Resummation for transverse observables at hadron colliders

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Based on
1604.02191 with E. Re and P. Torrielli
and
1705.09127 with W. Bizon, E. Re, L. Rottoli, and P. Torrielli

+ ongoing work with
W. Bizon, X. Chen, Gehrmann-De Ridder, Gehrmann, Glover, A. Huss,
E. Re, L. Rottoli, and P. Torrielli

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Outline

- Theory precision at colliders:
  - fixed-order vs. all-order perturbation theory
- Factorisation theorems and semi-numerical resummation
- Momentum-space resummation for transverse observables
- Predictions for differential distributions at $\mathrm{N^3LL}$+NNLO at the LHC
  - Higgs production
  - Drell-Yan production
- Conclusions

Fixed-order vs. All-order

- Fixed-order calculations of radiative corrections are formulated in a well established way (technically challenging, but well posed problem):
  - compute amplitudes at a given order
  - provide an effective subtraction of IRC divergences
  - compute any IRC-safe observable

$$\Sigma(v) = \int_0^v \frac{1}{\sigma_{\rm Born}} \frac{d\sigma}{dv'} dv' \sim \overset{\text{LO}}{1} + \overset{\text{NLO}}{\alpha_s} + \overset{\text{NNLO}}{\alpha_s^2} + \dots$$

- All-order calculations are still at an earlier stage of evolution
  - Each different observable has its own type of sensitivity to IRC physics, it is hard to formulate a general method that works for all at a generic perturbative order
  - Higher-order resummations are therefore often formulated in an observable-dependent way, for few well-behaved collider observables

$$\Sigma(v) = \int_0^v \frac{1}{\sigma_{\rm Born}} \frac{d\sigma}{dv'} dv' \underset{v \to 0}{\sim} e^{\overset{\text{LL}}{\alpha_s^n L^{n+1}} + \overset{\text{NLL}}{\alpha_s^n L^n} + \overset{\text{NNLL}}{\alpha_s^n L^{n-1}} + \dots}$$

Factorisation of the observable

- Factorisation of the amplitude is not enough as the all-order radiation is tangled by the observable

$$\Sigma(v) = \int d\Phi_{\rm rad} \sum_{n=0}^{\infty} |\mathcal{M}(k_1,\ldots,k_n)|^2 \Theta(v - V(k_1,\ldots,k_n))$$

- In order to perform an all-order calculation, one needs to ***break*** the observable too into hard, soft and collinear pieces. This can be done for some observables which treat the radiation rather inclusively
    - e.g. transverse momentum of a massive singlet

[Parisi, Petronzio '79]
[Collins et al. '85]
[Bozzi et al. '05]
[Becher et al. '10+'12]

$$\delta^{(2)}(\vec{p_t} - (\vec{k}_{t1} + \ldots + \vec{k}_{tn})) = \int \frac{d^2\vec{b}}{4\pi^2} e^{-i\vec{b}\cdot\vec{p_t}} \prod_{i=1}^{n} e^{i\vec{b}\cdot\vec{k}_{ti}},$$

$$\frac{d^2\Sigma(p_t)}{d\Phi_B\, dp_t} = \sum_{c_1,c_2} \frac{d|M_B|^2_{c_1c_2}}{d\Phi_B} \int b\, db\, p_t J_0(p_t b)\, \mathbf{f}^T(b_0/b) \mathbf{C}^{c_1;T}_{N_1}(\alpha_S(b_0/b)) H_{\rm CSS}(M) \mathbf{C}^{c_2}_{N_2}(\alpha_S(b_0/b)) \mathbf{f}(b_0/b)$$
$$\times \exp\left\{-\sum_{\ell=1}^{2} \int_{b_0/b}^{M} \frac{dk_t}{k_t} \mathbf{R}'_{{\rm CSS},\ell}(k_t)\right\}.$$

[Catani, Grazzini '11][Catani et al. '12][Gehrmann, Luebbert, Yang '14][Davies, Stirling '84]
[De Florian, Grazzini '01][Becher, Neubert '10][Li, Zhu '16][Vladimirov '16]

Eluding observable factorisation

- Factorisation is a powerful tool, but limited to observables that have a simple analytic expression in the relevant limits or do not mix soft and collinear radiation (e.g. jet rates)

- Ultimately, we want to use the modern knowledge of IRC dynamics to make more accurate generators. At present a general framework to assess the accuracy of Parton Showers is missing

    - It is of primary importance to formulate a link between higher-order resummation and PS

- Can we devise a formulation without a factorisation formula ?

    - ***recursive*** **IRC safety**: simple set of criteria for the observable that allows one to formulate the resummation at NLL for global observables without the need for an explicit factorisation. [Banfi, Salam, Zanderighi '01-'04]

    - Most of modern global observables fall into this category.

    - The method can be reformulated and extended at higher logarithmic orders

[Banfi, McAslan, PM, Zanderighi '14-'16]
[PM, Re, Torrielli '16]
[Bizon, PM, Re, Rottoli, Torrielli '17]

A case study: transverse observables

- Transverse and inclusive observables in colour-singlet production offer a clean experimental and theoretical environment for precision physics:

$$\boxed{V(\{\tilde{p}\},k) \equiv V(k) = d_\ell \, g_\ell(\phi) \left(\frac{k_t}{M}\right)^a}$$

$$\boxed{V(\{\tilde{p}\},k_1,\dots,k_n) = V(\{\tilde{p}\},k_1+\dots+k_n)}$$

  - SM measurements (e.g. W, Z, photon,…): parton distributions, strong coupling, W mass,…
    - sensitivity to non-perturbative effects (hadronisation, intrinsic kt) only through transverse recoil
    - very little/no sensitivity to multi-parton interactions
  - BSM measurements/constraints (e.g. Higgs): light/heavy NP, Yukawa couplings,…
  - Theoretically interesting:
    - clean environment to test/calibrate exclusive generators against high perturbative orders
    - Two mechanisms compete in the $p_t \to 0$ limit:
      - Sudakov (exponential) suppression when $k_{ti} \sim p_t \ll M$
      - Azimuthal cancellations (power suppression, dominant) when $p_t \ll k_{ti} \ll M$

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    - sensitivity to non-perturbative effects (hadronisation, intrinsic kt) only through
    -
  - BS

Can we build a more exclusive solution in momentum space ?

See also work in [Ebert, Tackmann '16][Kang, Lee, Vaidya '17]

  - Theoretically interesting:
    - clean environment to test/calibrate exclusive generators against high perturbative orders
    - Two mechanisms compete in the $p_t \to 0$ limit:
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Direct space: virtual corrections

- Write all-order cross section as ($V(\{\tilde{p}\}, k_1, \ldots, k_n) = |\vec{k}_{t1} + \cdots + \vec{k}_{tn}|$)

$$\Sigma(v) = \int d\Phi_B \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 \, \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_n)\right)$$

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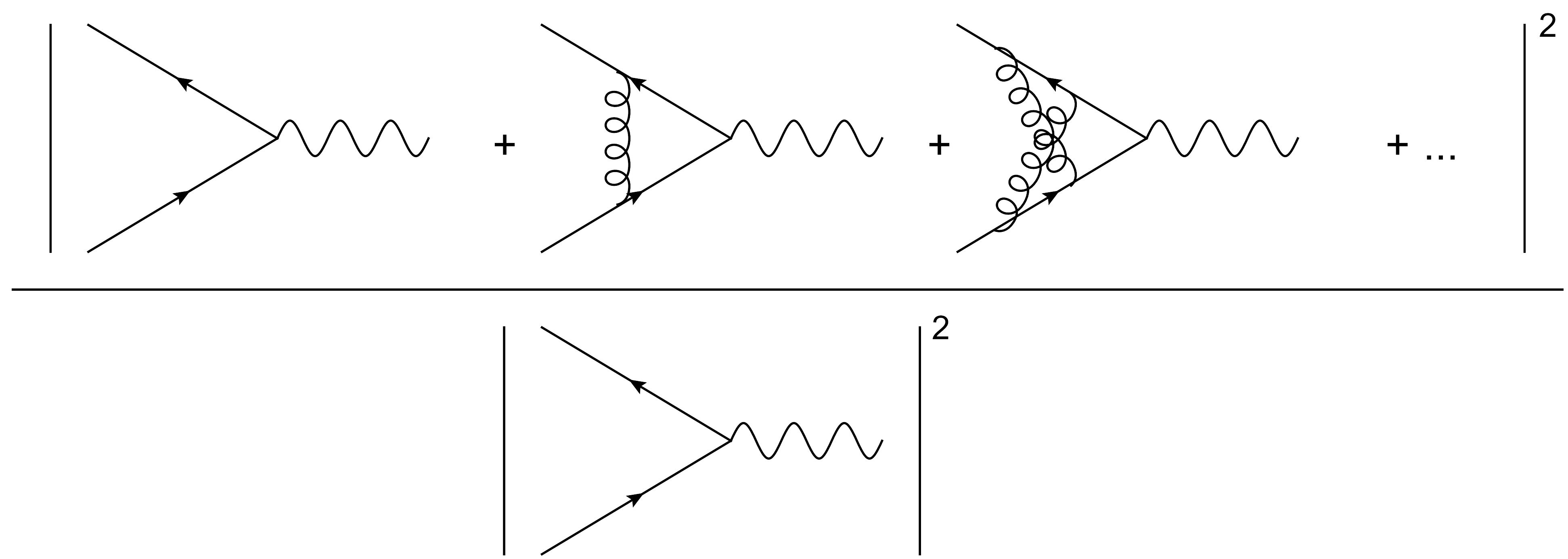
All-order form factor

e.g. [Dixon, Magnea, Sterman ’08]

$\mathcal{V}(\Phi_B) =$ 

Direct space: real radiation

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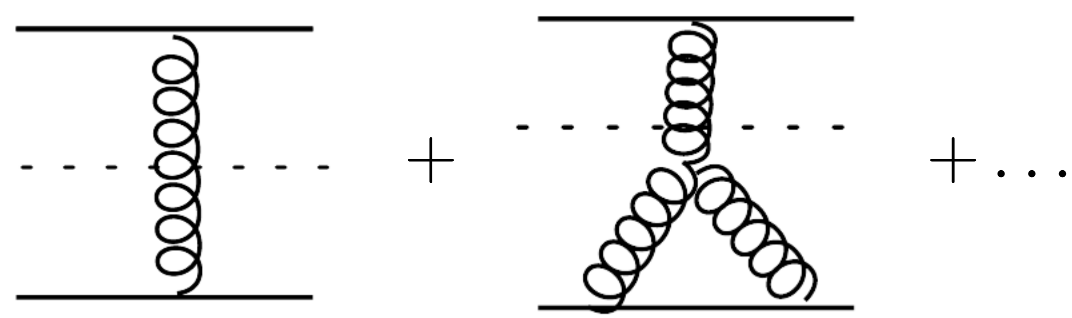
Real emissions

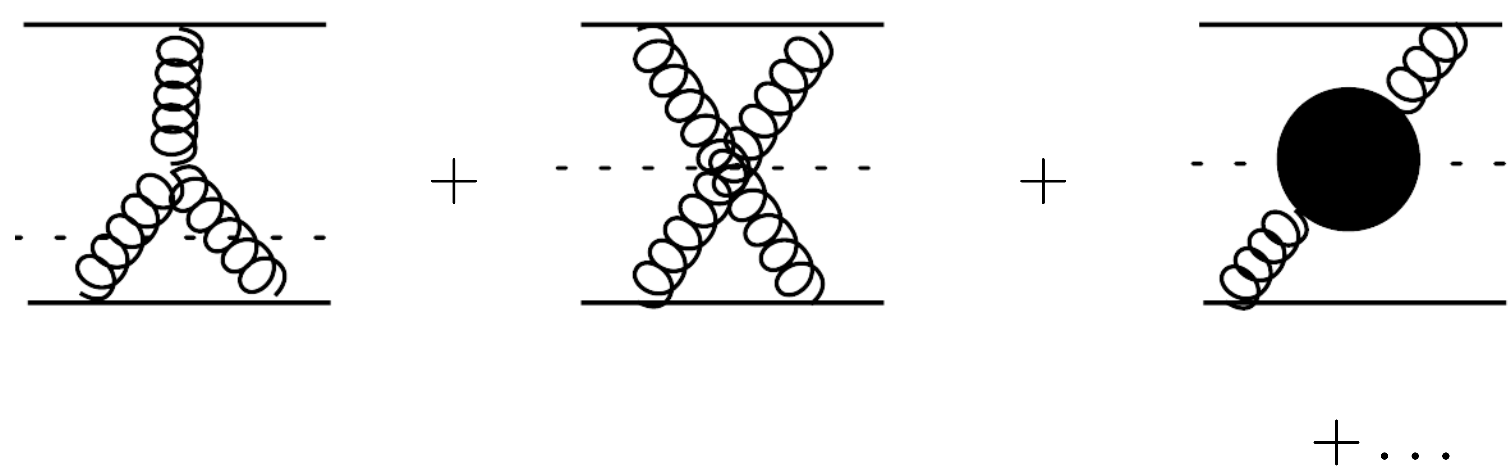
- Logarithmic counting: we need a logarithmic hierarchy in the squared amplitudes (resummation means iteration of lower-order amplitudes)

e.g.
soft radiation (one log down in hard-collinear case)

$$|\tilde{M}(k_a)|^2 = \frac{|M(\tilde{p}_1, \tilde{p}_2, k_a)|^2}{|M_B(\tilde{p}_1, \tilde{p}_2)|^2} = |M(k_a)|^2 \longrightarrow$$



$$|\tilde{M}(k_a, k_b)|^2 = \frac{|M(\tilde{p}_1, \tilde{p}_2, k_a, k_b)|^2}{|M_B(\tilde{p}_1, \tilde{p}_2)|^2} - \frac{1}{2!}|M(k_a)|^2 |M(k_b)|^2 \longrightarrow$$



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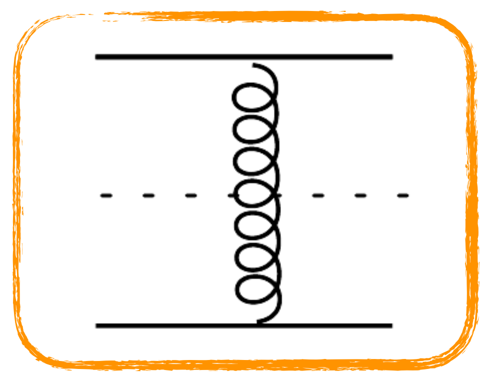
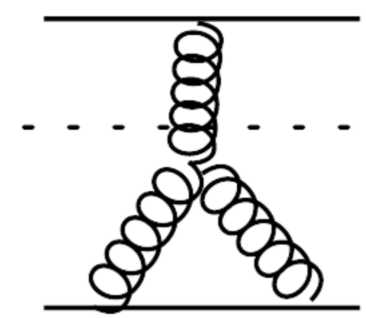


$\alpha_s L^2$

$+$



$+ \ldots$

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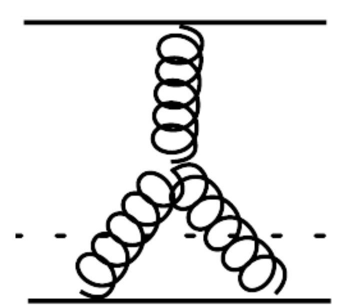

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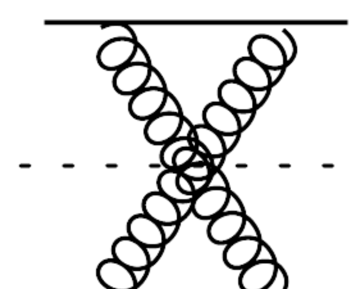

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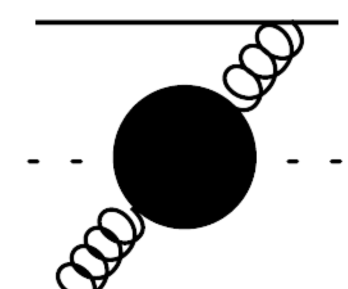

$+ \ldots$

Direct space: real radiation

- Write all-order cross section as ($V(\{\tilde p\},k_1,\dots,k_n) = |\vec{k}_{t1}+\dots+\vec{k}_{tn}|$)

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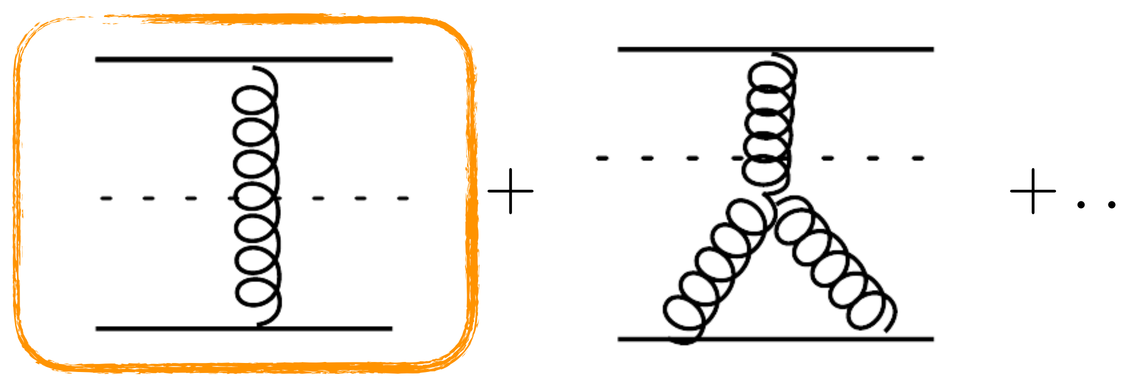

$\alpha_s L^2$

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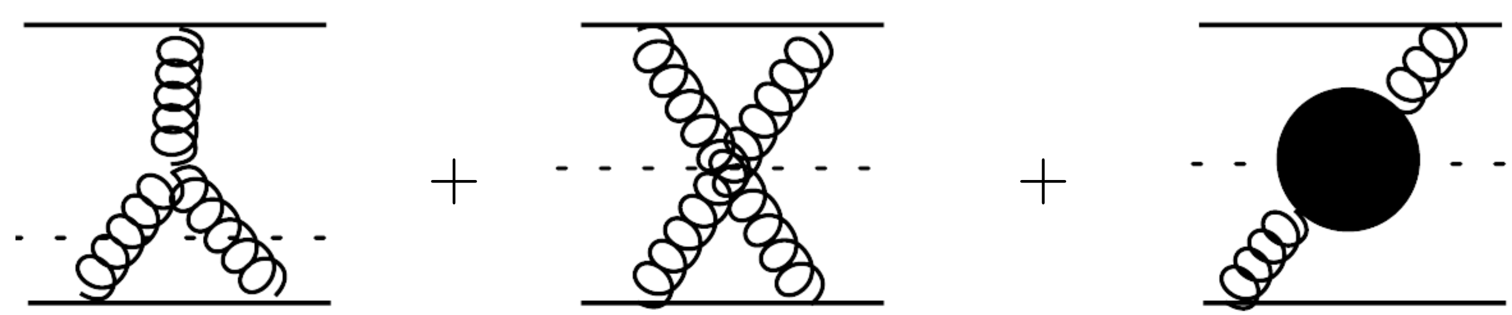

$+\dots$

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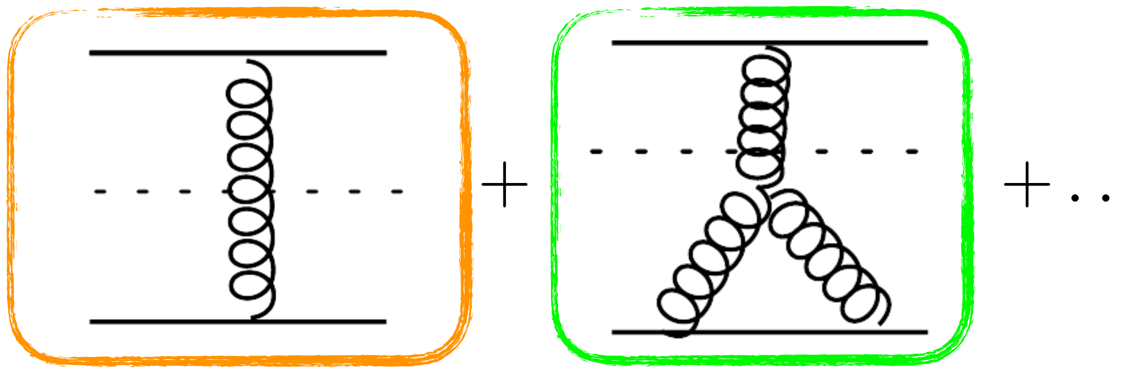

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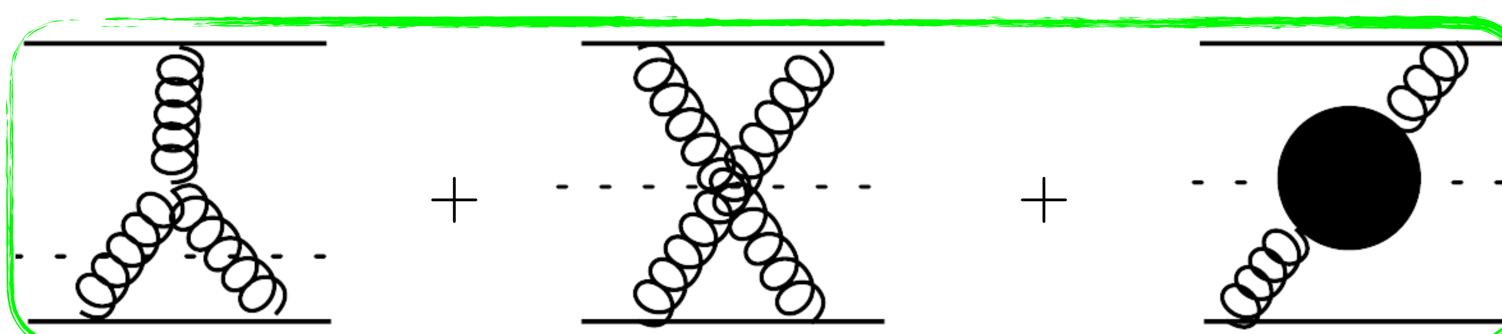

$\alpha_s^2 L^3 + \alpha_s^2 L^2$ $+ \ldots$

this LL is absorbed in the resummation of $|\mathrm{M(k)}|^2$

All-order subtraction of IRC singularities

- Write all-order cross section as ($V(\{\tilde{p}\}, k_1, \ldots, k_n) = |\vec{k}_{t1} + \cdots + \vec{k}_{tn}|$)

$$\Sigma(v) = \int d\Phi_B \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 \, \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_n)\right)$$

Subtraction of the IRC poles and computation of the observable

All-order subtraction of IRC singularities

- Subtraction of the IRC poles between $\sum_{n=0}^{\infty}\int\prod_{i=1}^{n}[dk_i]|M(\tilde{p}_1,\tilde{p}_2,k_1,\dots,k_n)|^2$ and $\mathcal{V}(\Phi_B)$:
  - introduce a phase-space resolution scale (slicing parameter) $Q_0 = \epsilon k_{t1}$
  - real correlated blocks with total transverse momentum $k_{ti} < \epsilon k_{t1}$(unresolved) do not modify the observable, and can be *ignored* in the measurement function
  - compute *unresolved* reals and *virtuals* analytically in D dimensions (*much* easier than full observable)

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$$
\begin{aligned}
\sum_{n=0}^{\infty}|M(\tilde{p}_1,\tilde{p}_2,k_1,\ldots,k_n)|^2 &\longrightarrow |M_B(\tilde{p}_1,\tilde{p}_2)|^2 \\
&\times\sum_{n=0}^{\infty}\frac{1}{n!}\left\{\prod_{i=1}^{n}\left(|M(k_i)|^2+\int[dk_a][dk_b]|\tilde{M}(k_a,k_b)|^2\delta^{(2)}(\vec{k}_{ta}+\vec{k}_{tb}-\vec{k}_{ti})\delta(Y_{ab}-Y_i)\right.\right. \\
&\left.\left.+\int[dk_a][dk_b][dk_c]|\tilde{M}(k_a,k_b,k_c)|^2\delta^{(2)}(\vec{k}_{ta}+\vec{k}_{tb}+\vec{k}_{tc}-\vec{k}_{ti})\delta(Y_{abc}-Y_i)+\ldots\right)\right\}
\end{aligned}
$$

All-order subtraction of IRC singularities

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$$\prod_{i=1}^{n} \int [dk_i] \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \prod_{i=1}^{n} \Big(|M(k_i)|^2 + \int [dk_a][dk_b] |\tilde{M}(k_a, k_b)|^2 \delta^{(2)}(\vec{k}_{ta} + \vec{k}_{tb} - \vec{k}_{ti}) \delta(Y_{ab} - Y_i) \right.$$
$$\left. + \int [dk_a][dk_b][dk_c] |\tilde{M}(k_a, k_b, k_c)|^2 \delta^{(2)}(\vec{k}_{ta} + \vec{k}_{tb} + \vec{k}_{tc} - \vec{k}_{ti}) \delta(Y_{abc} - Y_i) + \ldots \Big) \Theta(\epsilon k_{t1} - k_{ti}) \right\}$$

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$$\prod_{i=1}^{n}\int[dk_i]\mathcal{V}(\Phi_B)\sum_{n=0}^{\infty}\frac{1}{n!}\Bigg\{\prod_{i=1}^{n}\Bigg(|M(k_i)|^2+\int[dk_a][dk_b]|\tilde{M}(k_a,k_b)|^2\delta^{(2)}(\vec{k}_{ta}+\vec{k}_{tb}-\vec{k}_{ti})\delta(Y_{ab}-Y_i)$$
$$+\int[dk_a][dk_b][dk_c]|\tilde{M}(k_a,k_b,k_c)|^2\delta^{(2)}(\vec{k}_{ta}+\vec{k}_{tb}+\vec{k}_{tc}-\vec{k}_{ti})\delta(Y_{abc}-Y_i)+\ldots\Bigg)\Theta(\epsilon k_{t1}-k_{ti})\Bigg\}$$
$$\propto\int\frac{dk_{t1}}{k_{t1}}\frac{d\phi_1}{2\pi}e^{-R(\epsilon k_{t1})}R'(k_{t1})$$

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$$R(\epsilon k_{t1}) \equiv \sum_{\ell=1}^{2}\int_{\epsilon k_{t1}}^{M}\frac{dk_t}{k_t}R'_\ell(k_t) = \sum_{\ell=1}^{2}\int_{\epsilon k_{t1}}^{M}\frac{dk_t}{k_t}\left(A_\ell(\alpha_s(k_t))\ln\frac{M^2}{k_t^2} + B_\ell(\alpha_s(k_t))\right)$$

Anomalous dimensions start differing from b-space ones at N³LL

$$\prod_{i=1}^{n}\int[dk_i]\mathcal{V}(\Phi_B)\sum_{n=0}^{\infty}\frac{1}{n!}\left\{\prod_{i=1}^{n}\left(|M(k_i)|^2 + \int[dk_a][dk_b]|\tilde{M}(k_a,k_b)|^2\delta^{(2)}(\vec{k}_{ta}+\vec{k}_{tb}-\vec{k}_{ti})\delta(Y_{ab}-Y_i)\right.\right.$$
$$\left.\left.+\int[dk_a][dk_b][dk_c]|\tilde{M}(k_a,k_b,k_c)|^2\delta^{(2)}(\vec{k}_{ta}+\vec{k}_{tb}+\vec{k}_{tc}-\vec{k}_{ti})\delta(Y_{abc}-Y_i)+\ldots\right)\Theta(\epsilon k_{t1}-k_{ti})\right\}$$
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$$\hat{\mathbf{\Sigma}}^{c_1,c_2}_{N_1,N_2}(v) = \boxed{\begin{aligned}&\left[\mathbf{C}^{c_1;T}_{N_1}(\alpha_s(\mu_0))H(\mu_R)\mathbf{C}^{c_2}_{N_2}(\alpha_s(\mu_0))\right] \int_0^M \frac{dk_{t1}}{k_{t1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \\ &\times e^{-\mathbf{R}(\epsilon k_{t1})} \exp\left\{-\sum_{\ell=1}^{2}\left(\int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \frac{\alpha_s(k_t)}{\pi} \mathbf{\Gamma}_{N_\ell}(\alpha_s(k_t)) + \int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \mathbf{\Gamma}^{(C)}_{N_\ell}(\alpha_s(k_t))\right)\right\}\end{aligned}}$$

DGLAP anomalous dims

RGE evolution of coeff. functions

Sudakov radiator: integral of single inclusive block.

All-order subtraction of IRC singularities

- Subtraction of the IRC poles between $\sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2$ and $\mathcal{V}(\Phi_B)$:
  - introduce a phase-space resolution scale (slicing parameter) $Q_0 = \epsilon k_{t1}$
  - real correlated blocks with total transverse momentum $k_{ti} < \epsilon k_{t1}$(unresolved) do not modify the observable, and can be *ignored* in the measurement function
  - compute *unresolved* reals and *virtuals* analytically in D dimensions (*much* easier than full observable)

$$
\begin{aligned}
\hat{\mathbf{\Sigma}}^{c_1,c_2}_{N_1,N_2}(v) = & \left[\mathbf{C}^{c_1;T}_{N_1}(\alpha_s(\mu_0)) H(\mu_R) \mathbf{C}^{c_2}_{N_2}(\alpha_s(\mu_0))\right] \int_0^M \frac{dk_{t1}}{k_{t1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \\
& \times e^{-\mathbf{R}(\epsilon k_{t1})} \exp\left\{-\sum_{\ell=1}^{2}\left(\int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \frac{\alpha_s(k_t)}{\pi} \mathbf{\Gamma}_{N_\ell}(\alpha_s(k_t)) + \int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \mathbf{\Gamma}^{(C)}_{N_\ell}(\alpha_s(k_t))\right)\right\} \\
& \sum_{\ell_1=1}^{2}\left(\mathbf{R}'_{\ell_1}(k_{t1}) + \frac{\alpha_s(k_{t1})}{\pi} \mathbf{\Gamma}_{N_{\ell_1}}(\alpha_s(k_{t1})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_1}}(\alpha_s(k_{t1}))\right) \\
& \times \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_{\epsilon}^{1} \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \sum_{\ell_i=1}^{2}\left(\mathbf{R}'_{\ell_i}(k_{ti}) + \frac{\alpha_s(k_{ti})}{\pi} \mathbf{\Gamma}_{N_{\ell_i}}(\alpha_s(k_{ti})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_i}}(\alpha_s(k_{ti}))\right) \\
& \times \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right),
\end{aligned}
$$

DGLAP anomalous dims

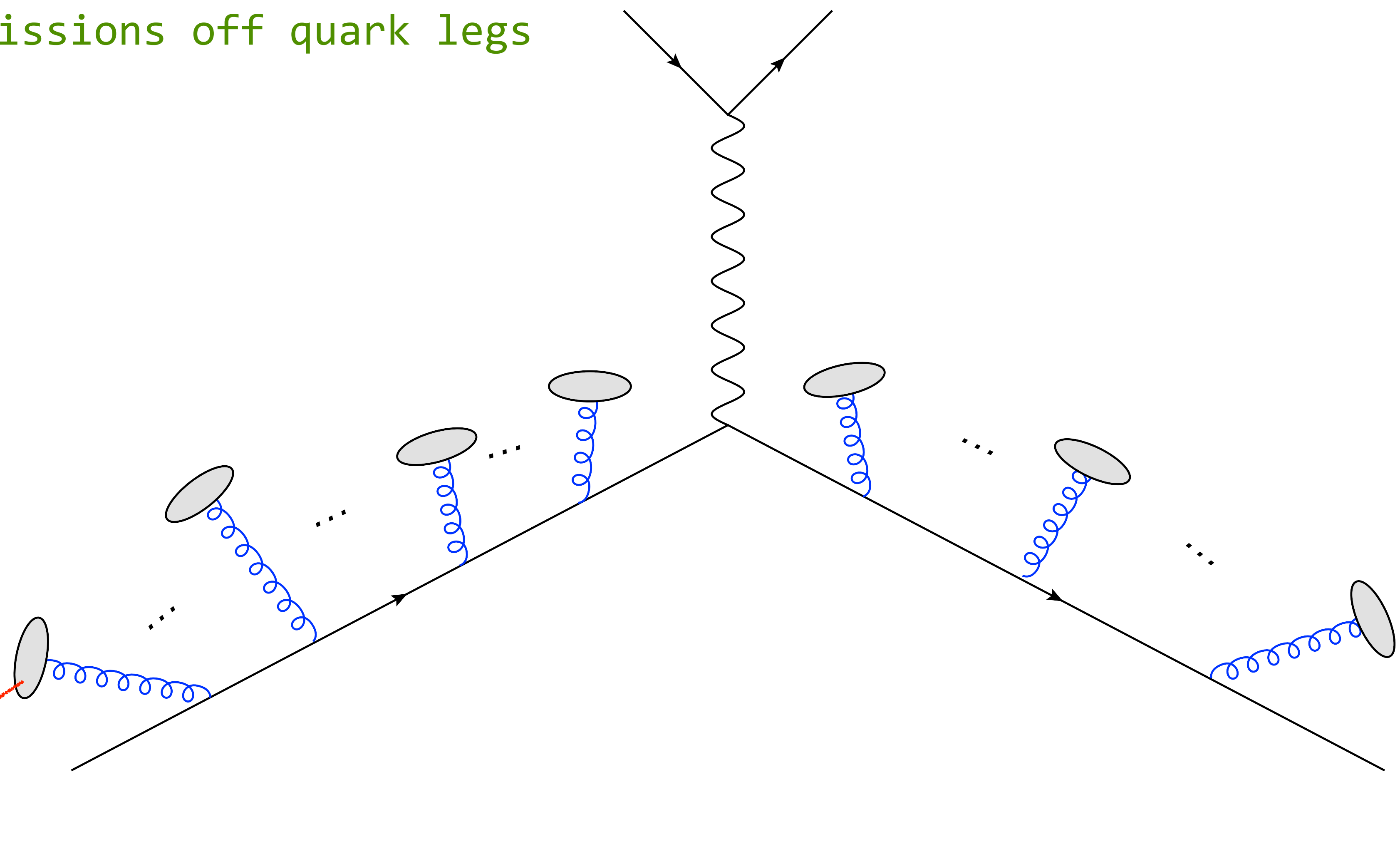
RGE evolution of coeff. functions

- compute *resolved* (reals only) in 4 dim. with $\epsilon \to 0$ (MC events !)

Physical picture: MC generator

- This is, essentially, a *quasi-exclusive generator* with higher logarithmic accuracy

➡ e.g. gluon emissions off quark legs



$$|M(k_i)|^2 + \int [dk_a][dk_b]|\tilde{M}(k_a,k_b)|^2\delta^{(2)}(\vec{k}_{ta}+\vec{k}_{tb}-\vec{k}_{ti})\delta(Y_{ab}-Y_i)$$

$$+\int [dk_a][dk_b][dk_c]|\tilde{M}(k_a,k_b,k_c)|^2\delta^{(2)}(\vec{k}_{ta}+\vec{k}_{tb}+\vec{k}_{tc}-\vec{k}_{ti})\delta(Y_{abc}-Y_i)+\dots$$

Small transverse momentum limit

- CSS result recovered by simply transforming observable into b-space and integrating over radiation (see backup material)

- Clear physical picture of the dynamics of azimuthal cancellations at small transverse momentum

e.g. NLL with $\mathcal{L}(\mathrm{k}_{\mathrm{t1}}) = 1$ for simplicity

$$\frac{d^3\Sigma(p_t)}{d^2p_t d\Phi_B} = \sigma^{(0)}(\Phi_B) \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(k_{t1})} R'(k_{t1}) \int d\mathcal{Z}[\{R'(k_{t1}), k_i\}] \delta^{(2)}(\vec{p_t} - \sum_{i=1}^{n+1} \vec{k}_{ti})$$

- Transition from exponential to a power-like suppression at small transverse momentum

$$\frac{d^2\Sigma(p_t)}{dp_t d\Phi_B} \simeq 4p_t\sigma^{(0)}(\Phi_B) \int_{\Lambda_{\mathrm{QCD}}}^{M} \frac{dk_{t1}}{k_{t1}^3} e^{-R(k_{t1})} \simeq 2p_t\sigma^{(0)}(\Phi_B) \left(\frac{\Lambda_{\mathrm{QCD}}^2}{M^2}\right)^{\frac{16}{25}\ln\frac{41}{16}}$$

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as $\mathrm{p_t} \to 0$ Sudakov is "frozen" at $\mathrm{k_{t1}} \gg \mathrm{p_t}$
(no exponential suppression)

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Random azimuthal orientation of momenta leads to scaling $\propto \; 1/\mathrm{k}_{\mathrm{t1}}^2$

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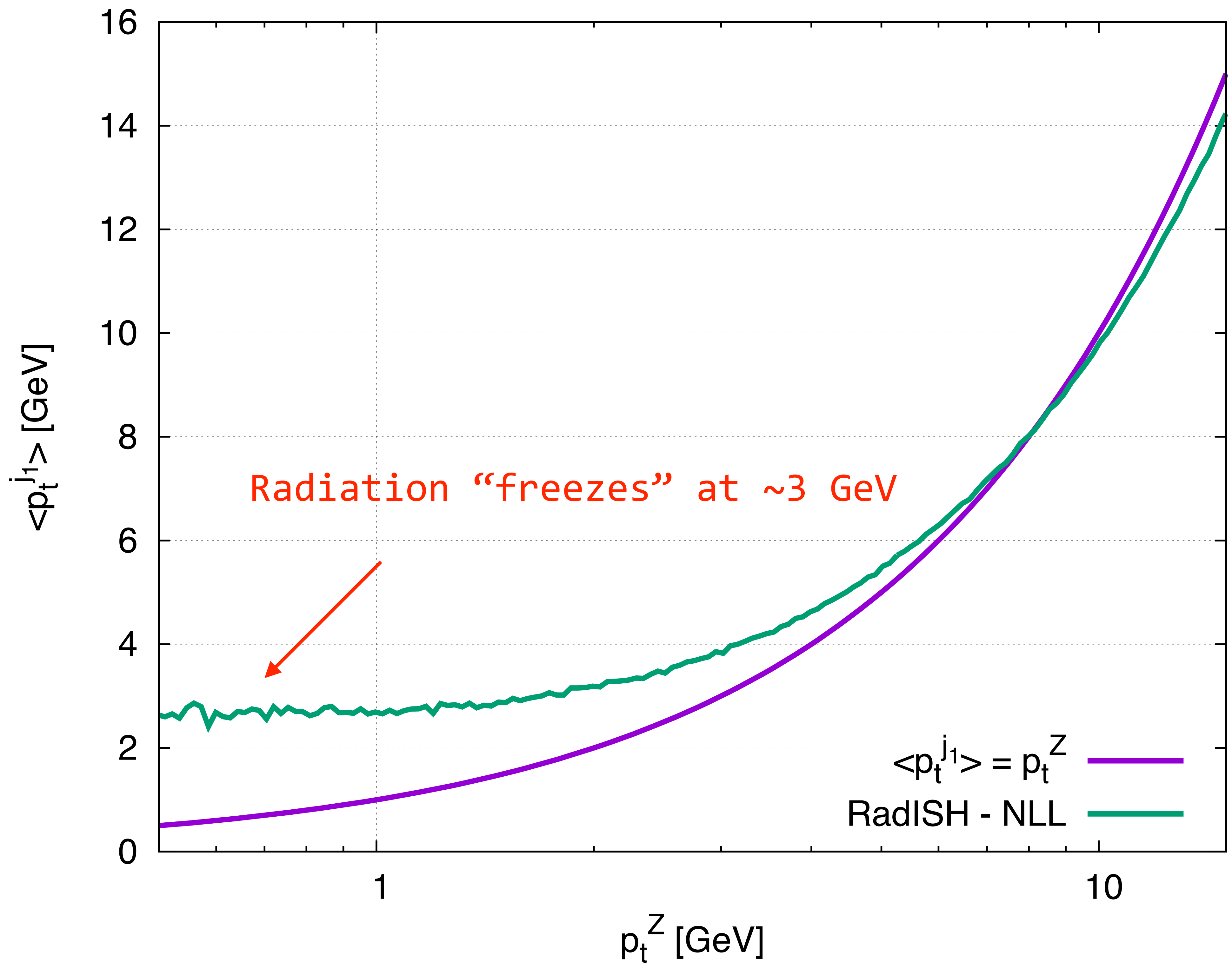
Small transverse momentum limit

➡ e.g. Z production at 14 TeV



Matching to Fixed Order

- Implementation in a MC code (RadISH) up to N3LL
  - fully differential in Born kinematics
  - matching to fixed order cumulative distribution, e.g. Higgs:

[Anastasiou et al. '15-'16] [Boughezal et al. '15] [Caola et al. '15] [Chen et al. '16]

$$\sigma^{\mathrm{N^3LO}}_{pp\to H} - \Sigma^{\mathrm{NNLO}}_{1-\mathrm{jet}}(p_t^H)$$

  - Additive vs. multiplicative schemes

OLD CHOICE :

$$\Sigma_{\mathrm{MAT}}(p_t) = (\Sigma_{\mathrm{RES}}(p_t))^{Z}\,\frac{\Sigma_{\mathrm{FO}}(p_t)}{(\Sigma_{\mathrm{EXP}}(p_t))^{Z}}$$

$$Z = \left(1-\left(\frac{p_t}{Q_{\mathrm{match}}}\right)\right)^{h}\Theta(Q_{\mathrm{match}}-p_t)$$

R – SCHEME :

$$\Sigma_{\mathrm{MAT}}(p_t) = \Sigma_{\mathrm{RES}}(p_t) + \Sigma_{\mathrm{FO}}(p_t) - \Sigma_{\mathrm{EXP}}(p_t)$$

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NEW CHOICE :

$$\Sigma_{\mathrm{MAT}}(p_t) = \frac{\Sigma_{\mathrm{RES}}(p_t)}{\mathcal{L}(\mu_F)} \left[\mathcal{L}(\mu_F) \frac{\Sigma_{\mathrm{FO}}(p_t)}{\Sigma_{\mathrm{EXP}}(p_t)}\right]_{\mathrm{EXPANDED}}$$

Matching to Fixed Order

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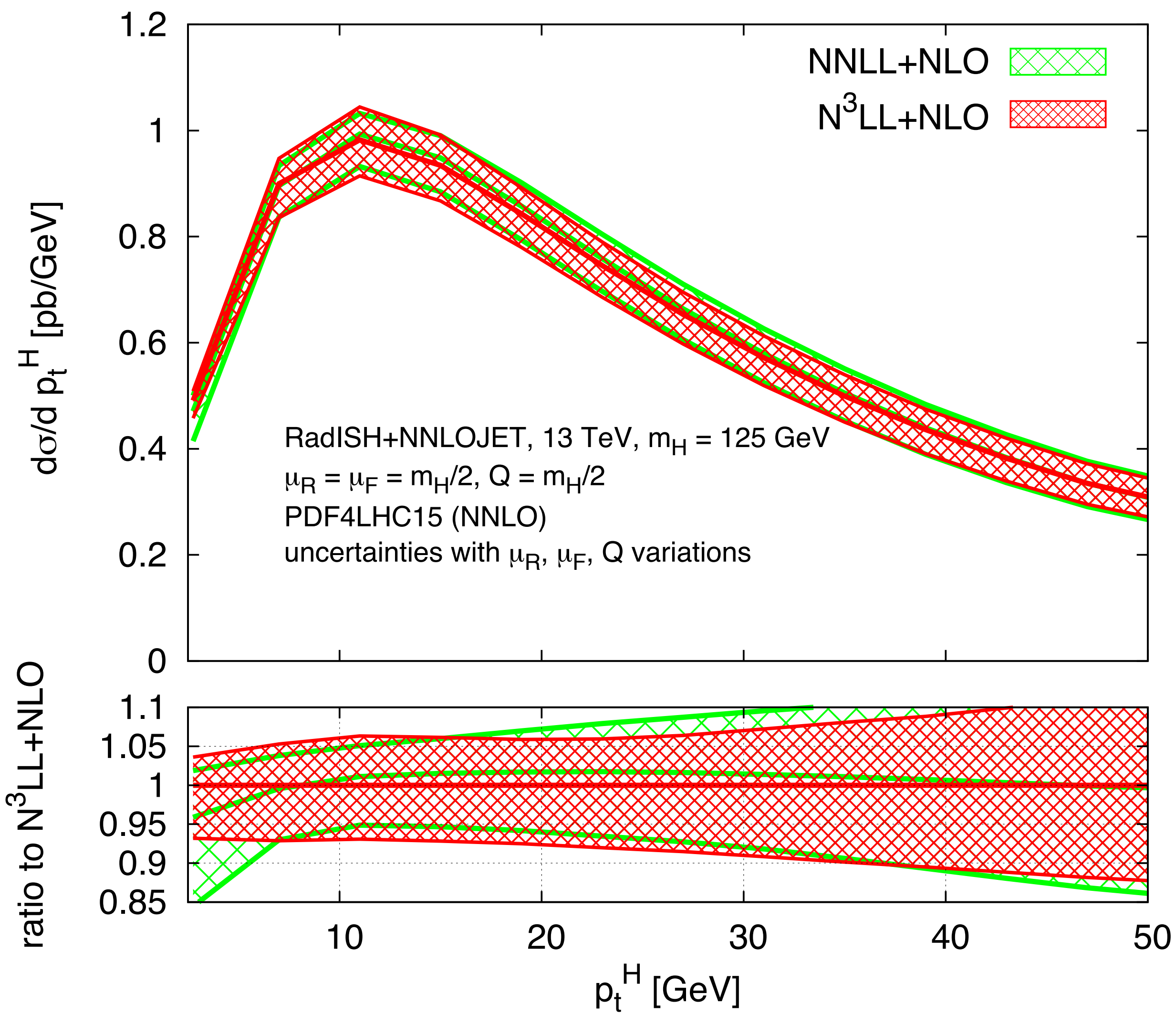
Higher-order (in a logarithmic sense) constants from FO in the multiplicative scheme. No extra parameters needed

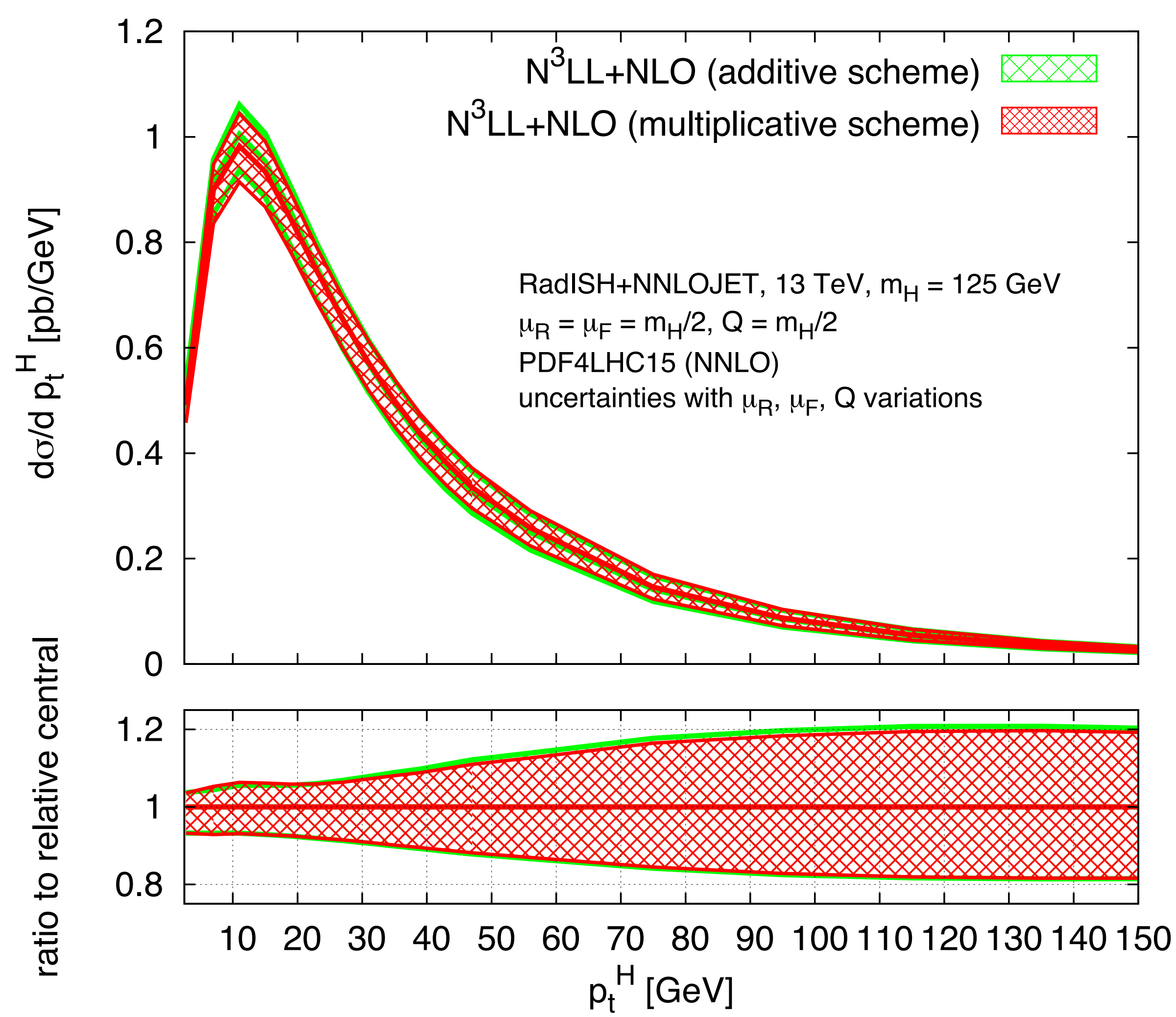
An example: Higgs pT spectrum

- Implementation in a MC code (`RadISH`) up to N^{3}LL
  - fully differential in Born kinematics

➡N^{3}LL corrections moderate, reduction of uncertainty at small pt

➡Good agreement between different matching schemes, choose multiplicative solution at higher order





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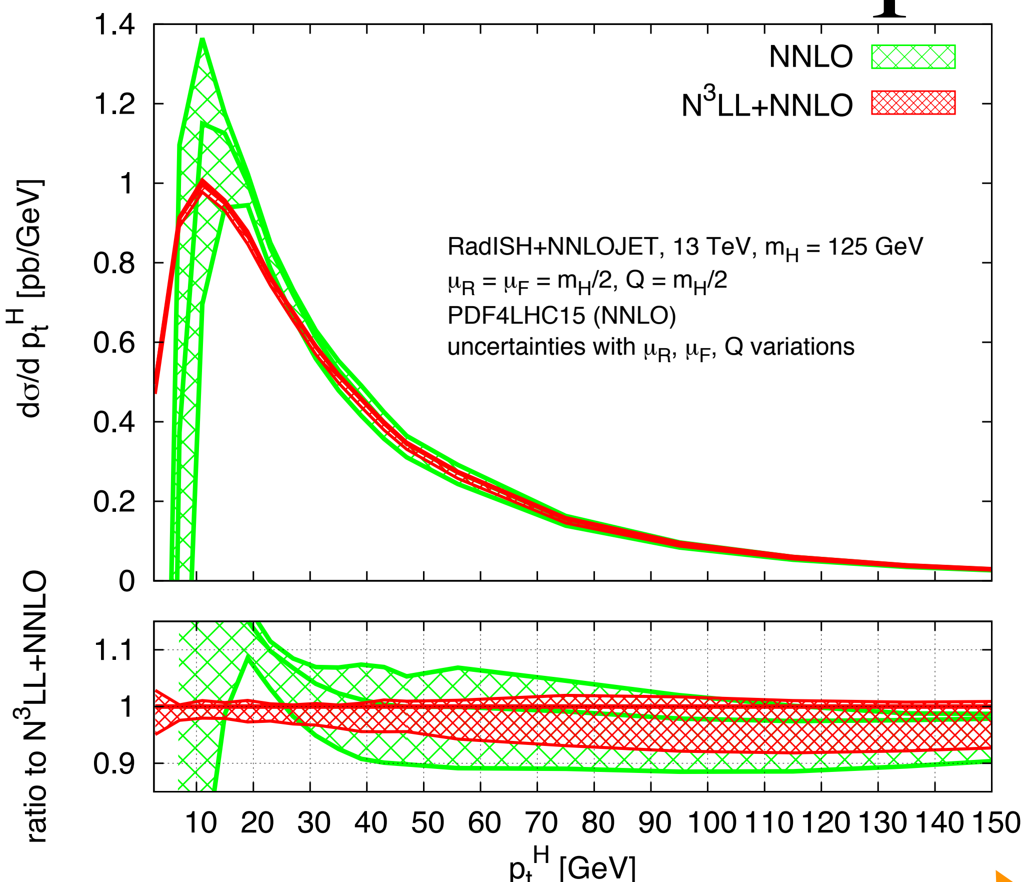
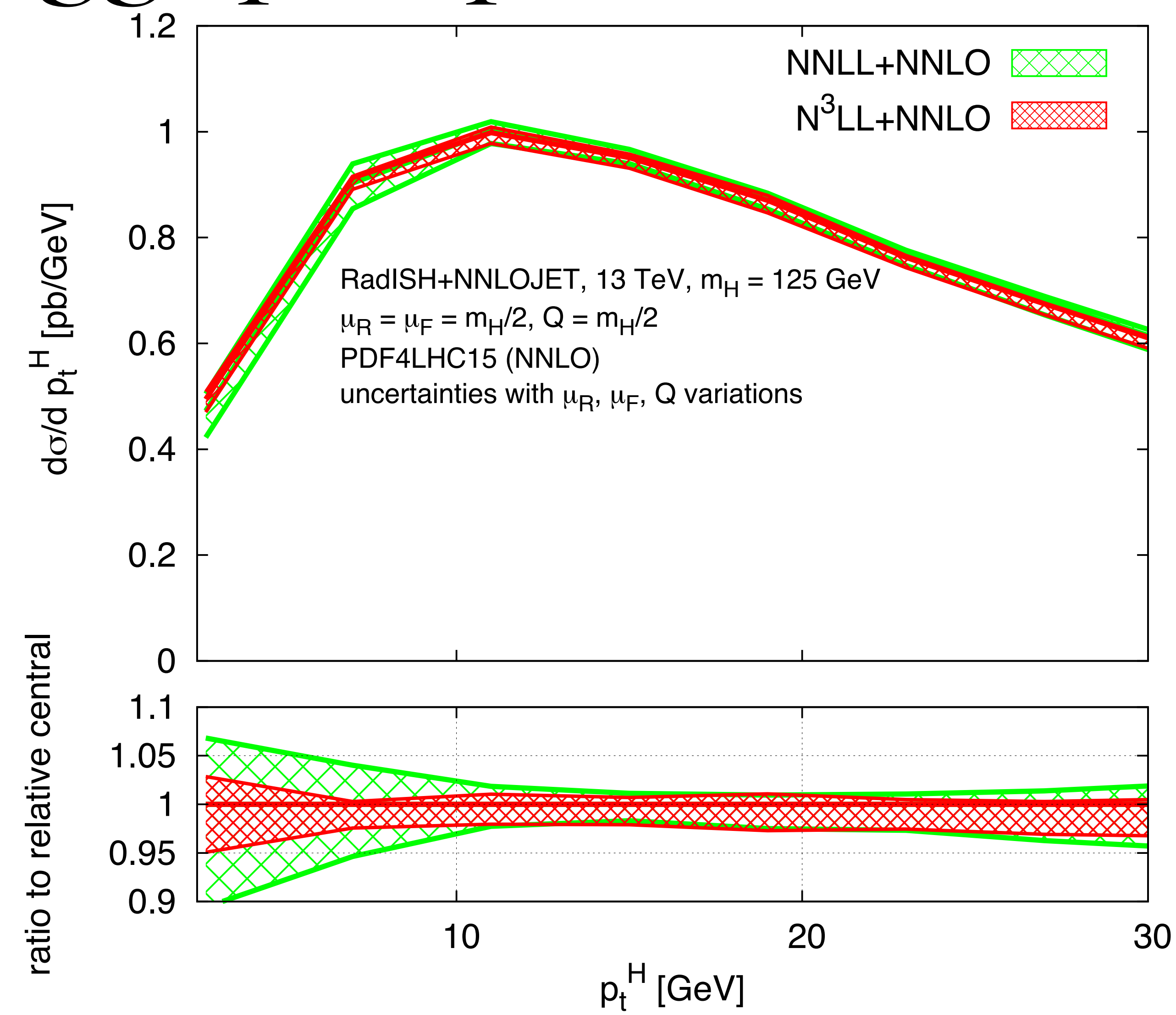
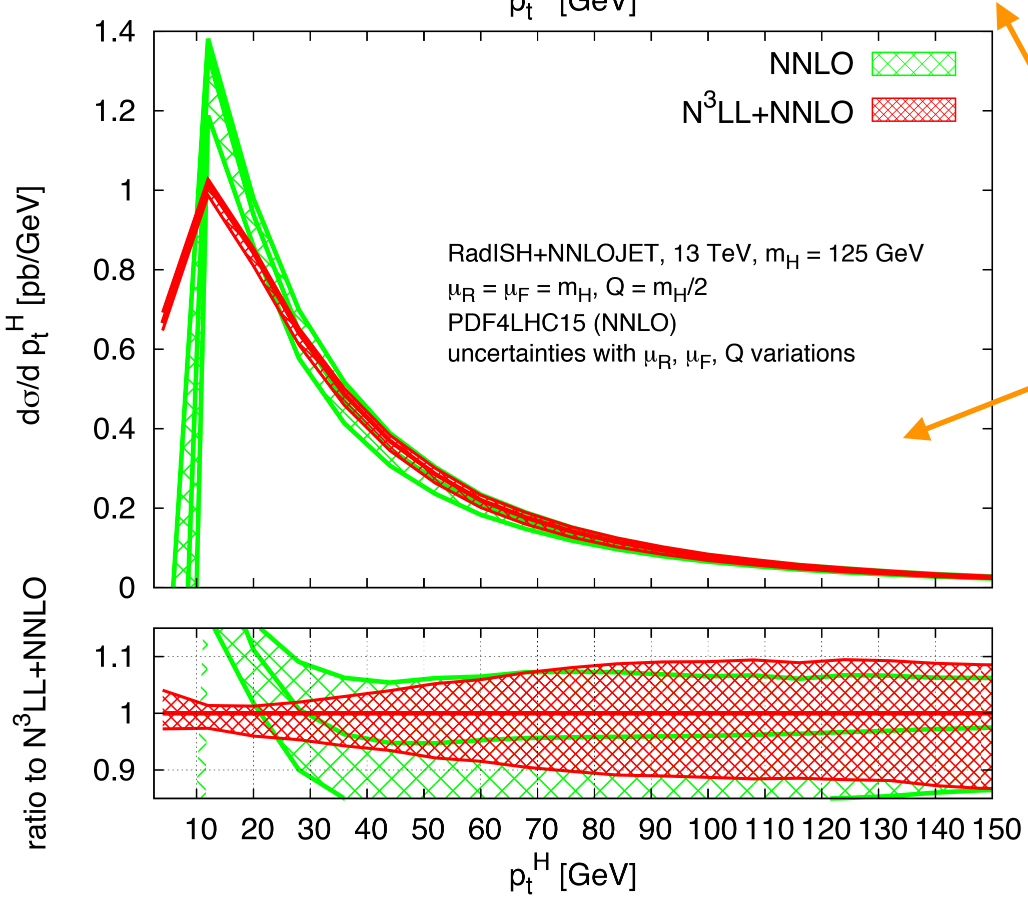




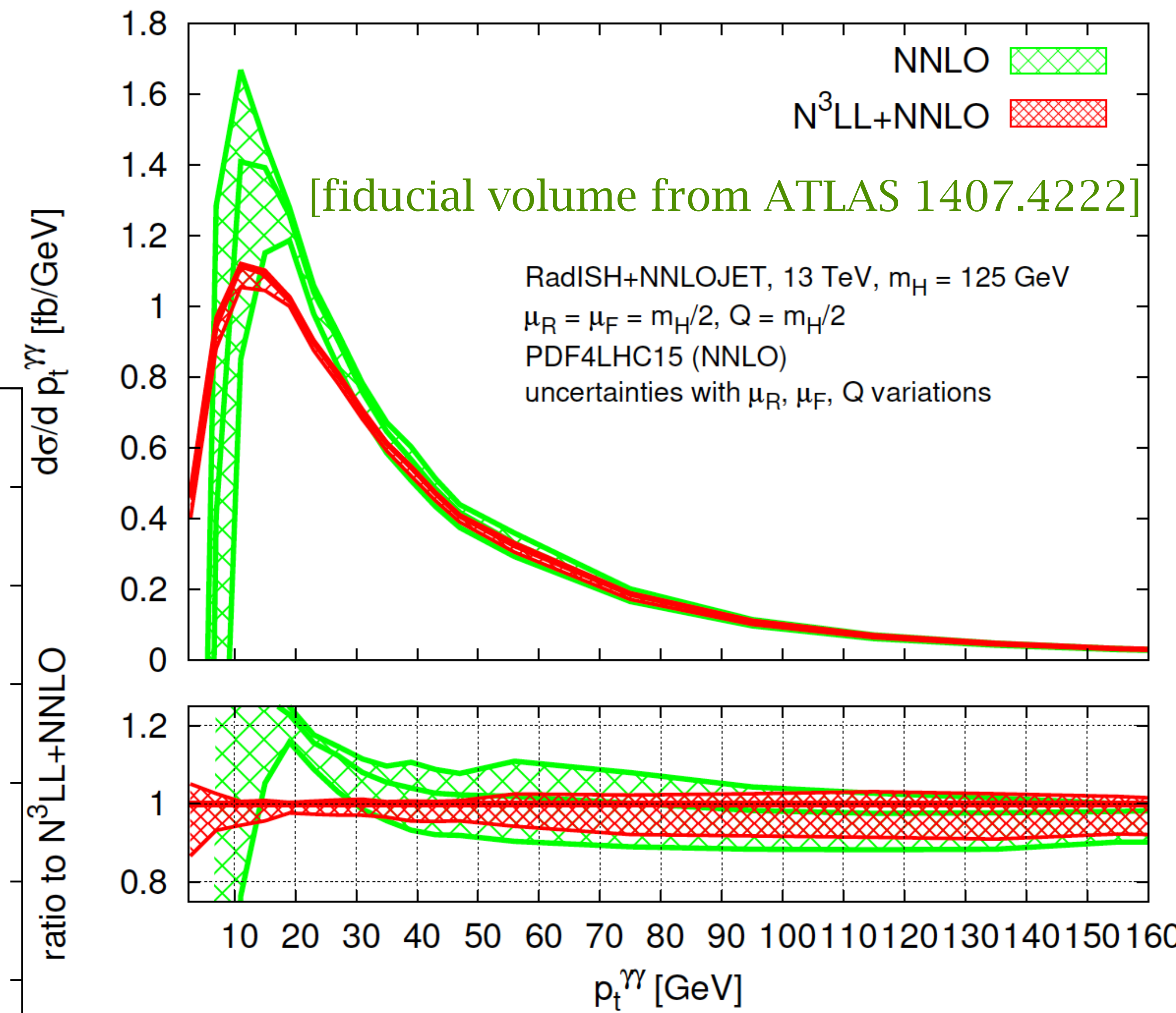


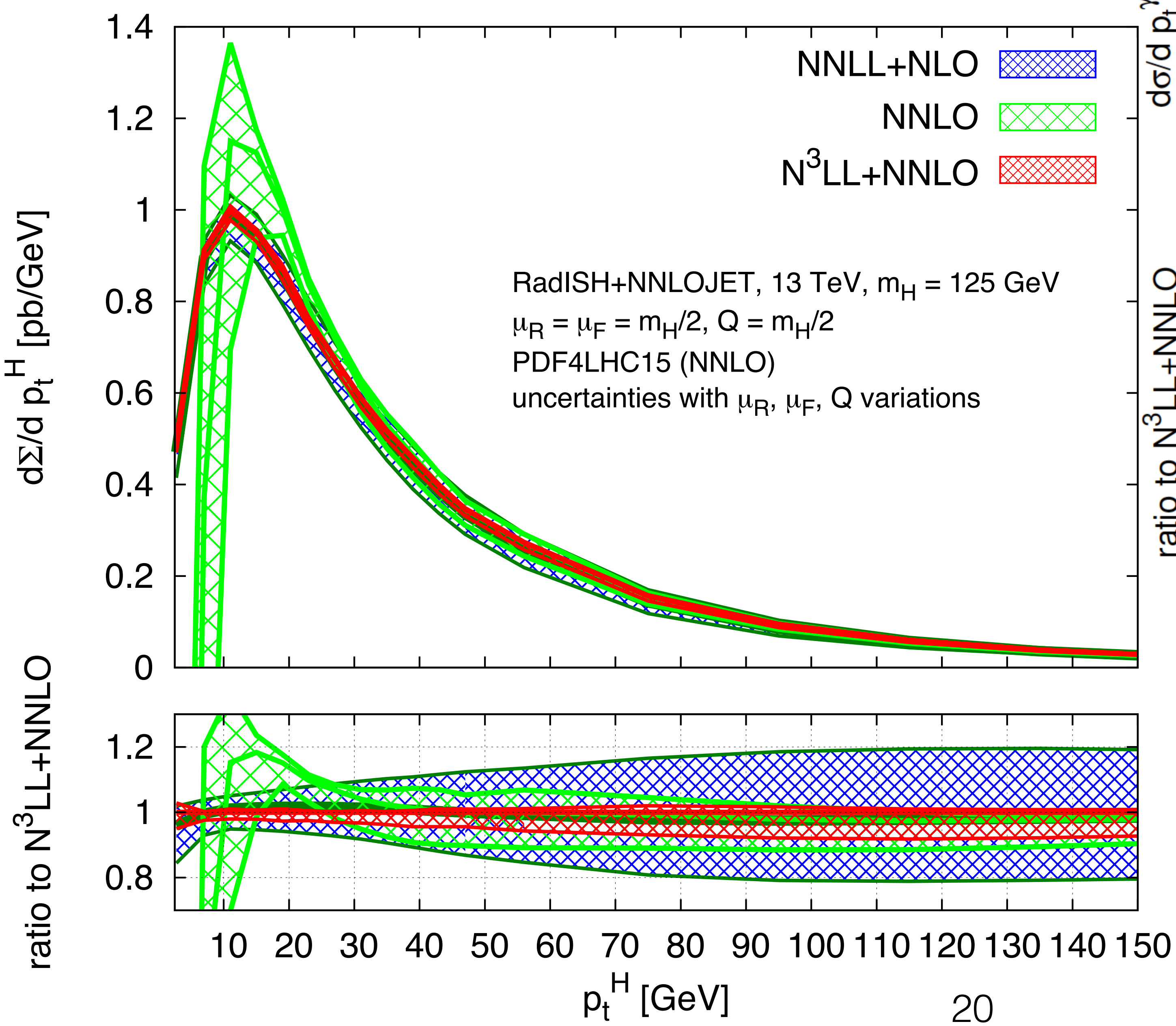
➡ Important cancellations at mH/2 (!), uncertainties likely underestimated at this scale (long known problem)

➡ N^{3}LL corrections amount to a few-% at small pt, reduction of band below 10 GeV consistently with NLO matching

An example: Higgs pT spectrum

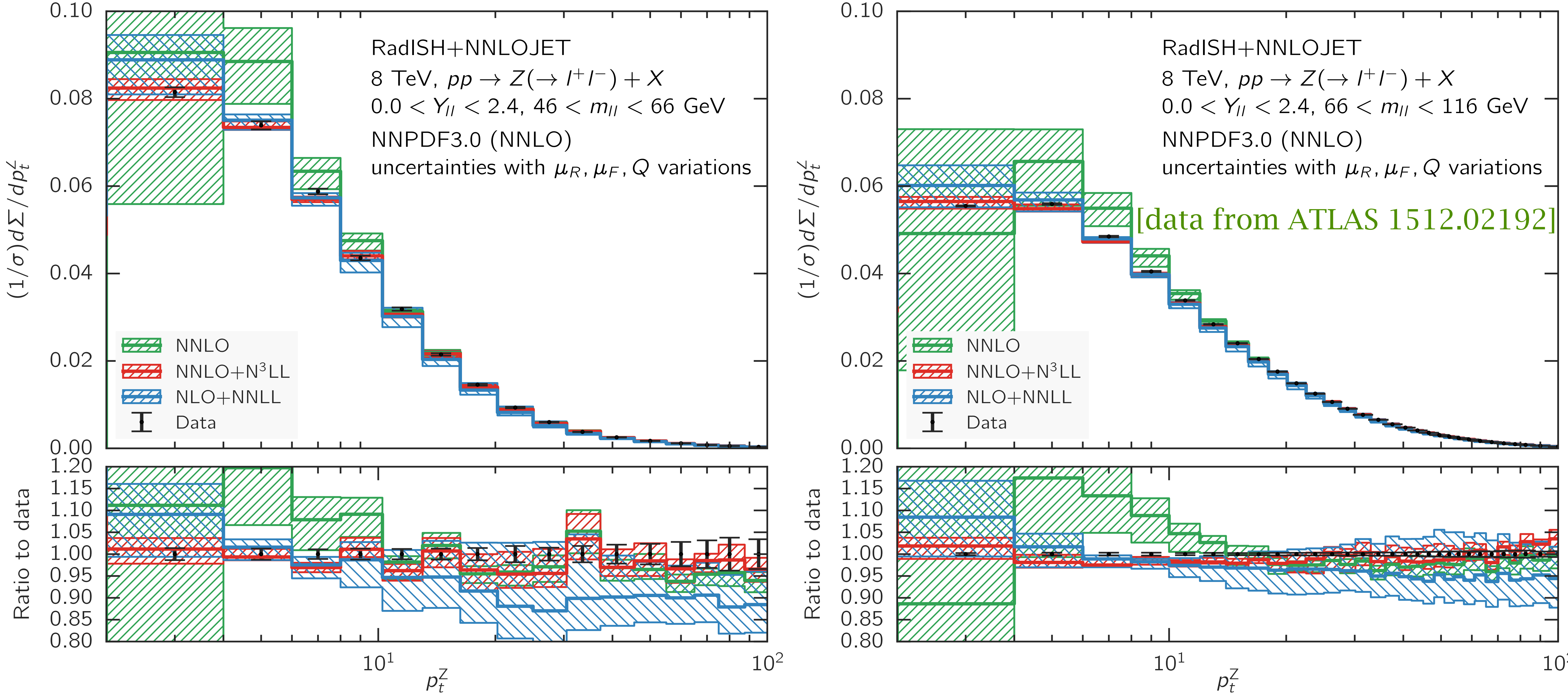
➡Simulation of fiducial distributions (e.g. H -> gamma gamma)





➡Good convergence across different perturbative orders

An example: DY distributions (pT)



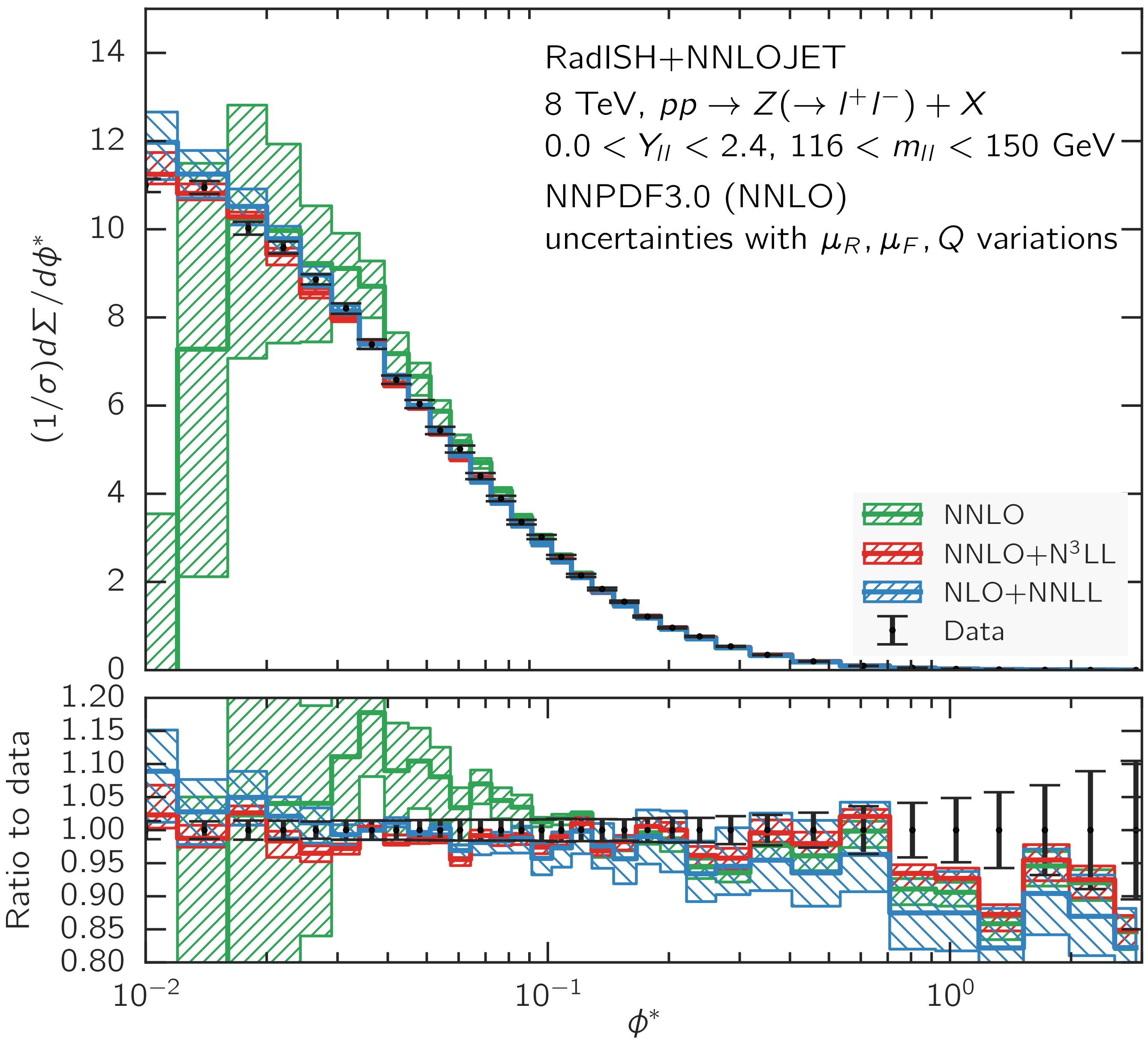
➡Matching to differential NNLO from NNLOJET, assume N³LO correction to total XS is zero (i.e. no as³ constant term included)

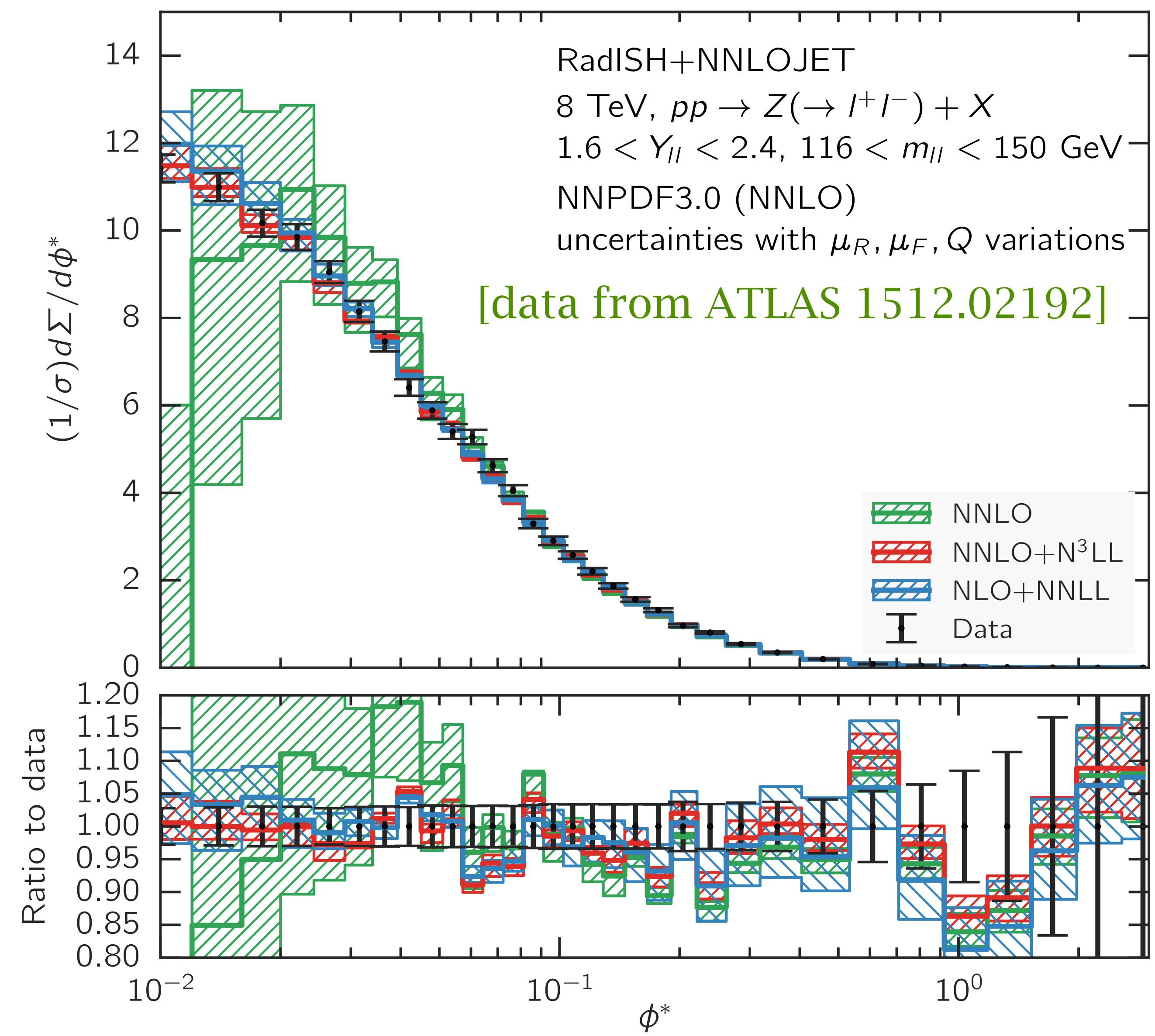
[Gehrmann-De Ridder, T. Gehrmann, E.W.N. Glover, A. Huss, T.A. Morgan '16]

➡(sub-)percent precision in data, theory can reach ~3-5% accuracy… Other effects important (QED, PDFs, quark masses, hadronisation)

➡Relevant for W-mass studies

An example: DY distributions (phi*)





➡Similar conclusions for angular distributions

Conclusions

- Higher-order resummation can be formulated directly in momentum space without the need for a factorisation for the considered observable
- The approach I briefly outlined is generalised to any rIRC safe observable in two-scale problems
  - Systematic extension to any logarithmic order
  - Efficient implementation in a computer code: e.g. `ARES`, `RadISH`
  - Analytic resummation formulated in a language closer to parton showers
- Differential distributions at N^{3}LL+NNLO
  - Higgs: uncertainties in the 5%-10% range - consistent inclusion of quark-mass effects necessary at this order of accuracy (ongoing study)
  - DY: uncertainties reduced to ~5% across the whole spectrum - good agreement with data in the large-invariant mass bins (study low invariant mass in progress)
    - Improving on this requires the assessment of several effects: NP corrections, quark-mass corrections, QED, theory uncertainties in PDFs, …

Thank you for listening

Squared amplitude decomposition

- Write all-order cross section as ($V(\{\tilde{p}\}, k_1, \ldots, k_n) = |\vec{k}_{t1} + \cdots + \vec{k}_{tn}|$)

$$\Sigma(v) = \int d\Phi_B \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] \underbrace{|M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2}_{\text{Real emissions}} \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_n)\right)$$

- Recast all-order squared ME for *n* real emissions as iteration of *correlated blocks*
- Scaling of the observable in the presence of radiation *must* preserve the above hierarchy

e.g. soft radiation (analogous considerations for hard-collinear)

$$|M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 = |M_B(\tilde{p}_1, \tilde{p}_2)|^2 \left\{ \left(\frac{1}{n!} \prod_{i=1}^{n} |M(k_i)|^2 \right) + \right.$$
$$\left[\sum_{a>b} \frac{1}{(n-2)!} \left(\prod_{\substack{i=1 \\ i \neq a,b}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b) \right|^2 + \right.$$
$$\left. \sum_{a>b} \sum_{\substack{c>d \\ c,d \neq a,b}} \frac{1}{(n-4)!2!} \left(\prod_{\substack{i=1 \\ i \neq a,b,c,d}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b) \right|^2 \left| \tilde{M}(k_c, k_d) \right|^2 + \ldots \right]$$
$$\left. + \left[\sum_{a>b>c} \frac{1}{(n-3)!} \left(\prod_{\substack{i=1 \\ i \neq a,b,c}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b, k_c) \right|^2 + \ldots \right] + \ldots \right\},$$

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$$\left[\sum_{a>b} \frac{1}{(n-2)!} \left(\prod_{\substack{i=1 \\ i \neq a,b}}^{n} |M(k_i)|^2 \right) \left|\tilde{M}(k_a, k_b)\right|^2 + \right.$$

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$$\left. + \left[\sum_{a>b>c} \frac{1}{(n-3)!} \left(\prod_{\substack{i=1 \\ i \neq a,b,c}}^{n} |M(k_i)|^2 \right) \left|\tilde{M}(k_a, k_b, k_c)\right|^2 + \ldots \right] + \ldots \right\},$$

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& \sum_{a>b} \sum_{\substack{c>d \\ c,d \neq a,b}} \underbrace{\frac{1}{(n-4)!2!} \left(\prod_{\substack{i=1 \\ i \neq a,b,c,d}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b) \right|^2 \left| \tilde{M}(k_c, k_d) \right|^2 + \ldots}_{\text{NLL}} \Bigg] \\
& + \Bigg[\sum_{a>b>c} \underbrace{\frac{1}{(n-3)!} \left(\prod_{\substack{i=1 \\ i \neq a,b,c}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b, k_c) \right|^2 + \ldots}_{\text{NNLL}} \Bigg] + \ldots \Bigg\},
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$$\Bigg[\sum_{a>b} \underbrace{\frac{1}{(n-2)!} \left(\prod_{\substack{i=1 \\ i \neq a,b}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b) \right|^2}_{\text{NLL}} +$$
$$\sum_{a>b} \sum_{\substack{c>d \\ c,d \neq a,b}} \underbrace{\frac{1}{(n-4)!2!} \left(\prod_{\substack{i=1 \\ i \neq a,b,c,d}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b) \right|^2 \left| \tilde{M}(k_c, k_d) \right|^2 + \dots}_{\text{NLL}} \Bigg]$$
$$+ \Bigg[\sum_{a>b>c} \underbrace{\frac{1}{(n-3)!} \left(\prod_{\substack{i=1 \\ i \neq a,b,c}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b, k_c) \right|^2 + \dots}_{\text{NNLL}} \Bigg] + \dots \Bigg\},$$

In addition to this counting, requiring that the observable is recursively IRC safe allows one to construct a (simpler) all-order subtraction scheme

Monte Carlo formulation

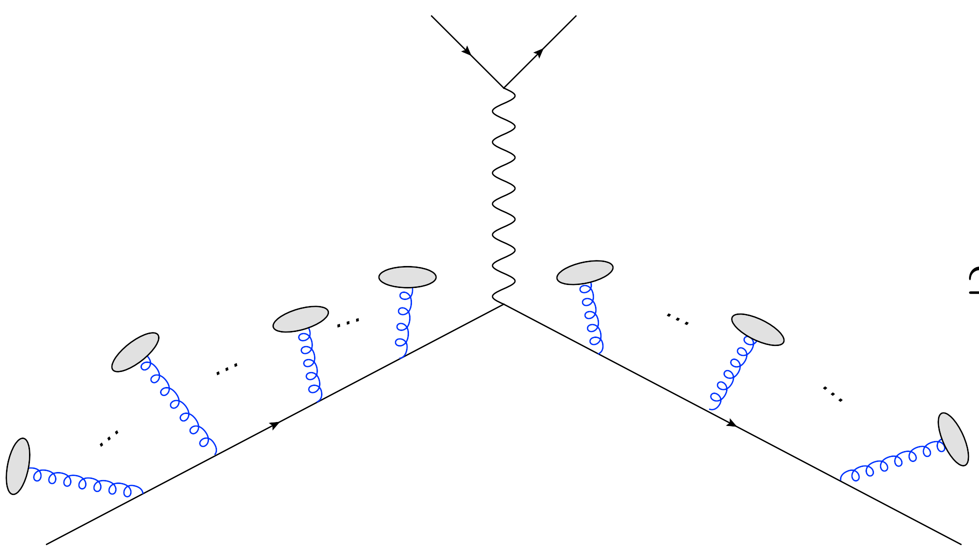
- One great simplification: choice of the resolution variable such that correlated blocks entering at N$^{\mathrm{k}}$LL in the unresolved radiation only contribute at N$^{\mathrm{k+1}}$LL in the resolved case
- i.e. we can expand out the cutoff dependence and retain in the Radiator only the terms necessary to cancel the singularities in the resolved radiation

$$R(\epsilon k_{t1}) = R(k_{t1}) + R'(k_{t1}) \ln\frac{1}{\epsilon} + \frac{1}{2} R''(k_{t1}) \ln^2\frac{1}{\epsilon} + \dots$$

$$R'(k_{ti}) = R'(k_{t1}) + R''(k_{t1}) \ln\frac{k_{t1}}{k_{ti}} + \dots$$

Expansion is safe since in the resolved radiation $k_{t1}/k_{ti} \sim 1$

e.g. at NLL



$$\simeq \int \frac{dk_{t1}}{k_{t1}} \partial_L \left(-e^{-R(k_{t1})}\mathcal{L}(k_{t1})\right) \int d\mathcal{Z}[\{R'(k_{t1}), k_i\}] \Theta(v - V(\{\tilde{p}\}, k_1, \dots, k_{n+1}))$$

$$\int d\mathcal{Z}[\{R'(k_{t1}), k_i\}] = \epsilon^{R'(k_{t1})} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_{\epsilon k_{t1}}^{k_{t1}} \frac{dk_{ti}}{k_{ti}} R'(k_{t1})$$

Monte Carlo formulation

- One great simplification: choice of the resolution variable such that correlated blocks entering at $\mathrm{N^kLL}$ in the unresolved radiation only contribute at $\mathrm{N^{k+1}LL}$ in the resolved case

- i.e. we can expand out the cutoff dependence and retain in the Radiator only the terms necessary to cancel the singularities in the resolved radiation

$$R(\epsilon k_{t1}) = R(k_{t1}) + R'(k_{t1}) \ln\frac{1}{\epsilon} + \frac{1}{2} R''(k_{t1}) \ln^2\frac{1}{\epsilon} + \dots$$

$$R'(k_{ti}) = R'(k_{t1}) + R''(k_{t1}) \ln\frac{k_{t1}}{k_{ti}} + \dots$$

Expansion is safe since in the resolved radiation $k_{t1}/k_{ti} \sim 1$

- Corrections beyond NLL are obtained as follows
  - Add subleading effects in the Sudakov radiator and constants
  - Correct *a fixed number* of the NLL resolved emissions:
    - only one at NNLL
    - two at $\mathrm{N^3LL}$
    - …

e.g. at NNLL see:
[Banfi, PM, Salam, Zanderighi ’12]
[Banfi, McAslan, PM, Zanderighi ’14-’16]

Numerical implementation: RadISH

- Since the transverse momenta of the *resolved* reals are of the same order, we can expand the whole integrand about $k_{ti} \sim k_{t1}$ up to the desired logarithmic accuracy
- This expansion allows us to compute higher-order corrections to the NLL *resolved* reals by simply including one correction at a time

e.g. expansion up to NLL

$$\frac{d\Sigma(v)}{d\Phi_B} = \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} \partial_L \left(-e^{-R(k_{t1})} \mathcal{L}_{\mathrm{N^3LL}}(k_{t1}) \right) \int d\mathcal{Z}[\{R', k_i\}] \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right)$$

$$\mathcal{L}_{\mathrm{N^3LL}}(k_{t1}) = \sum_{c,c'} \frac{d|M_B|^2_{cc'}}{d\Phi_B} \sum_{i,j} \int_{x_1}^{1} \frac{dz_1}{z_1} \int_{x_2}^{1} \frac{dz_2}{z_2} f_i\left(k_{t1}, \frac{x_1}{z_1}\right) f_j\left(k_{t1}, \frac{x_2}{z_2}\right)$$
$$\Bigg\{ \delta_{ci}\delta_{c'j}\delta(1-z_1)\delta(1-z_2) \left(1 + \frac{\alpha_s(\mu_R)}{2\pi} H^{(1)}(\mu_R) + \frac{\alpha_s^2(\mu_R)}{(2\pi)^2} H^{(2)}(\mu_R)\right)$$
$$+ \frac{\alpha_s(\mu_R)}{2\pi} \frac{1}{1-2\alpha_s(\mu_R)\beta_0 L} \left(1 - \alpha_s(\mu_R)\frac{\beta_1}{\beta_0} \frac{\ln\left(1-2\alpha_s(\mu_R)\beta_0 L\right)}{1-2\alpha_s(\mu_R)\beta_0 L}\right)$$
$$\times \left(C^{(1)}_{ci}(z_1)\delta(1-z_2)\delta_{c'j} + \{z_1 \leftrightarrow z_2; c, i \leftrightarrow c', j\}\right)$$
$$+ \frac{\alpha_s^2(\mu_R)}{(2\pi)^2} \frac{1}{(1-2\alpha_s(\mu_R)\beta_0 L)^2} \Bigg(\left(C^{(2)}_{ci}(z_1) - 2\pi\beta_0 C^{(1)}_{ci}(z_1) \ln\frac{M^2}{\mu_R^2}\right) \delta(1-z_2)\delta_{c'j}$$
$$+ \{z_1 \leftrightarrow z_2; c, i \leftrightarrow c', j\} \Bigg) + \frac{\alpha_s^2(\mu_R)}{(2\pi)^2} \frac{1}{(1-2\alpha_s(\mu_R)\beta_0 L)^2} \left(C^{(1)}_{ci}(z_1) C^{(1)}_{c'j}(z_2) + G^{(1)}_{ci}(z_1) G^{(1)}_{c'j}(z_2)\right)$$
$$+ \frac{\alpha_s^2(\mu_R)}{(2\pi)^2} H^{(1)}(\mu_R) \frac{1}{1-2\alpha_s(\mu_R)\beta_0 L} \left(C^{(1)}_{ci}(z_1)\delta(1-z_2)\delta_{c'j} + \{z_1 \leftrightarrow z_2; c, i \leftrightarrow c', j\}\right) \Bigg\}$$

- Coefficient functions and hard-virtual corrections absorbed into effective parton luminosities

Numerical implementation: RadISH

- Since the transverse momenta of the *resolved* reals are of the same order, we can expand the whole integrand about $k_{ti} \sim k_{t1}$ up to the desired logarithmic accuracy
- This expansion allows us to compute higher-order corrections to the NLL *resolved* reals by simply including one correction at a time

e.g. expansion up to NLL

$$\frac{d\Sigma(v)}{d\Phi_B} = \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} \partial_L \left(-e^{-R(k_{t1})} \mathcal{L}_{\mathrm{N^3LL}}(k_{t1}) \right) \int d\mathcal{Z}[\{R', k_i\}] \Theta \left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}) \right)$$

$$k_{ti}/k_{t1} = \zeta_i = \mathcal{O}(1)$$

$$\boxed{\int d\mathcal{Z}[\{R', k_i\}] G(\{\tilde{p}\}, \{k_i\}) = \epsilon^{R'(k_{t1})} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_{\epsilon}^{1} \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} R'(k_{t1}) G(\{\tilde{p}\}, k_1, \ldots, k_{n+1})}$$

- Coefficient functions and hard-virtual corrections absorbed into effective parton luminosities
- The ensemble of NLL real emissions dZ is generated as a parton shower. Fast numerical evaluation with Monte-Carlo methods.

Numerical implementation: RadISH

- Since the transverse momenta of the *resolved* reals are of the same order, we can expand the whole integrand about $k_{ti} \sim k_{t1}$ up to the desired logarithmic accuracy
- This expansion allows us to compute higher-order corrections to the NLL *resolved* reals by simply including one correction at a time

e.g. expansion up to NNLL

$$\frac{d\Sigma(v)}{d\Phi_B} = \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} \partial_L \left(-e^{-R(k_{t1})} \mathcal{L}_{\mathrm{N^3LL}}(k_{t1}) \right) \int d\mathcal{Z}[\{R', k_i\}] \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right)$$

$$+ \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(k_{t1})} \int d\mathcal{Z}[\{R', k_i\}] \int_0^1 \frac{d\zeta_s}{\zeta_s} \frac{d\phi_s}{2\pi} \Bigg\{ \left(R'(k_{t1}) \mathcal{L}_{\mathrm{NNLL}}(k_{t1}) - \partial_L \mathcal{L}_{\mathrm{NNLL}}(k_{t1}) \right)$$
$$\times \left(R''(k_{t1}) \ln \frac{1}{\zeta_s} + \frac{1}{2} R'''(k_{t1}) \ln^2 \frac{1}{\zeta_s} \right) - R'(k_{t1}) \left(\partial_L \mathcal{L}_{\mathrm{NNLL}}(k_{t1}) - 2 \frac{\beta_0}{\pi} \alpha_s^2(k_{t1}) \hat{P}^{(0)} \otimes \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \ln \frac{1}{\zeta_s} \right)$$
$$+ \frac{\alpha_s^2(k_{t1})}{\pi^2} \hat{P}^{(0)} \otimes \hat{P}^{(0)} \otimes \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \Bigg\} \Bigg\{ \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_s)\right) - \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right) \Bigg\}$$

- Coefficient functions and hard-virtual corrections absorbed into effective parton luminosities
- The ensemble of NLL real emissions $d\mathcal{Z}$ is generated as a parton shower. Fast numerical evaluation with Monte-Carlo methods.

Numerical implementation: RadISH

- Since the transverse momenta of the *resolved* reals are of the same order, we can expand the whole integrand about $k_{ti} \sim k_{t1}$ up to the desired logarithmic accuracy
- This expansion allows us to compute higher-order corrections to the NLL *resolved* reals by simply including one correction at a time

e.g. expansion up to N³LL

$$\frac{d\Sigma(v)}{d\Phi_B} = \int \frac{dk_{t1}}{k_{t1}}\frac{d\phi_1}{2\pi}\partial_L\left(-e^{-R(k_{t1})}\mathcal{L}_{\mathrm{N^3LL}}(k_{t1})\right)\int d\mathcal{Z}[\{R',k_i\}]\Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1})\right)$$

$$+\int \frac{dk_{t1}}{k_{t1}}\frac{d\phi_1}{2\pi}e^{-R(k_{t1})}\int d\mathcal{Z}[\{R',k_i\}]\int_0^1\frac{d\zeta_s}{\zeta_s}\frac{d\phi_s}{2\pi}\left\{\left(R'(k_{t1})\mathcal{L}_{\mathrm{NNLL}}(k_{t1}) - \partial_L\mathcal{L}_{\mathrm{NNLL}}(k_{t1})\right)\right.$$
$$\times\left(R''(k_{t1})\ln\frac{1}{\zeta_s}+\frac{1}{2}R'''(k_{t1})\ln^2\frac{1}{\zeta_s}\right) - R'(k_{t1})\left(\partial_L\mathcal{L}_{\mathrm{NNLL}}(k_{t1}) - 2\frac{\beta_0}{\pi}\alpha_s^2(k_{t1})\hat{P}^{(0)}\otimes\mathcal{L}_{\mathrm{NLL}}(k_{t1})\ln\frac{1}{\zeta_s}\right)$$
$$\left.+\frac{\alpha_s^2(k_{t1})}{\pi^2}\hat{P}^{(0)}\otimes\hat{P}^{(0)}\otimes\mathcal{L}_{\mathrm{NLL}}(k_{t1})\right\}\left\{\Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1},k_s)\right) - \Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1})\right)\right\}$$

$$+\frac{1}{2}\int \frac{dk_{t1}}{k_{t1}}\frac{d\phi_1}{2\pi}e^{-R(k_{t1})}\int d\mathcal{Z}[\{R',k_i\}]\int_0^1\frac{d\zeta_{s1}}{\zeta_{s1}}\frac{d\phi_{s1}}{2\pi}\int_0^1\frac{d\zeta_{s2}}{\zeta_{s2}}\frac{d\phi_{s2}}{2\pi}R'(k_{t1})$$
$$\times\left\{\mathcal{L}_{\mathrm{NLL}}(k_{t1})\left(R''(k_{t1})\right)^2\ln\frac{1}{\zeta_{s1}}\ln\frac{1}{\zeta_{s2}} - \partial_L\mathcal{L}_{\mathrm{NLL}}(k_{t1})R''(k_{t1})\left(\ln\frac{1}{\zeta_{s1}}+\ln\frac{1}{\zeta_{s2}}\right)\right.$$
$$\left.+\frac{\alpha_s^2(k_{t1})}{\pi^2}\hat{P}^{(0)}\otimes\hat{P}^{(0)}\otimes\mathcal{L}_{\mathrm{NLL}}(k_{t1})\right\}$$
$$\times\left\{\Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1},k_{s1},k_{s2})\right) - \Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1},k_{s1})\right) -\right.$$
$$\left.\Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1},k_{s2})\right) + \Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1})\right)\right\} + \mathcal{O}\left(\alpha_s^n\ln^{2n-6}\frac{1}{v}\right)$$

- Coefficient functions and hard-virtual corrections absorbed into effective parton luminosities
- The ensemble of NLL real emissions *dZ* is generated as a parton shower. Fast numerical evaluation with Monte-Carlo methods.

Equivalence to CSS formula

- Hard-collinear emissions off initial-state legs require some care in the treatment of kinematics. Final result reads

$$\frac{d\Sigma(v)}{dp_t d\Phi_B} = \int_{\mathcal{C}_1} \frac{dN_1}{2\pi i} \int_{\mathcal{C}_2} \frac{dN_2}{2\pi i} x_1^{-N_1} x_2^{-N_2} \sum_{c_1,c_2} \frac{d|M_B|^2_{c_1 c_2}}{d\Phi_B} \mathbf{f}^T_{N_1}(\mu_0) \frac{d\hat{\mathbf{\Sigma}}^{c_1,c_2}_{N_1,N_2}(v)}{dp_t} \mathbf{f}_{N_2}(\mu_0)$$

$$\begin{aligned}
\hat{\mathbf{\Sigma}}^{c_1,c_2}_{N_1,N_2}(v) = {} & \left[\mathbf{C}^{c_1;T}_{N_1}(\alpha_s(\mu_0)) H(\mu_R) \mathbf{C}^{c_2}_{N_2}(\alpha_s(\mu_0))\right] \int_0^M \frac{dk_{t1}}{k_{t1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \\
& \times e^{-\mathbf{R}(\epsilon k_{t1})} \exp\left\{-\sum_{\ell=1}^{2} \left(\int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \frac{\alpha_s(k_t)}{\pi} \mathbf{\Gamma}_{N_\ell}(\alpha_s(k_t)) + \int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \mathbf{\Gamma}^{(C)}_{N_\ell}(\alpha_s(k_t))\right)\right\} \\
& \sum_{\ell_1=1}^{2} \left(\mathbf{R}'_{\ell_1}(k_{t1}) + \frac{\alpha_s(k_{t1})}{\pi} \mathbf{\Gamma}_{N_{\ell_1}}(\alpha_s(k_{t1})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_1}}(\alpha_s(k_{t1}))\right) \\
& \times \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_\epsilon^1 \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \sum_{\ell_i=1}^{2} \left(\mathbf{R}'_{\ell_i}(k_{ti}) + \frac{\alpha_s(k_{ti})}{\pi} \mathbf{\Gamma}_{N_{\ell_i}}(\alpha_s(k_{ti})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_i}}(\alpha_s(k_{ti}))\right) \\
& \times \Theta\left(v - V(\{\tilde{p}\}, k_1, \dots, k_{n+1})\right),
\end{aligned}$$

- Formulation equivalent to b-space result, up to a scheme change. Using the delta representation for the distribution one finds

$$\boxed{\delta^{(2)}(\vec{p_t} - (\vec{k}_{t1} + \dots + \vec{k}_{tn})) = \int \frac{d^2\vec{b}}{4\pi^2} e^{-i\vec{b}\cdot\vec{p_t}} \prod_{i=1}^{n} e^{i\vec{b}\cdot\vec{k}_{ti}}}$$

$$\frac{d\Sigma(v)}{dp_t d\Phi_B} = \int_{\mathcal{C}_1} \frac{dN_1}{2\pi i} \int_{\mathcal{C}_2} \frac{dN_2}{2\pi i} x_1^{-N_1} x_2^{-N_2} \sum_{c_1,c_2} \frac{d|M_B|^2_{c_1 c_2}}{d\Phi_B} \mathbf{f}^T_{N_1}(\mu_0) \frac{d\hat{\mathbf{\Sigma}}^{c_1,c_2}_{N_1,N_2}(v)}{dp_t} \mathbf{f}_{N_2}(\mu_0) =$$

$$\boxed{(1 - J_0(bk_t)) \simeq \Theta(k_t - \frac{b_0}{b}) + \frac{\zeta_3}{12} \frac{\partial^3}{\partial \ln(Mb/b_0)^3} \Theta(k_t - \frac{b_0}{b}) + \dots}$$

$$\begin{aligned}
= {} & \sum_{c_1,c_2} \frac{d|M_B|^2_{c_1 c_2}}{d\Phi_B} \int b\, db\, p_t J_0(p_t b)\, \mathbf{f}^T(b_0/b) \mathbf{C}^{c_1;T}_{N_1}(\alpha_s(b_0/b)) H(M) \mathbf{C}^{c_2}_{N_2}(\alpha_s(b_0/b)) \mathbf{f}(b_0/b) \\
& \times \exp\left\{-\sum_{\ell=1}^{2} \int_0^M \frac{dk_t}{k_t} \mathbf{R}'_\ell(k_t) (1 - J_0(bk_t))\right\}.
\end{aligned}$$