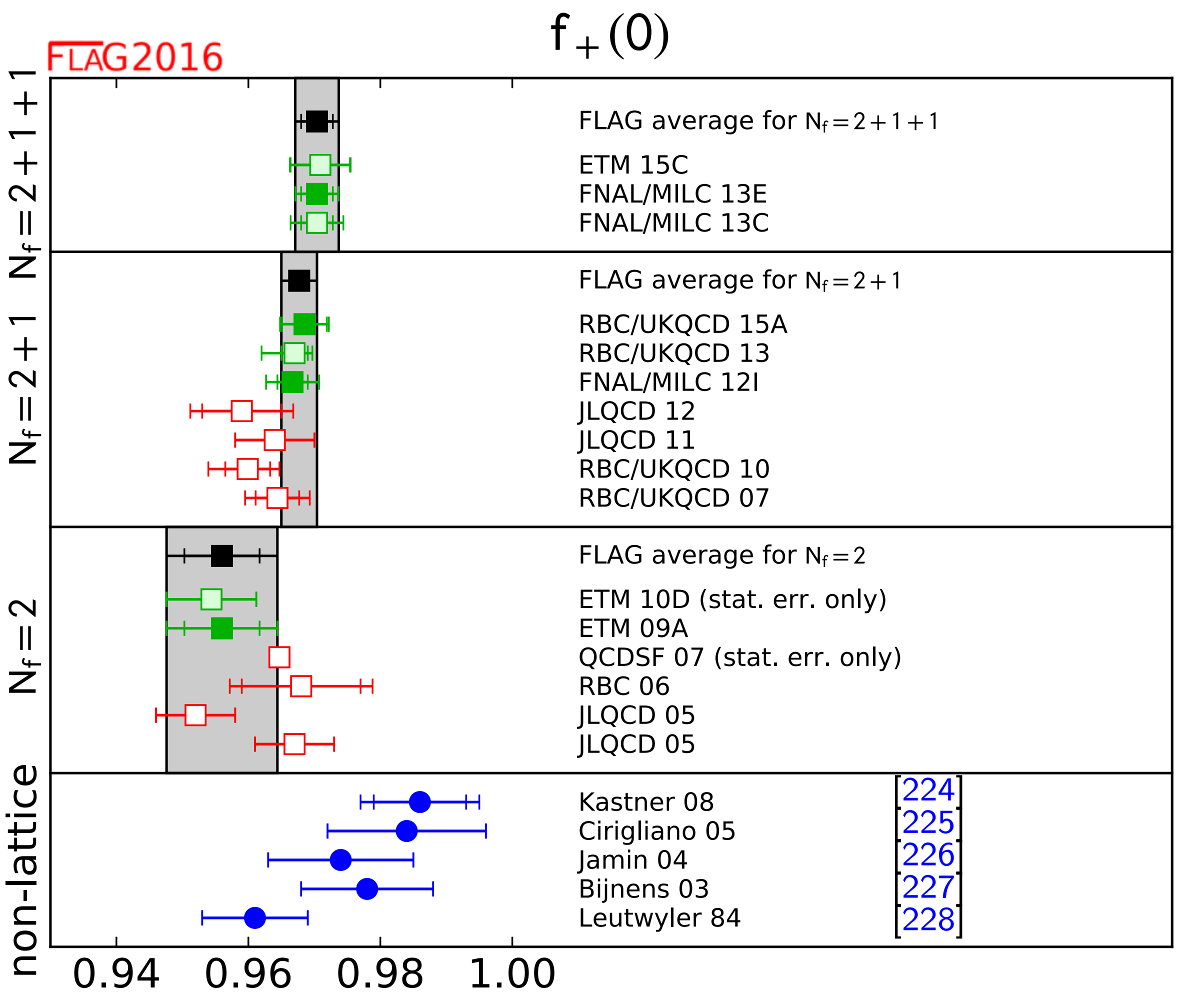
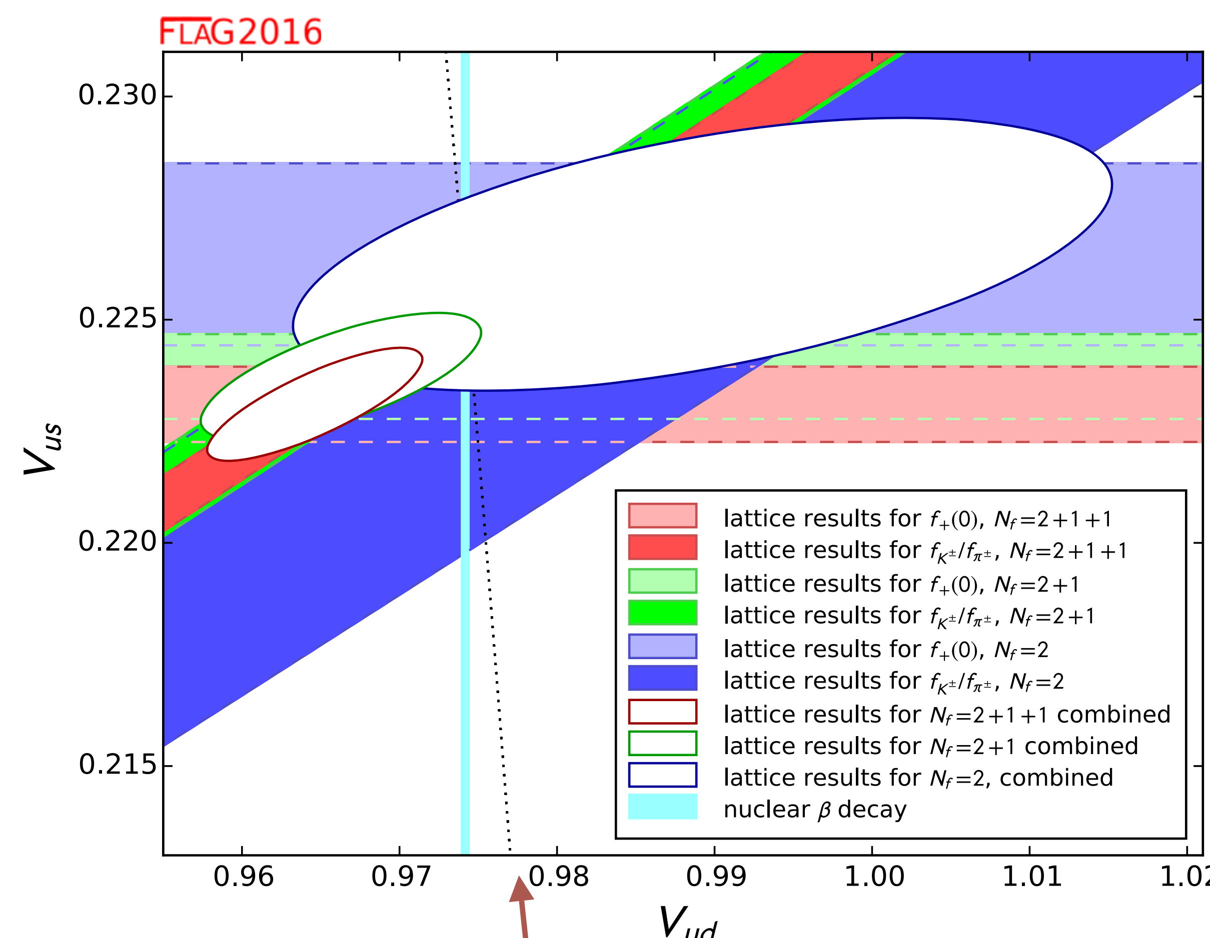
Flavour **L**attice **A**veraging **G**roup

WG to provide average of lattice results
issues: 2010, **2013, 2016**,
so far does not contain any baryonic quantities
nucleon axial charge to be included in the next

form factor calculation: one example
to flavor physics pheno





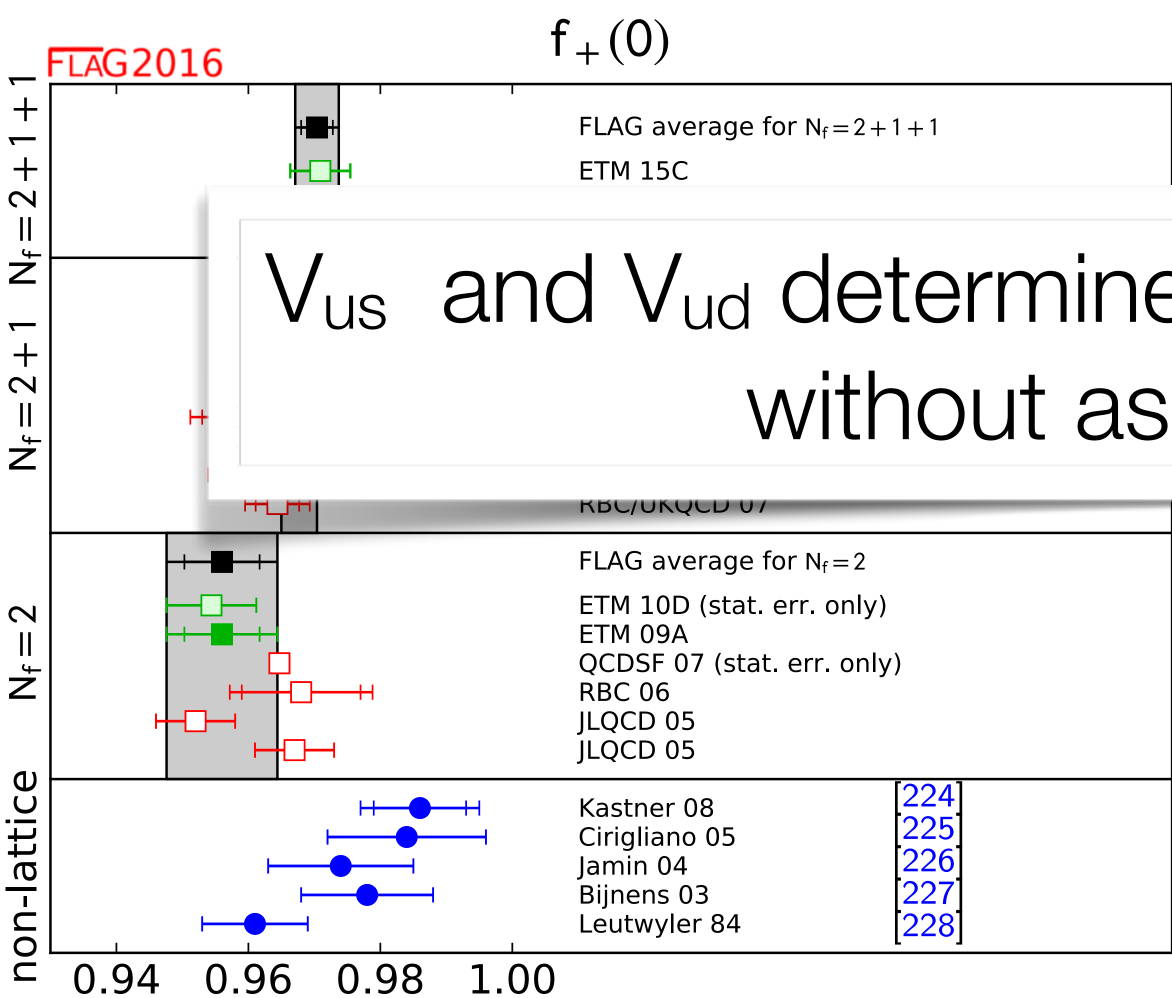
unitary line with $|V_{ud}|^2+|V_{us}|^2+|V_{ub}|^2=1$
with better determined $|V_{ub}|$ input

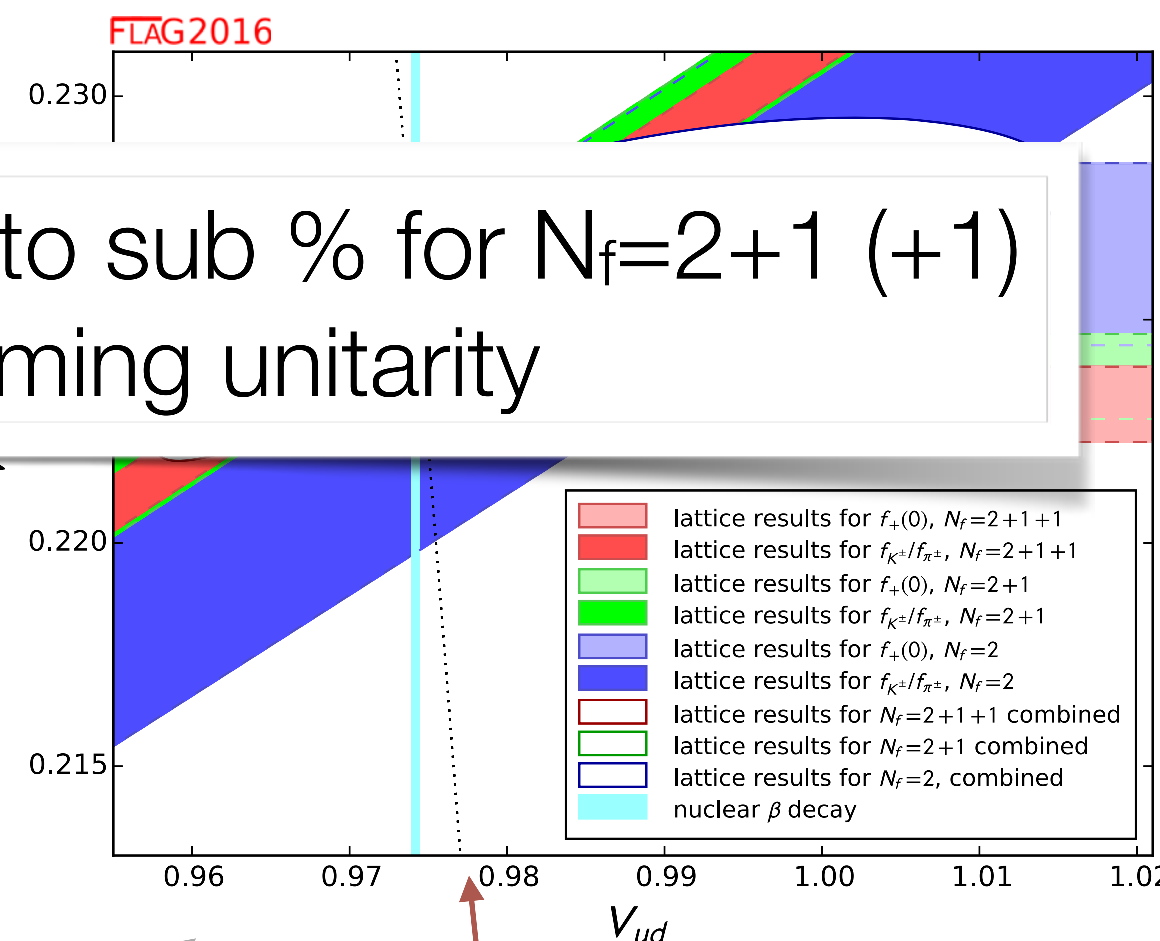
$$|V_{us}| f_+(0) = 0.2165(4)$$

and using a similar relation with π and K decay constants (leptonic decay) on

$$\left|\frac{V_{us}}{V_{ud}}\right| \frac{f_K}{f_\pi}$$

form factor calculation: one example to flavor physics pheno





V_{us} and V_{ud} determined to sub % for $N_f=2+1$ (+1) without assuming unitarity

unitary line with $|V_{ud}|^2+|V_{us}|^2+|V_{ub}|^2=1$ with better determined $|V_{ub}|$ input

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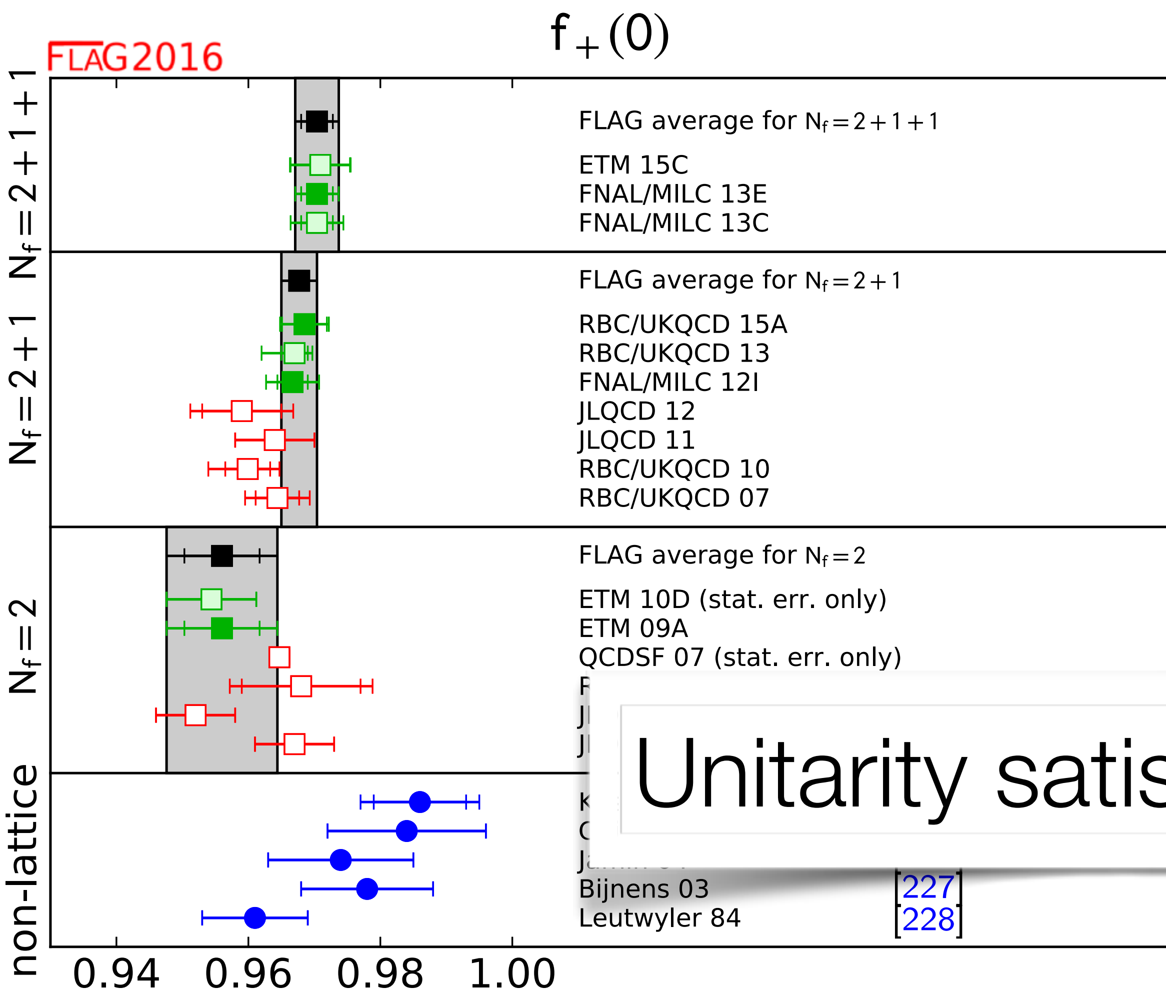
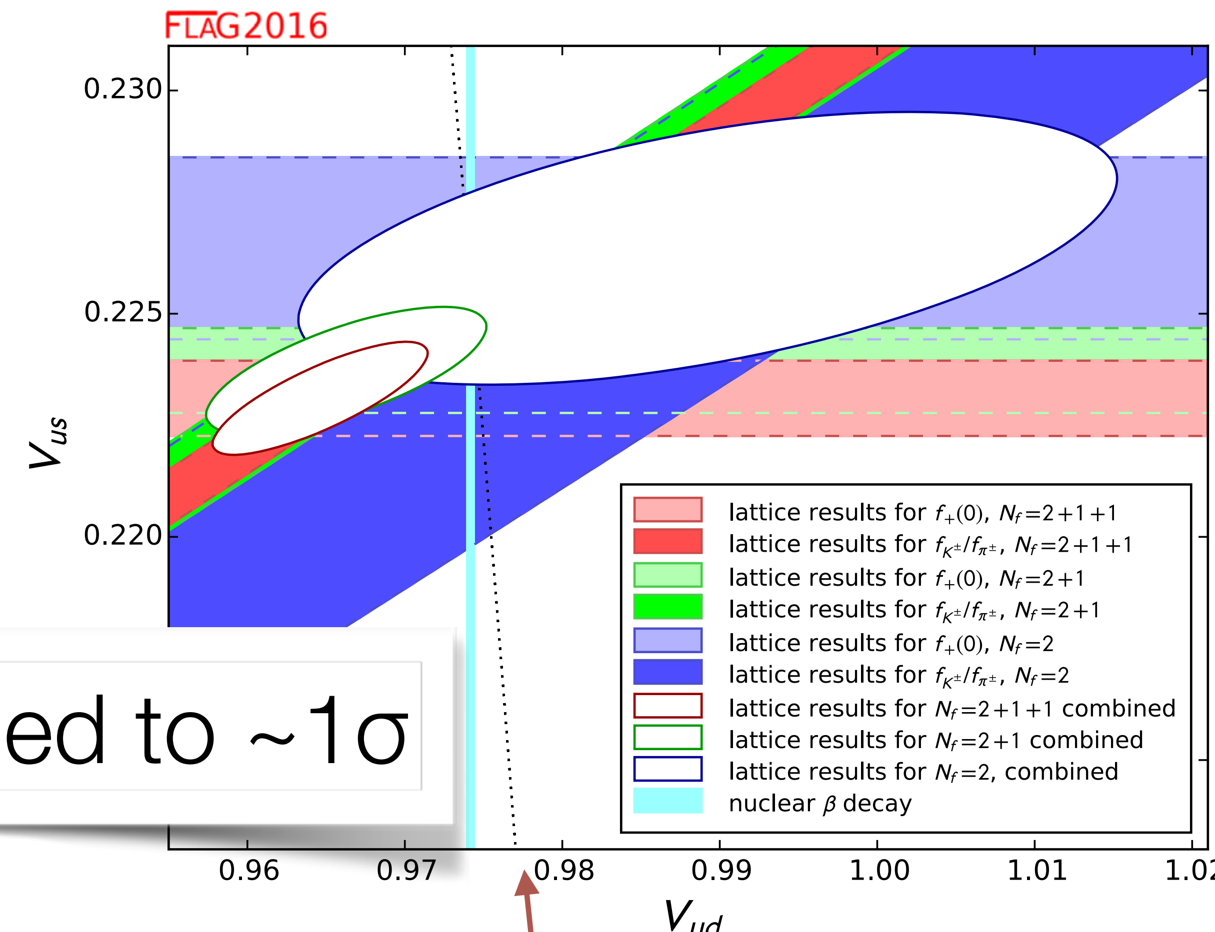
form factor calculation: one example to flavor physics pheno





Unitarity satisfied to ~1σ

unitary line with $|V_{ud}|^2+|V_{us}|^2+|V_{ub}|^2=1$
with better determined $|V_{ub}|$ input

$$|V_{us}|f_+(0) = 0.2165(4)$$

and using a similar relation with π and K decay constants (leptonic decay) on

$$\left|\frac{V_{us}}{V_{ud}}\right| \frac{f_K}{f_\pi}$$

FLAG 2016 coverage I

Table 1 Summary of the main results of this review, grouped in terms of N_f, the number of dynamical quark flavours in lattice simulations. Quark masses and the quark condensate are given in the $\overline{\text{MS}}$ scheme at running scale $\mu = 2$ GeV or as indicated; the other quantities listed are specified in the quoted sections. For each result we list the references that entered the FLAG average or estimate. From the entries in this column one can also read off the number of results that enter our averages for each quantity. We emphasize that these numbers only give a very rough indication of how thoroughly the quantity in question has been explored on the lattice and recommend to consult the detailed tables and figures in the relevant section for more significant information and for explanations on the source of the quoted errors

Quantity	Sects.	$N_f = 2+1+1$	Refs.	$N_f = 2+1$	Refs.	$N_f = 2$	Refs.
m_s [MeV]	3.1.3	93.9(1.1)	[4,5]	92.0(2.1)	[6–10]	101(3)	[11,12]
m_{ud} [MeV]	3.1.3	3.70(17)	[4]	3.373(80)	[7–10,13]	3.6(2)	[11]
m_s/m_{ud}	3.1.4	27.30(34)	[4,14]	27.43(31)	[6–8,10]	27.3(9)	[11]
m_u [MeV]	3.1.5	2.36(24)	[4]	2.16(9)(7)	[a]	2.40(23)	[16]
m_d [MeV]	3.1.5	5.03(26)	[4]	4.68(14)(7)	[a]	4.80(23)	[16]
m_u/m_d	3.1.5	0.470(56)	[4]	0.46(2)(2)	[a]	0.50(4)	[16]
$\overline{m}_c$(3 GeV) [GeV]	3.2	0.996(25)	[4,5]	0.987(6)	[9,17]	1.03(4)	[11]
m_c/m_s	3.2.4	11.70(6)	[4,5,14]	11.82(16)	[17,18]	11.74(35)	[11,132]
$\overline{m}_b(\overline{m}_b)$ [GeV]	3.3.4	4.190(21)	[5,19]	4.164(23)	[9]	4.256(81)	[20,21]
$f_+(0)$	4.3	0.9704(24)(22)	[22]	0.9677(27)	[23,24]	0.9560(57)(62)	[25]
$f_{K^\pm}/f_{\pi^\pm}$	4.3	1.193(3)	[14,26,27]	1.192(5)	[28–31]	1.205(6)(17)	[32]
$f_{\pi^\pm}$ [MeV]	4.6			130.2(1.4)	[28,29,31]		
$f_{K^\pm}$ [MeV]	4.6	155.6(4)	[14,26,27]	155.9(9)	[28,29,31]	157.5(2.4)	[32]
$\Sigma^{1/3}$ [MeV]	5.2.1	280(8)(15)	[33]	274(3)	[10,13,34,35]	266(10)	[33,36–38]
F_π/F	5.2.1	1.076(2)(2)	[39]	1.064(7)	[10,29,34,35,40]	1.073(15)	[36–38,41]
$\bar{\ell}_3$	5.2.2	3.70(7)(26)	[39]	2.81(64)	[10,29,34,35,40]	3.41(82)	[36,37,41]
$\bar{\ell}_4$	5.2.2	4.67(3)(10)	[39]	4.10(45)	[10,29,34,35,40]	4.51(26)	[36,37,41]
$\bar{\ell}_6$	5.2.2					15.1(1.2)	[37,41]
$\hat{B}_K$	6.1	0.717(18)(16)	[42]	0.7625(97)	[10,43–45]	0.727(22)(12)	[46]

[a] This is a FLAG estimate, based on χPT and the isospin averaged up- and down-quark mass m_{ud} [7–10,13]

FLAG 2016 coverage II

Table 2 Summary of the main results of this review, grouped in terms of N_f, the number of dynamical quark flavours in lattice simulations. The quantities listed are specified in the quoted sections. For each result we list the references that entered the FLAG average or estimate. From the entries in this column one can also read off the number of results that enter our averages for each quantity. We emphasize that these numbers only give a very rough indication of how thoroughly the quantity in question has been explored on the lattice and recommend to consult the detailed tables and figures in the relevant section for more significant information and for explanations on the source of the quoted errors

Quantity	Sects.	$N_f = 2+1+1$	Refs.	$N_f = 2+1$	Refs.	$N_f = 2$	Refs.
f_D [MeV]	7.1	212.15(1.45)	[14,27]	209.2(3.3)	[47,48]	208(7)	[20]
f_{D_s} [MeV]	7.1	248.83(1.27)	[14,27]	249.8(2.3)	[17,48,49]	250(7)	[20]
f_{D_s}/f_D	7.1	1.1716(32)	[14,27]	1.187(12)	[47,48]	1.20(2)	[20]
$f_+^{D\pi}(0)$	7.2			0.666(29)	[50]		
$f_+^{DK}(0)$	7.2			0.747(19)	[51]		
f_B [MeV]	8.1	186(4)	[52]	192.0(4.3)	[48,53–56]	188(7)	[20,57,58]
f_{B_s} [MeV]	8.1	224(5)	[52]	228.4(3.7)	[48,53–56]	227(7)	[20,57,58]
f_{B_s}/f_B	8.1	1.205(7)	[52]	1.201(16)	[48,53–55]	1.206(23)	[20,57,58]
$f_{B_d}\sqrt{\hat{B}_{B_d}}$ [MeV]	8.2			219(14)	[54,59]	216(10)	[20]
$f_{B_s}\sqrt{\hat{B}_{B_s}}$ [MeV]	8.2			270(16)	[54,59]	262(10)	[20]
$\hat{B}_{B_d}$	8.2			1.26(9)	[54,59]	1.30(6)	[20]
$\hat{B}_{B_s}$	8.2			1.32(6)	[54,59]	1.32(5)	[20]
ξ	8.2			1.239(46)	[54,60]	1.225(31)	[20]
B_{B_s}/B_{B_d}	8.2			1.039(63)	[54,60]	1.007(21)	[20]

Quantity	Sects.	$N_f = 2+1$ and $N_f = 2+1+1$	Refs.
$\alpha_{\overline{\mathrm{MS}}}^{(5)}(M_Z)$	9.9	0.1182(12)	[5,9,61–63]
$\Lambda_{\overline{\mathrm{MS}}}^{(5)}$ [MeV]	9.9	211(14)	[5,9,61–63]

FLAG 2016 coverage II

Eur. Phys. J. C (2017) 77:112

Table 2 Summary of the main results of this review, grouped in terms of N_f, the number of dynamical quark flavours in lattice simulations. The quantities listed are specified in the quoted sections. For each result we list the references that entered the FLAG average or estimate. From the entries in this column one can also read off the number of results that enter our averages for each quantity. We emphasize that these numbers only give a very rough indication of how thoroughly the quantity in question has been explored on the lattice and recommend to consult the detailed tables and figures in the relevant section for more significant information and for explanations on the source of the quoted errors

aiming to provide convenient summary for non expert

Quantity	Sects.	$N_f = 2+1+1$	Refs.	$N_f = 2+1$	Refs.	$N_f = 2$	Refs.
f_D [MeV]							[20]
f_{D_s} [MeV]							[20]
f_{D_s}/f_D							[20]
$f_+^{D\pi}(0)$							
$f_+^{DK}(0)$							
f_B [MeV]	8.1	186(4)	[52]	192.0(4.3)	[48,53–56]	188(7)	[20,57,58]
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$\alpha^{(5)}_{\overline{\rm MS}}(M_Z)$	9.9	0.1182(12)	[5,9,61–63]
$\Lambda^{(5)}_{\overline{\rm MS}}$ [MeV]	9.9	211(14)	[5,9,61–63]

Nucleon form factors and parton distributions

recent developments

recent developments

- nucleon axial charge g_A

recent developments

- nucleon axial charge g_A
- proton spin

recent developments

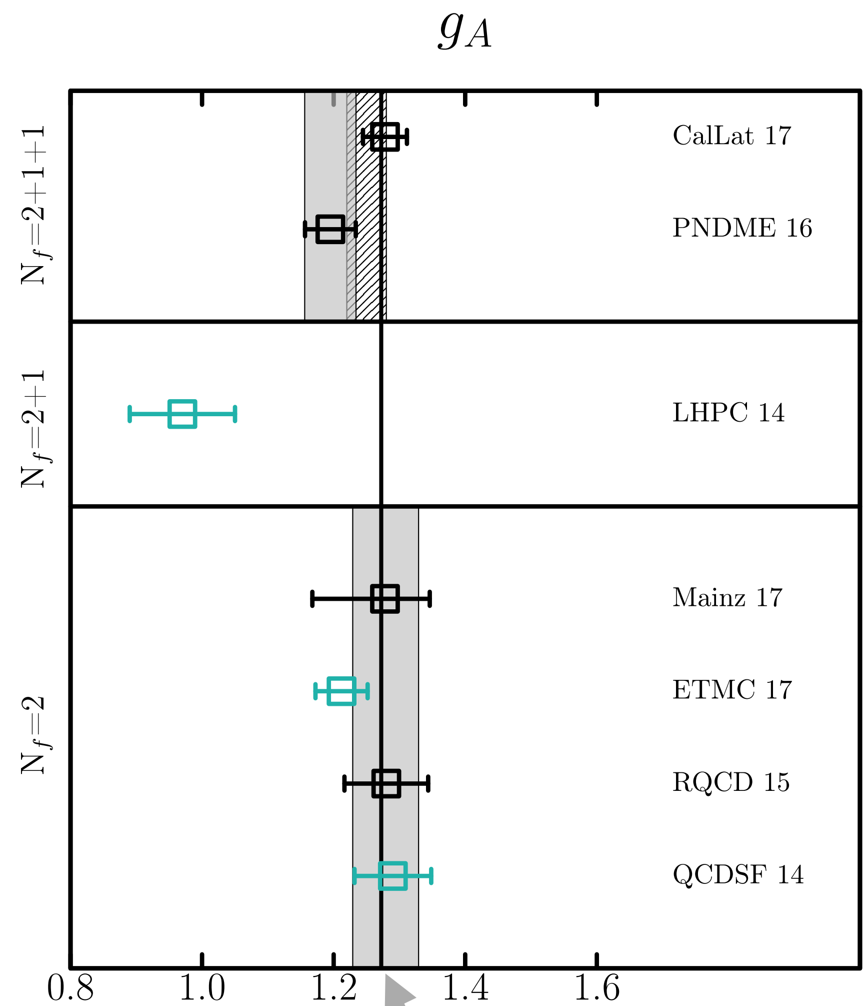
- nucleon axial charge g_A
- proton spin
- parton distribution in proton

recent developments

- nucleon axial charge g_A
- proton spin
- parton distribution in proton

- Note:
  - all are computed from hadronic matrix elements obtained in a similar way as Kaon semi leptonic decay
    - calculated from hadronic 3-point function
  - difference
    - external states → here in/out states are nucleons
    - operator
    - complexity of the scheme / additional parameters

Lattice g_A summary from PDF "white paper"



$$g_A^{\mathrm{exp}} = 1.2723(23)$$

$$g_A \equiv \langle 1 \rangle_{\Delta u^+ - \Delta d^+}$$

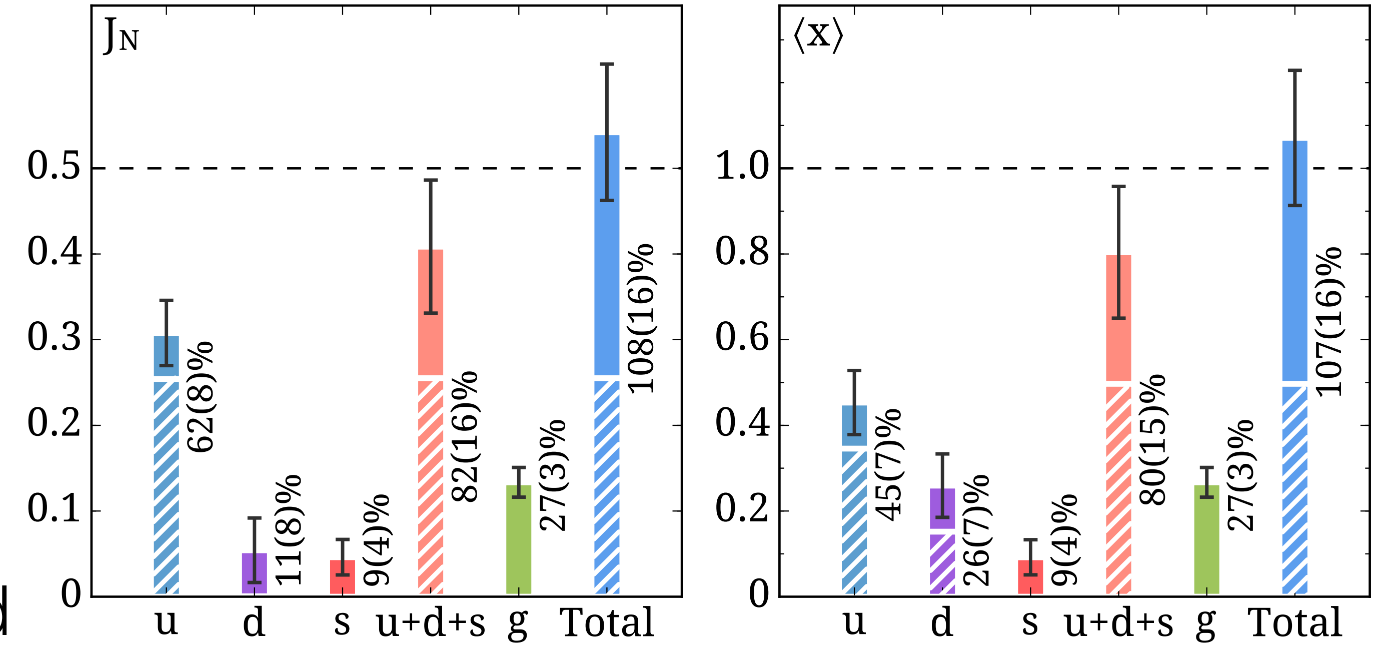
[1711.07916]

only those which clear certain quality criteria are shown
cyan omitted from average due to worse quality score
lattice g_A used to be smaller than experiment,
state-of-the art calculations are consistent with exp.

Proton spin "puzzle"

- Nucleon Spin and Momentum Decomposition Using Lattice QCD Simulations
    - ETMC: C. Alexandrou et al [PRL 119 (2017) 142002]
    - using Ji's gauge invariant decomposition

$$J_N = \sum_{q=u,d,s,c\cdots} \left(\frac{1}{2}\Delta\Sigma_q + L_q \right) + J_g$$



- on-physical point simulation
- only one lattice spacing
- total J consistent with 1/2
  claiming
  "resolving a long standing puzzle"
- systematic errors not fully investigated
    - twisted mass → isospin violation should be investigated
    - adding another lattice spacing will help!

parton distribution and quasi distribution

unpolarized quark distribution

$$q(x,\mu) \equiv \int \frac{d\xi^-}{4\pi} e^{-ixP^+\xi^-} \langle P|\bar{\psi}(\xi^-)\gamma^+ W(\xi^-,0)\psi(0)|P\rangle$$

matrix element of a non-local operator, extended in light cone

X. Ji's idea of relating this with a quasi distribution

$$\tilde{q}(x,P_z,\tilde{\mu}) = \int_{-\infty}^{\infty} \frac{dz}{4\pi} e^{ixP_z z} \tilde{h}(z,P_z,\tilde{\mu})$$

$$\tilde{h}(z,P_z,\tilde{\mu}) = \langle P|O_\Gamma|P\rangle$$

$$O_\Gamma = \bar{\psi}(z)\Gamma W_z(z,0)\psi(0) \qquad\qquad \Gamma = \gamma_z \ \text{ or } \ \tfrac{P_z}{P_t}\gamma_t$$

extended in equal time plane → Euclidian definition is possible

infinite momentum boost is needed

← Feynman's original idea was to use infinite momentum frame

parton distribution and quasi distribution

unpolarized quark distrib

$$q(x, \mu) \equiv \int \quad)\psi(0)|P\rangle$$

matrix element o ed in light cone

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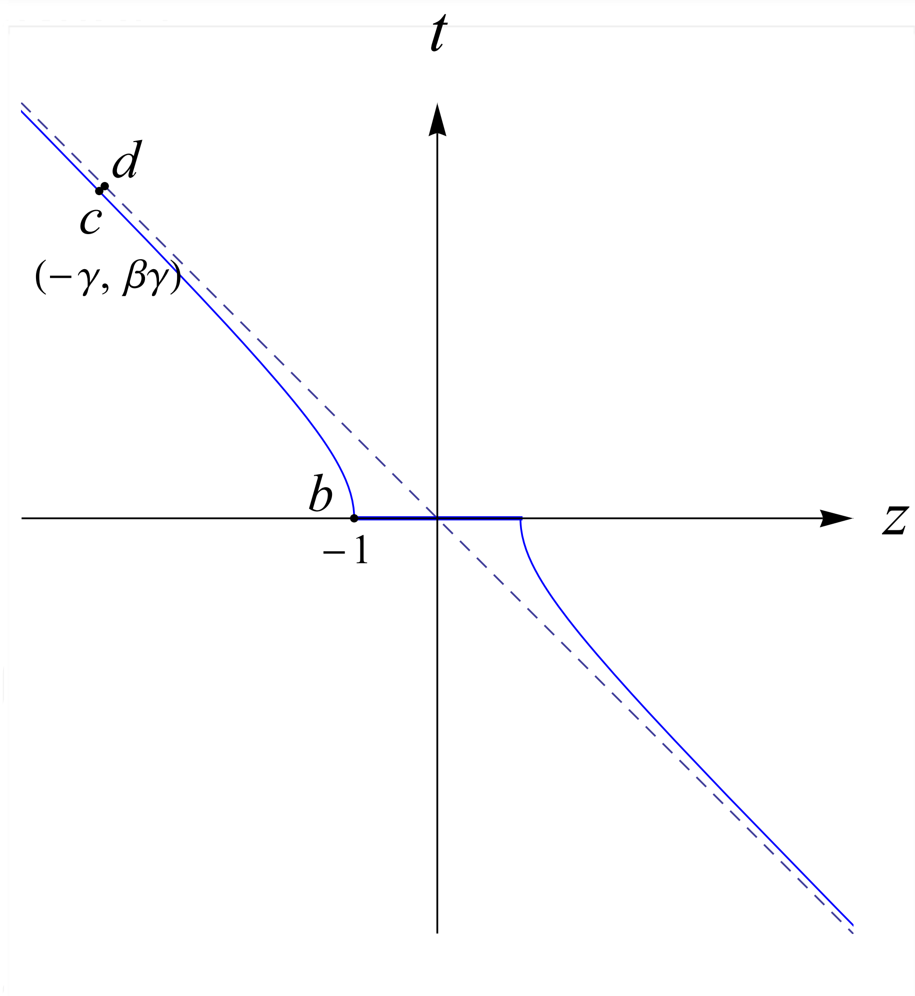

Chen et al NPB911 (2016) 246

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infinite momentum boost is needed

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parton distribution and quasi distribution

unpolarized quark distribution

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parton distribution from quasi distribution on lattice

unpolarized quark distribution

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parton distribution from quasi distribution on lattice

unpolarized quark distribution

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1) compute on the **lattice**
with hadron with momentum P_z

$$O_\Gamma = \bar{\psi}(z)\Gamma W_z(z,0)\psi(0)$$

parton distribution from quasi distribution on lattice

unpolarized quark distribution

$$q(x,\mu) \equiv \int \frac{d\xi^-}{4\pi} e^{-ixP^+\xi^-} \langle P|\bar{\psi}(\xi^-)\gamma^+ W(\xi^-,0)\psi(0)|P\rangle$$

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$$O_\Gamma = \bar{\psi}(z)\Gamma W_z(z,0)\psi(0)$$

1) compute on the **lattice** with hadron with momentum P_z

2) non-perturbative renormalization
UV **power divergence** subtracted
mixing resolved

parton distribution from quasi distribution on lattice

unpolarized quark distribution

$$q(x,\mu) \equiv \int \frac{d\xi^-}{4\pi} e^{-ixP^+\xi^-} \langle P|\bar{\psi}(\xi^-)\gamma^+ W(\xi^-,0)\psi(0)|P\rangle$$

matrix element of a non-local operator, extended in light cone

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$$\tilde{q}(x,P_z,\tilde{\mu}) = \int_{-\infty}^{\infty} \frac{dz}{4\pi} e^{ixP_z z}\tilde{h}(z,P_z,\tilde{\mu})$$

3) lattice to continuum matching; $a \to 0$

$$\tilde{h}(z,P_z,\tilde{\mu}) = \langle P|O_\Gamma|P\rangle$$

1) compute on the **lattice** with hadron with momentum P_z

$$O_\Gamma = \bar{\psi}(z)\Gamma W_z(z,0)\psi(0)$$

2) non-perturbative renormalization UV **power divergence** subtracted mixing resolved

parton distribution from quasi distribution on lattice

unpolarized quark distribution

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matrix element of a non-local operator, extended in light cone

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3) lattice to continuum

no $a \to 0$ results available now

$$\tilde{h}(z,P_z,\tilde{\mu}) = \langle P|O_\Gamma|P\rangle$$

1) compute on the **lattice**
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$$O_\Gamma = \bar{\psi}(z)\Gamma W_z(z,0)\psi(0)$$

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parton distribution from quasi distribution on lattice

unpolarized quark distribution

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matrix element of a non-loca

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$$\tilde{h}(z,P_z,\tilde{\mu}) = \langle P|O_\Gamma|P\rangle$$

$$O_\Gamma = \bar{\psi}(z)\Gamma W_z(z,0)\psi(0)$$

4) factorization/matching with
$P_z \to \infty$

3) lattice to continuum

no $a \to 0$ results available now

1) compute on the **lattice**
with hadron with momentum P_z

2) non-perturbative renormalization
UV **power divergence** subtracted
mixing resolved

parton distribution from quasi distribution with LaMET (Large-Momentum Effective Theory)

unpolarized quark distribution

$$q(x,\mu) \equiv \int \frac{d\xi^-}{4\pi} e^{-ixP^+\xi^-} \langle P|\bar{\psi}(\xi^-)\gamma^+ W(\xi^-,0)\psi(0)|P\rangle$$

matrix element of a non-loca

4) factorization/matching with matching: $\boldsymbol{P_z \to \infty}$

X. Ji's idea of relating this with a qu

$$\tilde{q}(x,P_z,\tilde{\mu}) = \int_{-\infty}^{\infty} \frac{dz}{4\pi} e^{ixP_z z}\tilde{h}(z,P_z,\tilde{\mu})$$

Factorization formula

$$\tilde{q}(x,\Lambda,P_z) = \int \frac{dy}{|y|} Z\left(\frac{x}{y},\frac{\mu}{P_z},\frac{\Lambda}{P_z}\right) q(y,\mu) + \mathcal{O}\left(\frac{\Lambda^2_{\rm QCD}}{P_z^2},\frac{M^2}{P_z^2}\right) + \ldots.$$

Ji, Xion et al, Ma and Qiu, Ishikawa et al, Lin et al, Chen et al

$O(1/P_z^2)$ subtracted or removed by $P_z \to \infty$

Nucleon parton distribution - status -

physical quark mass already be in use !

1. Improved Parton Distribution Functions at Physical Pion Mass
   - *LP^3: Lin et al [1708.05301] (unpolarized and helicity)*

2. Reconstruction of light-cone parton distribution functions from lattice QCD simulations at the physical point
   - *ETMC: Alexandrou et al [1803.02685] (unpolarized and helicity)*

3. Lattice Calculation of Parton Distribution Function from LaMET at Physical Pion Mass with Large Nucleon Momentum
   - *LP^3: Chen et al [1803.04393] (unpolarized with larger momentum)*

all are computed on single lattice spacing

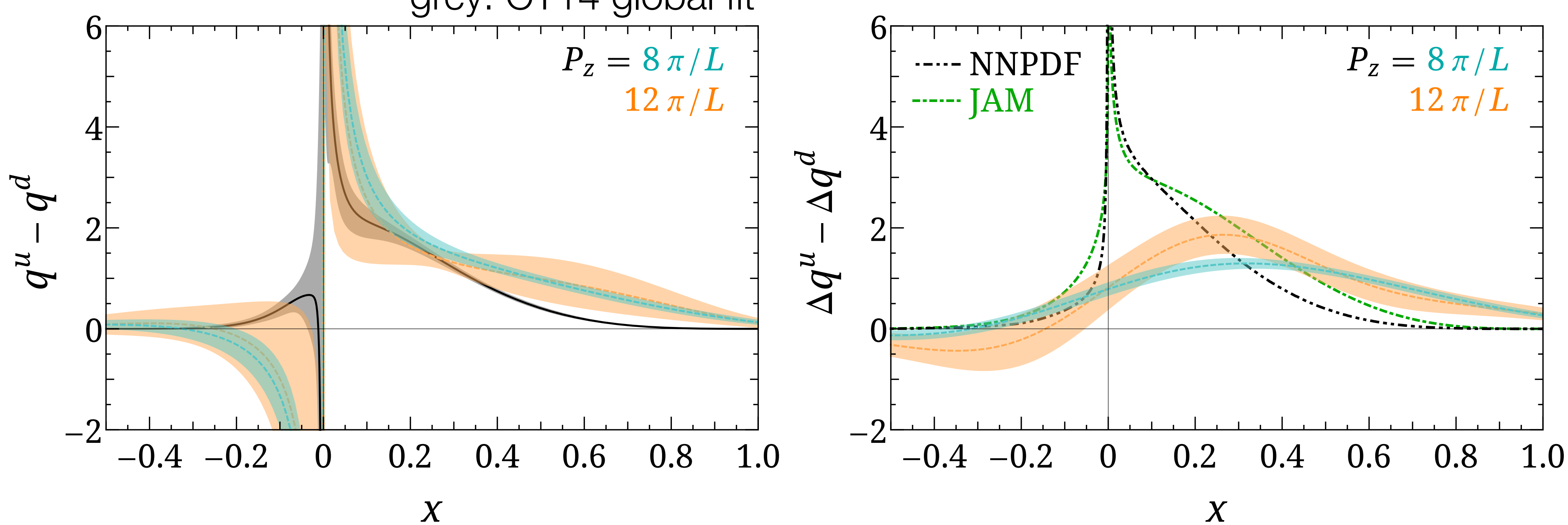
- $a \to 0$ study yet to be done
- this is very important: power divergence properly removed ? etc.

Nucleon parton distribution - status -

1. Improved Parton Distribution Functions at Physical Pion Mass
   - LP[3]: Lin et al [1708.05301] (unpolarized and helicity)
   - HISQ sea quark (N_f=2+1+1) with Clover-Wilson valence quark

   - in-depth investigation of the unphysical “oscillation” observed before
     - likely due to truncation of Fourier transformation
   - comes up with 2 methods to ease the problem (derivative / filter)

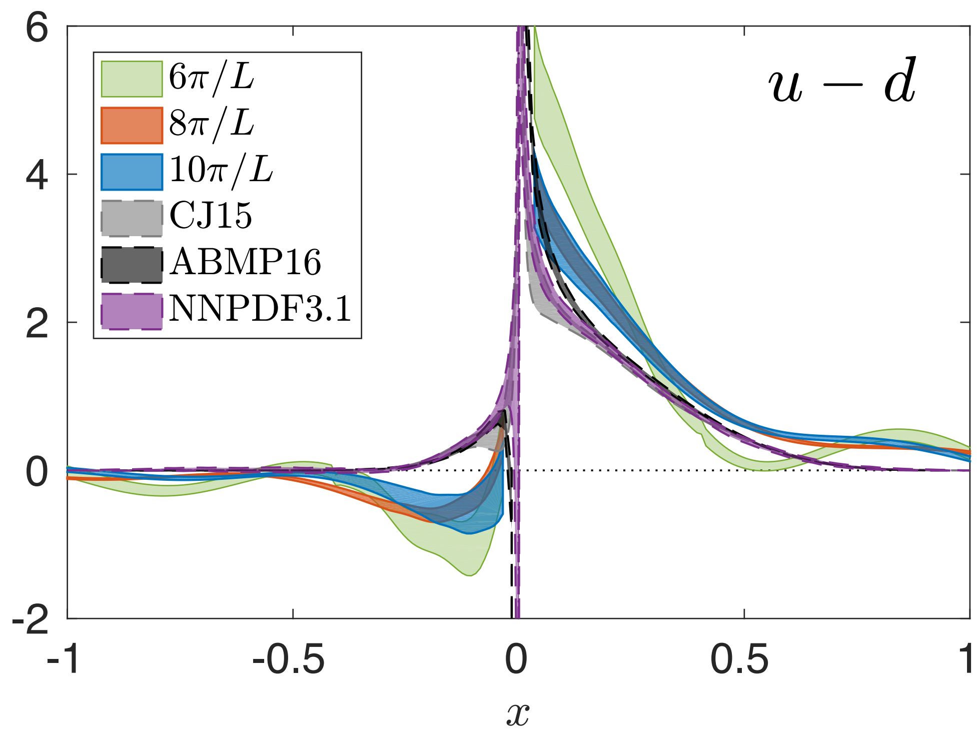
grey: CT14 global fit

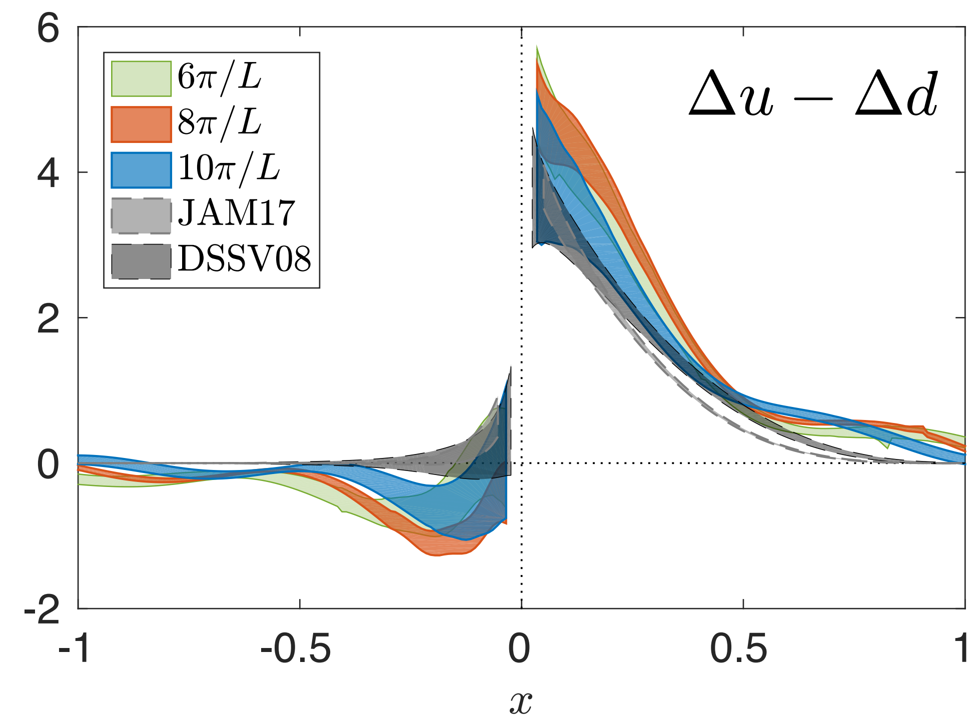


$q(-x) = \overline{q}(x)$ $\overline{MS}$, 2GeV

Nucleon parton distribution - status -

2. Reconstruction of light-cone parton distribution functions from lattice QCD simulations at the physical point
   - ETMC: Alexandrou et al [1803.02685] (unpolarized and helicity)
   - twisted mass sea (N_f=2) twisted mass valence quark
   - $P_z \leq 1.4$ GeV

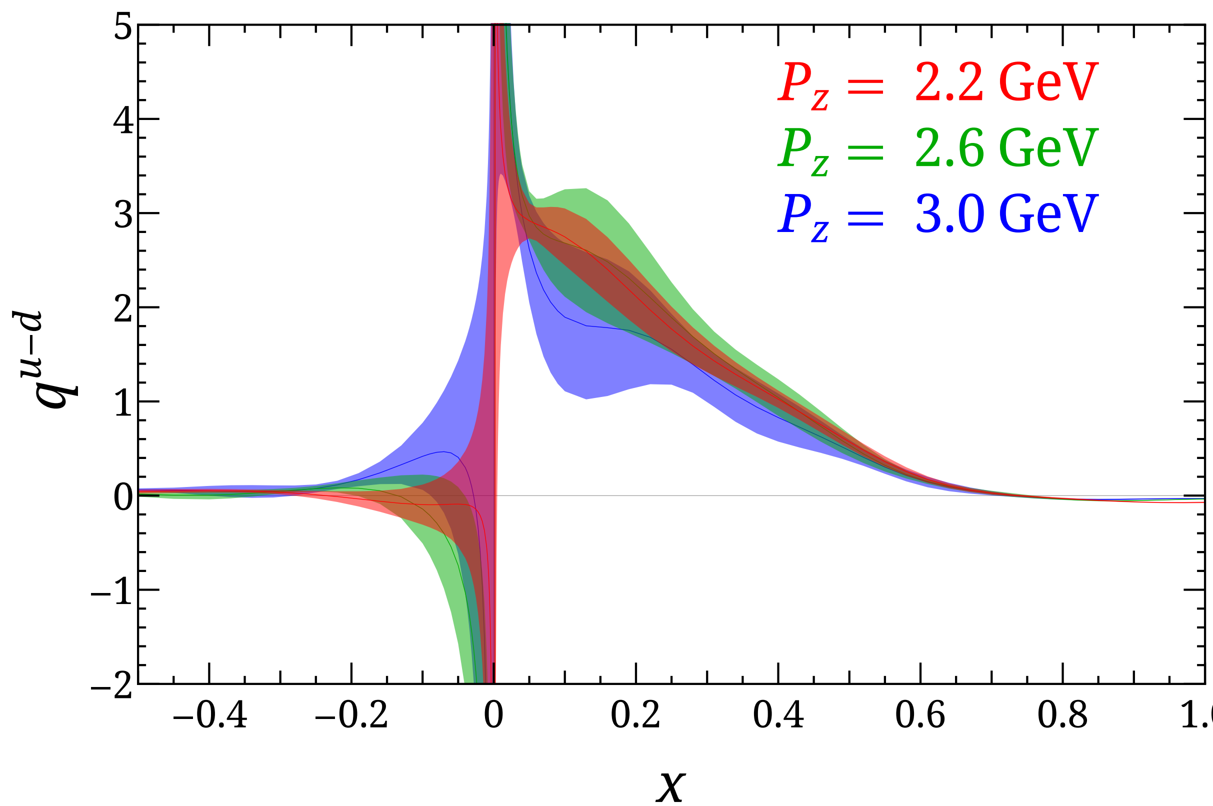


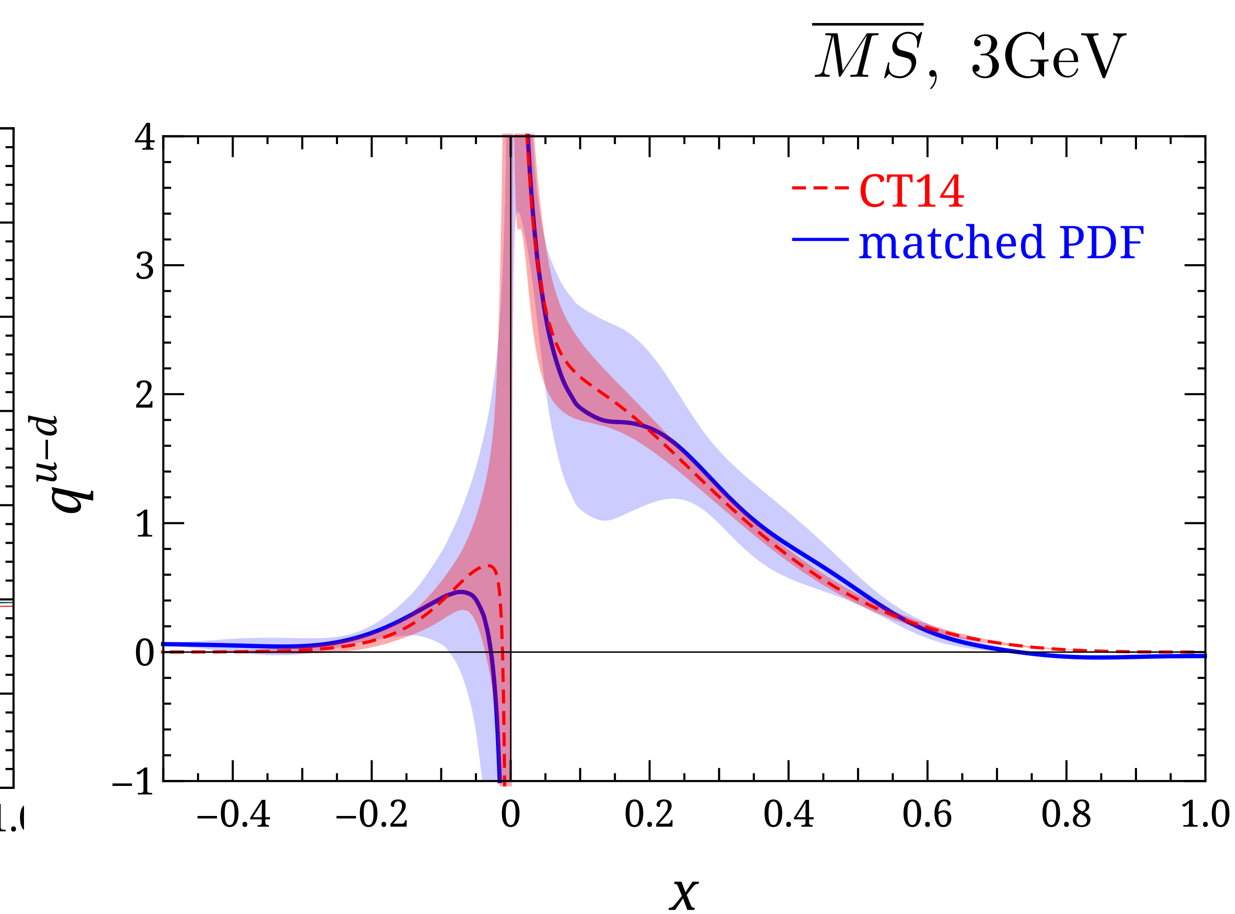


- oscillation behavior persists → authors admit they are unphysical
  - larger P_z makes less problematic
    → hoping it to disappear for even larger momentum → future

Nucleon parton distribution - status -

3. Lattice Calculation of Parton Distribution Function from LaMET at Physical Pion Mass with Large Nucleon Momentum
   - LP[3]: Chen et al [1803.04393] (unpolarized)
   - same set up as 1, but, larger statistics and larger momentum
   - $P_z \leq 3$ GeV
   - other improved items, see paper





- lattice result now reproduces global fit including x~-0.1 asymmetry

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    - especially physical quark mass simulations in hand
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    - gold plated quantities obtained with sub-percent total error

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- More difficult quantities with involved procedures under development
- PDF is one of them
  - lots of efforts poured into developments of quasi PDF
  - two groups so far leading the computation (LP^3 and ETMC)
    - seemingly very good and healthy competition
      - involved steps critically reviewed each other
  - other activities are underway → see “white paper”
  - further efforts underway aiming for more precision
    - larger nucleon momentum with multiple lattice spacing

Related reviews on PDF from Lattice

- White paper
  - Parton distributions and lattice QCD calculations: a community white paper
    - H-W. Lin et al (32 lattice & non-lattice authors) [1711.07916]
  - how lattice calculation could make impact is also discussed

- Lattice 2016 review
  - From C to Parton Sea: Bjorken-x Dependence of the PDFs
    - H-W. Lin [1612.09366]

- Lattice 2017 review
  - Parton distributions in the LHC era
    - L. Del Debbio [EPJ Web Conf. 175 (2018) 01006]
      - global fit and lattice