Resummation for transverse observables at hadron colliders

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Based on
1604.02191 with E. Re and P. Torrielli
and
1705.09127 with W. Bizon, E. Re, L. Rottoli, and P. Torrielli

+ ongoing work with
W. Bizon, X. Chen, Gehrmann-De Ridder, Gehrmann, Glover, A. Huss,
E. Re, L. Rottoli, and P. Torrielli

DIS 2018, Kobe, Japan - 18 April 2018
Outline

- Theory precision at colliders:
  - fixed-order vs. all-order perturbation theory
- Factorisation theorems and semi-numerical resummation
- Momentum-space resummation for transverse observables
- Predictions for differential distributions at $N^3LL+NNLO$ at the LHC
  - Higgs production
  - Drell-Yan production
- Conclusions
Fixed-order vs. All-order

- Fixed-order calculations of radiative corrections are formulated in a well established way (technically challenging, but well posed problem):
  - compute amplitudes at a given order
  - provide an effective subtraction of IRC divergences
  - compute any IRC-safe observable

$$\Sigma(v) = \int_0^v \frac{1}{\sigma_{\text{Born}}} \frac{d\sigma}{dv'} dv' \sim 1 + \alpha_s + \alpha_s^2 + \ldots$$

- All-order calculations are still at an earlier stage of evolution
  - Each different observable has its own type of sensitivity to IRC physics, it is hard to formulate a general method that works for all at a generic perturbative order
  - Higher-order resummations are therefore often formulated in an observable-dependent way, for few well-behaved collider observables

$$\Sigma(v) = \int_0^v \frac{1}{\sigma_{\text{Born}}} \frac{d\sigma}{dv'} dv' \sim e^{\alpha_s L^{n+1}} + \alpha_s L^n + \alpha_s L^{n-1} + \ldots$$

$$v \to 0$$
Factorisation of the observable

- Factorisation of the amplitude is not enough as the all-order radiation is tangled by the observable

\[ \Sigma(v) = \int d\Phi_{\text{rad}} \sum_{n=0}^{\infty} |\mathcal{M}(k_1, \ldots, k_n)|^2 \Theta(v - V(k_1, \ldots, k_n)) \]

- In order to perform an all-order calculation, one needs to **break** the observable too into hard, soft and collinear pieces. This can be done for some observables which treat the radiation rather inclusively

  - e.g. transverse momentum of a massive singlet

\[ \delta^{(2)}(p_t - (\vec{k}_{t1} + \ldots + \vec{k}_{tn})) = \int \frac{d^2b}{4\pi^2} e^{-ib\cdot \vec{p}_t} \prod_{i=1}^{n} e^{ib\cdot \vec{k}_{ti}}, \]

\[
\frac{d^2\Sigma(p_t)}{d\Phi_B dp_t} = \sum_{c_1, c_2} \frac{d|M_B|^2_{c_1 c_2}}{d\Phi_B} \int b \, db \, p_t J_0(p_t b) f^T(b_0/b) C_{N_1}^{c_1 T} (\alpha_S(b_0/b)) H_{\text{CSS}}(M) C_{N_2}^{c_2} (\alpha_S(b_0/b)) f(b_0/b) \\
\times \exp \left\{- \sum_{\ell=1}^{2} \int_{b_0/b}^{M} \frac{dk_t}{k_t} R'_{\text{CSS},\ell}(k_t) \right\}.
\]
Eluding observable factorisation

- Factorisation is a powerful tool, but limited to observables that have a simple analytic expression in the relevant limits or do not mix soft and collinear radiation (e.g. jet rates)

- Ultimately, we want to use the modern knowledge of IRC dynamics to make more accurate generators. At present a general framework to assess the accuracy of Parton Showers is missing
  - It is of primary importance to formulate a link between higher-order resummation and PS

- Can we devise a formulation without a factorisation formula?
  - *recursive IRC safety*: simple set of criteria for the observable that allows one to formulate the resummation at NLL for global observables without the need for an explicit factorisation.  
    - Most of modern global observables fall into this category.
  - The method can be reformulated and extended at higher logarithmic orders

[Banfi, Salam, Zanderighi '01-'04]
[Banfi, McAslan, PM, Zanderighi ’14–’16]
[PM, Re, Torrielli ’16]
[Bizon, PM, Re, Rottoli, Torrielli ’17]
A case study: transverse observables

- **Transverse and inclusive** observables in colour-singlet production offer a clean experimental and theoretical environment for precision physics:

  \[ V(\{\tilde{p}\}, k) \equiv V(k) = d_\ell \, g_\ell (\phi) \left( \frac{k_t}{M} \right)^a \]

  \[ V(\{\tilde{p}\}, k_1, \ldots, k_n) = V(\{\tilde{p}\}, k_1 + \cdots + k_n) \]

- **SM measurements** (e.g. W, Z, photon, ...): parton distributions, strong coupling, W mass, ...
  - sensitivity to non-perturbative effects (hadronisation, intrinsic kt) only through transverse recoil
  - very little/no sensitivity to multi-parton interactions

- **BSM measurements/constraints** (e.g. Higgs): light/heavy NP, Yukawa couplings, ...

- **Theoretically interesting**:
  - clean environment to test/calibrate exclusive generators against high perturbative orders
  - **Two mechanisms compete** in the \( p_t \to 0 \) limit:
    - Sudakov (exponential) suppression when \( k_{t_i} \sim p_t \ll M \)
    - Azimuthal cancellations (power suppression, dominant) when \( p_t \ll k_{t_i} \ll M \)
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Can we build a more exclusive solution in momentum space?

See also work in [Ebert, Tackmann ’16][Kang, Lee, Vaidya ’17]
Direct space: virtual corrections

- Write all-order cross section as \(( V(\{\tilde{p}\}, k_1, \ldots, k_n) = |\tilde{k}_{t1} + \cdots + \tilde{k}_{tn}| )\)

\[
\Sigma(v) = \int d\Phi_B \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} |dk_i| M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)^2 \Theta(v - V(\{\tilde{p}\}, k_1, \ldots, k_n))
\]
Direct space: virtual corrections

- Write all-order cross section as \( V(\{\hat{p}\}, k_1, \ldots, k_n) = |\bar{k}_{t1} + \cdots + \bar{k}_{tn}| \)

\[
\Sigma(v) = \int d\Phi_B \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] |M(\hat{p}_1, \hat{p}_2, k_1, \ldots, k_n)|^2 \Theta(v - V(\{\hat{p}\}, k_1, \ldots, k_n))
\]

All-order form factor
e.g. [Dixon, Magnea, Sterman ’08]
Direct space: real radiation

- Write all-order cross section as \( V(\{\tilde{p}\}, k_1, \ldots, k_n) = |\tilde{k}_{t1} + \cdots + \tilde{k}_{tn}| \)

\[
\Sigma(v) = \int d\Phi_B \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 \Theta(v - V(\{\tilde{p}\}, k_1, \ldots, k_n))
\]

- Logarithmic counting: we need a logarithmic hierarchy in the squared amplitudes (resummation means iteration of lower-order amplitudes)

\[
|M(\tilde{k}_a)|^2 = \frac{|M(\tilde{p}_1, \tilde{p}_2, k_a)|^2}{|M_B(\tilde{p}_1, \tilde{p}_2)|^2} = |M(k_a)|^2
\]

\[
|M(\tilde{k}_a, k_b)|^2 = \frac{|M(\tilde{p}_1, \tilde{p}_2, k_a, k_b)|^2}{|M_B(\tilde{p}_1, \tilde{p}_2)|^2} - \frac{1}{2!} |M(k_a)|^2 |M(k_b)|^2
\]

Real emissions

E.g., soft radiation (one log down in hard-collinear case)

\[\ldots + \cdots + \ldots\]
Direct space: real radiation

- Write all-order cross section as \((V(\{\tilde{p}\},k_1,\ldots,k_n) = |\tilde{k}_{t1} + \cdots + \tilde{k}_{tn}|)\)

\[
\Sigma(v) = \int d\Phi_B \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \prod_{i=1}^{n} [dk_i] |M(\tilde{p}_1,\tilde{p}_2,k_1,\ldots,k_n)|^2 \Theta(v - V(\{\tilde{p}\},k_1,\ldots,k_n))
\]

- Logarithmic counting: we need a logarithmic hierarchy in the squared amplitudes (resummation means iteration of lower-order amplitudes)

E.g.
soft radiation (one log down in hard-collinear case)

\[
|\tilde{M}(k_a)|^2 = \frac{|M(\tilde{p}_1,\tilde{p}_2,k_a)|^2}{|M_B(\tilde{p}_1,\tilde{p}_2)|^2} = |M(k_a)|^2
\]

\[
|\tilde{M}(k_a,k_b)|^2 = \frac{|M(\tilde{p}_1,\tilde{p}_2,k_a,k_b)|^2}{|M_B(\tilde{p}_1,\tilde{p}_2)|^2} - \frac{1}{2!} |M(k_a)|^2 |M(k_b)|^2
\]

\[
\alpha_s L^2
\]
Direct space: real radiation

- Write all-order cross section as \( V(\{\tilde{p}\}, k_1, \ldots, k_n) = |\tilde{k}_{t1} + \cdots + \tilde{k}_{tn}| \)

\[
\Sigma(v) = \int d\Phi_B \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 \Theta(v - V(\{\tilde{p}\}, k_1, \ldots, k_n))
\]

- Logarithmic counting: we need a logarithmic hierarchy in the squared amplitudes (resummation means iteration of lower-order amplitudes)

e.g.
soft radiation (one log down in hard-collinear case)

\[
|M(\tilde{p}_1, \tilde{p}_2, k_a)|^2 \quad \text{and} \quad \frac{|M(\tilde{p}_1, \tilde{p}_2, k_a)|^2}{|M_{B}(\tilde{p}_1, \tilde{p}_2)|^2} = |M(k_a)|^2
\]

\[
|M(\tilde{p}_1, \tilde{p}_2, k_a, k_b)|^2 = \frac{|M(\tilde{p}_1, \tilde{p}_2, k_a, k_b)|^2}{|M_{B}(\tilde{p}_1, \tilde{p}_2)|^2} - \frac{1}{2!} |M(k_a)|^2 |M(k_b)|^2
\]

\[
\alpha_s L^2
\]

\[
\alpha_s^2 L^4
\]

\[
\alpha_s L^2
\]

\[
\alpha_s^2 L^4
\]
Direct space: real radiation

- Write all-order cross section as \( V(\{\vec{p}\}, k_1, \ldots, k_n) = |\vec{k}_{t1} + \cdots + \vec{k}_{tn}| \)

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\]

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\[
|\tilde{M}(k_a)|^2 = \frac{|M(\vec{p}_1, \vec{p}_2, k_a)|^2}{|M_B(\vec{p}_1, \vec{p}_2)|^2} = |M(k_a)|^2
\]

\[
|\tilde{M}(k_a, k_b)|^2 = \frac{|M(\vec{p}_1, \vec{p}_2, k_a, k_b)|^2}{|M_B(\vec{p}_1, \vec{p}_2)|^2} - \frac{1}{2!} |M(k_a)|^2 |M(k_b)|^2
\]

\(\alpha_s L^2\) \quad \alpha_s^2 L^4

this LL is absorbed in the resummation of \(|M(k)|^2\)
All-order subtraction of IRC singularities

- Write all-order cross section as 
  \[ V(\{\tilde{p}\}, k_1, \ldots, k_n) = |\tilde{k}_{t1} + \cdots + \tilde{k}_{tn}| \]

\[
\Sigma(v) = \int d\Phi_B \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} (dk_i) |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 \Theta(v - V(\{\tilde{p}\}, k_1, \ldots, k_n))
\]

Subtraction of the IRC poles and computation of the observable
All-order subtraction of IRC singularities

- Subtraction of the IRC poles between \[ \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] |M(\vec{p}_1, \vec{p}_2, k_1, \ldots, k_n)|^2 \] and \( \mathcal{V}(\Phi_B) \):
  - introduce a phase-space resolution scale (slicing parameter) \( Q_0 = \epsilon k_{t1} \)
  - real correlated blocks with total transverse momentum \( k_{ti} < \epsilon k_{t1} \) (unresolved) do not modify the observable, and can be ignored in the measurement function
  - compute unresolved reals and virtuals analytically in D dimensions (much easier than full observable)
All-order subtraction of IRC singularities

- Subtraction of the IRC poles between \( \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 \) and \( \nu(\Phi_B) \):
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\[
\sum_{n=0}^{\infty} |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 \rightarrow |M_B(\tilde{p}_1, \tilde{p}_2)|^2
\]

\[
\times \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \prod_{i=1}^{n} \left( |M(k_i)|^2 + \int [dk_a][dk_b] |M(k_a, k_b)|^2 \delta^{(2)}(\vec{k}_{ta} + \vec{k}_{tb} - \vec{k}_{ti}) \delta(Y_{ab} - Y_i) \\
+ \int [dk_a][dk_b][dk_c] |M(k_a, k_b, k_c)|^2 \delta^{(2)}(\vec{k}_{ta} + \vec{k}_{tb} + \vec{k}_{tc} - \vec{k}_{ti}) \delta(Y_{abc} - Y_i) + \ldots \right) \right\}
\]
All-order subtraction of IRC singularities

- Subtraction of the IRC poles between $\sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2$ and $\mathcal{V}(\Phi_B)$:

  - introduce a phase-space resolution scale (slicing parameter) $Q_0 = \epsilon k_{t1}$

  - real correlated blocks with total transverse momentum $k_{ti} < \epsilon k_{t1}$ (unresolved) do not modify the observable, and can be ignored in the measurement function

  - compute unresolved reals and virtuals analytically in D dimensions (much easier than full observable)

$$\prod_{i=1}^{n} \int [dk_i] \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \prod_{i=1}^{n} \left( |M(k_i)|^2 + \int [dk_a][dk_b] |M(k_a, k_b)|^2 \delta^{(2)}(\tilde{k}_{ta} + \tilde{k}_{tb} - \tilde{k}_{ti}) \delta(Y_{ab} - Y_i) \right) \right.\left. + \int [dk_a][dk_b][dk_c] |M(k_a, k_b, k_c)|^2 \delta^{(2)}(\tilde{k}_{ta} + \tilde{k}_{tb} + \tilde{k}_{tc} - \tilde{k}_{ti}) \delta(Y_{abc} - Y_i) + \ldots \right\} \Theta(\epsilon k_{t1} - k_{ti})$$
All-order subtraction of IRC singularities

- Subtraction of the IRC poles between \( \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 \) and \( \mathcal{V}(\Phi_B) \):
  - introduce a phase-space resolution scale (slicing parameter) \( Q_0 = \epsilon k_{t1} \)
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\[
\prod_{i=1}^{n} \int [dk_i] \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \prod_{i=1}^{n} \left( |M(k_i)|^2 + \int [dk_a][dk_b] |\tilde{M}(k_a, k_b)|^2 \delta^{(2)}(\tilde{k}_{ta} + \tilde{k}_{tb} - \tilde{k}_{ti}) \delta(Y_{ab} - Y_i) \right) \right. \\
\left. \quad + \int [dk_a][dk_b][dk_c] |\tilde{M}(k_a, k_b, k_c)|^2 \delta^{(2)}(\tilde{k}_{ta} + \tilde{k}_{tb} + \tilde{k}_{tc} - \tilde{k}_{ti}) \delta(Y_{abc} - Y_i) + \cdots \right) \Theta(\epsilon k_{t1} - k_{ti}) \\
\propto \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(\epsilon k_{t1})} R'(k_{t1})
\]
All-order subtraction of IRC singularities

- Subtraction of the IRC poles between \( \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] |M(\bar{p}_1, \bar{p}_2, k_1, \ldots, k_n)|^2 \) and \( \mathcal{V}(\Phi_B) \):
  - introduce a phase-space resolution scale (slicing parameter) \( Q_0 = \epsilon k_{t1} \)
  - real correlated blocks with total transverse momentum \( k_{ti} < \epsilon k_{t1} \) (unresolved) do not modify the observable, and can be ignored in the measurement function
  - compute unresolved reals and virtuals analytically in \( D \) dimensions (much easier than full observable)

\[
R(\epsilon k_{t1}) = \sum_{\ell=1}^{2} \int_{\epsilon k_{t1}}^{M} \frac{dk_t}{k_t} R'_\ell(k_t) = \sum_{\ell=1}^{2} \int_{\epsilon k_{t1}}^{M} \frac{dk_t}{k_t} \left( A_\ell(\alpha_s(k_t)) \ln \frac{M^2}{k_t^2} + B_\ell(\alpha_s(k_t)) \right)
\]

- Anomalous dimensions start differing from b-space ones at N^3LL

\[
\prod_{i=1}^{n} \int [dk_i] \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \prod_{i=1}^{n} \left( |M(k_i)|^2 + \int [dk_a][dk_b] |M(k_a, k_b)|^2 \delta^{(2)}(\bar{k}_{ta} + \bar{k}_{tb} - \bar{k}_{ti}) \delta(Y_{ab} - Y_i) \right. \\
\left. + \int [dk_a][dk_b][dk_c] |\tilde{M}(k_a, k_b, k_c)|^2 \delta^{(2)}(\bar{k}_{ta} + \bar{k}_{tb} + \bar{k}_{tc} - \bar{k}_{ti}) \delta(Y_{abc} - Y_i) + \ldots \right\} \Theta(\epsilon k_{t1} - k_{ti})
\]

\[
\propto \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(\epsilon k_{t1})} R'(k_{t1})
\]
All-order subtraction of IRC singularities

- Subtraction of the IRC poles between \( \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 \) and \( \mathcal{V}(\Phi_B) \):
  - introduce a phase-space resolution scale (slicing parameter) \( Q_0 = \epsilon k_{t1} \)
  - real correlated blocks with total transverse momentum \( k_{ti} < \epsilon k_{t1} \) (unresolved) do not modify the observable, and can be ignored in the measurement function
  - compute unresolved reals and virtuals analytically in D dimensions (much easier than full observable)

\[
\Sigma_{N_1, N_2}^{c_1, c_2} (v) = \left[ C_{N_1}^{c_1; T} (\alpha_s(\mu_0)) H(\mu_R) C_{N_2}^{c_2} (\alpha_s(\mu_0)) \right] \int_0^M \frac{dk_{t1}}{k_{t1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi}
\times e^{-R(\epsilon k_{t1})} \exp \left\{ - \sum_{\ell=1}^{2} \left( \int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \frac{\alpha_s(k_t)}{\pi} \Gamma_{N_\ell} (\alpha_s(k_t)) + \int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \Gamma_{N_\ell}^{(C)} (\alpha_s(k_t)) \right) \right\}
\]

- DGLAP anomalous dims
- RGE evolution of coeff. functions
- Sudakov radiator: integral of single inclusive block.
All-order subtraction of IRC singularities

- Subtraction of the IRC poles between \( \sum_{n=0}^{\infty} \prod_{i=1}^{n} [dk_i] M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)^2 \) and \( \mathcal{V}(\Phi_B) \):
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\[
\Sigma_{N_1, N_2}^{c_1, c_2}(v) = \left[ \mathcal{C}_{N_1}^{c_1:T}(\alpha_s(\mu_0)) H(\mu_R) \mathcal{C}_{N_2}^{c_2}(\alpha_s(\mu_0)) \right] \int_0^M \frac{dk_{t1}}{k_{t1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \exp \left\{ - \sum_{\ell=1}^D \left( \int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \frac{\alpha_s(k_t)}{\pi} \Gamma_{N_\ell}(\alpha_s(k_t)) + \int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \Gamma_{N_\ell}^{(C)}(\alpha_s(k_t)) \right) \right\}
\]

- compute resolved (reals only) in 4 dim. with \( \epsilon \to 0 \) (MC events !)
This is, essentially, a *quasi-exclusive generator* with higher logarithmic accuracy

- e.g. gluon emissions off quark legs

\[
|M(k_i)|^2 + \int [dk_a][dk_b] |\tilde{M}(k_a, k_b)|^2 \delta^{(2)}(\vec{k}_{ta} + \vec{k}_{tb} - \vec{k}_{ti}) \delta(Y_{ab} - Y_i) \\
+ \int [dk_a][dk_b][dk_c] |\tilde{M}(k_a, k_b, k_c)|^2 \delta^{(2)}(\vec{k}_{ta} + \vec{k}_{tb} + \vec{k}_{tc} - \vec{k}_{ti}) \delta(Y_{abc} - Y_i) + \ldots
\]
Small transverse momentum limit

- CSS result recovered by simply transforming observable into b-space and integrating over radiation (see backup material)

- Clear physical picture of the dynamics of azimuthal cancellations at small transverse momentum
e.g. NLL with $\mathcal{L}(k_{t1}) = 1$ for simplicity

\[
\frac{d^3 \Sigma(p_t)}{d^2 p_t d\Phi_B} = \sigma^{(0)}(\Phi_B) \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(k_{t1})} R'(k_{t1}) \int dZ \{ R'(k_{t1}), k_i \} \delta^{(2)}(p_t - \sum_{i=1}^{n+1} \vec{k}_{ti})
\]

- Transition from exponential to a power-like suppression at small transverse momentum

\[
\frac{d^2 \Sigma(p_t)}{dp_t d\Phi_B} \approx 4p_t \sigma^{(0)}(\Phi_B) \int_{\Lambda_{QCD}}^{M} \frac{dk_{t1}}{k_{t1}^3} e^{-R(k_{t1})} \approx 2p_t \sigma^{(0)}(\Phi_B) \left( \frac{\Lambda_{QCD}^2}{M^2} \right)^{\frac{16}{25}} \ln \frac{41}{16}
\]
Small transverse momentum limit

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$$

as $p_t \to 0$ Sudakov is ''frozen'' at $k_{t1} \gg p_t$
(no exponential suppression)

- Transition from exponential to a power-like suppression at small transverse momentum

$$
\frac{d^2 \Sigma(p_t)}{dp_t d\Phi_B} \simeq 4p_t \sigma^{(0)}(\Phi_B) \int_{\Lambda_{QCD}}^{M} \frac{dk_{t1}}{k_{t1}^3} e^{-R(k_{t1})} \simeq 2p_t \sigma^{(0)}(\Phi_B) \left( \frac{\Lambda_{QCD}^2}{M^2} \right)^{16 \ln(41/16)^{25}} ^{16 \ln(41/16)}
$$
Small transverse momentum limit

- CSS result recovered by simply transforming observable into b-space and integrating over radiation (see backup material)

- Clear physical picture of the dynamics of azimuthal cancellations at small transverse momentum

\[ \frac{d^3 \Sigma(p_t)}{d^2 p_t d \Phi_B} = \sigma^{(0)}(\Phi_B) \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(k_{t1})} \int d\mathcal{Z} \left[ \{ R'(k_{t1}), k_i \} \right] \delta^{(2)}(\vec{p}_t - \sum_{i=1}^{n+1} \vec{k}_{ti}) \]

as \( p_t \to 0 \) Sudakov is ”frozen” at \( k_{t1} \gg p_t \) (no exponential suppression)

- Random azimuthal orientation of momenta leads to scaling \( \propto 1/k_{t1}^2 \)

- Transition from exponential to a power-like suppression at small transverse momentum

\[ \frac{d^2 \Sigma(p_t)}{dp_t d \Phi_B} \simeq 4p_t \sigma^{(0)}(\Phi_B) \int_{\Lambda_{QCD}}^M \frac{dk_{t1}}{k_{t1}^3} e^{-R(k_{t1})} \simeq 2p_t \sigma^{(0)}(\Phi_B) \left( \frac{\Lambda_{QCD}^2}{M^2} \right)^{16/25} \ln^{41/16} \]
Small transverse momentum limit

- e.g. Z production at 14 TeV

Radiation “freezes” at ~3 GeV
Matching to Fixed Order

- Implementation in a MC code (RadISH) up to N^3LL
  - fully differential in Born kinematics
  - matching to fixed order cumulative distribution, e.g. Higgs:
    \[ \sigma_{pp\rightarrow H}^{N^3LO} - \sum_{1-jet}^{\text{NNLO}} (p_t^H) \]
    - [Anastasiou et al. '15-'16]
    - [Boughezal et al. '15]
    - [Caola et al. '15]
    - [Chen et al. '16]

- Additive vs. multiplicative schemes

OLD CHOICE :
\[
\Sigma_{\text{MAT}}(p_t) = (\Sigma_{\text{RES}}(p_t))^{Z} \frac{\Sigma_{\text{FO}}(p_t)}{(\Sigma_{\text{EXP}}(p_t))^{Z}}
\]
\[ Z = \left(1 - \left(\frac{p_t}{Q_{\text{match}}}\right)^h\right) \cdot \Theta(Q_{\text{match}} - p_t) \]

R - SCHEME :
\[
\Sigma_{\text{MAT}}(p_t) = \Sigma_{\text{RES}}(p_t) + \Sigma_{\text{FO}}(p_t) - \Sigma_{\text{EXP}}(p_t)
\]
Matching to Fixed Order

- Implementation in a MC code (**RadISH**) up to N^3LL
  - fully differential in Born kinematics
  - matching to fixed order cumulative distribution, e.g. Higgs:

\[
\sigma_{pp\to H}^{N^3LO} - \sum_{1-jet}^{NNLO} (p_t^H)
\]

- Additive vs. multiplicative schemes

**OLD CHOICE:**
\[
\Sigma_{\text{MAT}}(p_t) = \left( \Sigma_{\text{RES}}(p_t) \right)^Z \left( \Sigma_{\text{FO}}(p_t) \right)^{Z^+} \left( \Sigma_{\text{EXP}}(p_t) \right)^{Z^-}
\]
\[
Z = \left( 1 - \frac{p_t}{Q_{\text{match}}} \right)^h \Theta(Q_{\text{match}} - p_t)
\]

**NEW CHOICE:**
\[
\Sigma_{\text{MAT}}(p_t) = \frac{\Sigma_{\text{RES}}(p_t)}{\mathcal{L}(\mu_F)} \left[ \mathcal{L}(\mu_F) \frac{\Sigma_{\text{FO}}(p_t)}{\Sigma_{\text{EXP}}(p_t)} \right]_{\text{EXPANDED}}
\]

**OLD CHOICE:**
\[
\Sigma_{\text{MAT}}(p_t) = \Sigma_{\text{RES}}(p_t) + \Sigma_{\text{FO}}(p_t) - \Sigma_{\text{EXP}}(p_t)
\]

**R – SCHEME:**
\[
\Sigma_{\text{MAT}}(p_t) = \Sigma_{\text{RES}}(p_t) + \Sigma_{\text{FO}}(p_t) - \Sigma_{\text{EXP}}(p_t)
\]

References:
- [Boughezal et al. ’15]
- [Caola et al. ’15]
- [Chen et al. ‘16]
Matching to Fixed Order

- Implementation in a MC code (RadISH) up to $N^3LL$
  - fully differential in Born kinematics
  - matching to fixed order cumulative distribution, e.g. Higgs:
    \[
    \sigma_{pp\rightarrow H}^{N^3LO} - \sum_{1-jet}^{NNLO} (p_t^H)
    \]
    
- Additive vs. multiplicative schemes

**OLD CHOICE:**
\[
\Sigma_{\text{MAT}}(p_t) = (\Sigma_{\text{RES}}(p_t))^Z \frac{\Sigma_{\text{FO}}(p_t)}{(\Sigma_{\text{EXP}}(p_t))^Z}
\]
\[
Z = \left(1 - \left(\frac{p_t}{Q_{\text{match}}}\right)^h\right) \Theta(Q_{\text{match}} - p_t)
\]

**NEW CHOICE:**
\[
\Sigma_{\text{MAT}}(p_t) = \frac{\Sigma_{\text{RES}}(p_t)}{\mathcal{L}(\mu_F)} \left[\mathcal{L}(\mu_F) \frac{\Sigma_{\text{FO}}(p_t)}{\Sigma_{\text{EXP}}(p_t)}\right]_{\text{EXPANDED}}
\]

Higher-order (in a logarithmic sense) constants from FO in the multiplicative scheme. No extra parameters needed
An example: Higgs pT spectrum

- Implementation in a MC code (RadISH) up to $N^3LL$
  - fully differential in Born kinematics

$N^3LL$ corrections moderate, reduction of uncertainty at small pt

- Good agreement between different matching schemes, choose multiplicative solution at higher order
An example: Higgs pT spectrum

- Important cancellations at $m_H/2$ (!), uncertainties likely underestimated at this scale (long known problem)

- $N^3LL$ corrections amount to a few-% at small $p_t$, reduction of band below 10 GeV consistently with NLO matching
An example: Higgs pT spectrum

Simulation of fiducial distributions (e.g. H -> gamma gamma)

RadISH+NNLOJET, 13 TeV, m_H = 125 GeV
\[ \mu_R = \mu_F = m_H/2, \quad Q = m_H/2 \]
PDF4LHC15 (NNLO) uncertainties with \( \mu_R, \mu_F, Q \) variations

Good convergence across different perturbative orders

[fiducial volume from ATLAS 1407.4222]
An example: DY distributions (pT)

Matching to differential NNLO from NNLOJET, assume N^3LO correction to total XS is zero (i.e. no as^3 constant term included)


(sub-)percent precision in data, theory can reach ~3-5% accuracy...
Other effects important (QED, PDFs, quark masses, hadronisation)

Relevant for W-mass studies
An example: DY distributions (phi*)

RadISH+NNLOJET
8 TeV, pp → Z(→ l⁺l⁻) + X
0.0 < Y_II < 2.4, 116 < m_II < 150 GeV
NNPDF3.0 (NNLO)
uncertainties with \( \mu_R, \mu_F, Q \) variations

\[
\frac{(1/\sigma)d\Sigma}{d\phi^*}
\]

10 \( ^2 \) → 10 \( ^0 \)

RadISH+NNLOJET
8 TeV, pp → Z(→ l⁺l⁻) + X
1.6 < Y_II < 2.4, 116 < m_II < 150 GeV
NNPDF3.0 (NNLO)
uncertainties with \( \mu_R, \mu_F, Q \) variations

\[
\frac{(1/\sigma)d\Sigma}{d\phi^*}
\]

10 \( ^2 \) → 10 \( ^0 \)

Similar conclusions for angular distributions

[Data from ATLAS 1512.02192]
Conclusions

• Higher-order resummation can be formulated directly in momentum space without the need for a factorisation for the considered observable

• The approach I briefly outlined is generalised to any rIRC safe observable in two-scale problems
  • Systematic extension to any logarithmic order
  • Efficient implementation in a computer code: e.g. ARES, RadISH
  • Analytic resummation formulated in a language closer to parton showers

• Differential distributions at $N^3$LL+NNLO
  • Higgs: uncertainties in the 5%-10% range - consistent inclusion of quark-mass effects necessary at this order of accuracy (ongoing study)
  • DY: uncertainties reduced to ~5% across the whole spectrum - good agreement with data in the large-invariant mass bins (study low invariant mass in progress)
    • Improving on this requires the assessment of several effects: NP corrections, quark-mass corrections, QED, theory uncertainties in PDFs, …
Thank you for listening
Squared amplitude decomposition

• Write all-order cross section as \( V(\{\tilde{p}\}, k_1, \ldots, k_n) = |\tilde{k}_{t1} + \cdots + \tilde{k}_{tn}| \)

\[
\Sigma(v) = \int d\Phi_B V(\Phi_B) \sum_{n=0}^{\infty} \prod_{i=1}^{n} [dk_i] |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 \Theta(v - V(\{\tilde{p}\}, k_1, \ldots, k_n))
\]

• Recast all-order squared ME for \( n \) real emissions as iteration of correlated blocks

• Scaling of the observable in the presence of radiation must preserve the above hierarchy

e.g. soft radiation (analogous considerations for hard-collinear)

\[
|M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 = |M_B(\tilde{p}_1, \tilde{p}_2)|^2 \left\{ \left( \frac{1}{n!} \prod_{i=1}^{n} |M(k_i)|^2 \right) + \right. \\
\sum_{a>b} \frac{1}{(n-2)!} \left( \prod_{i\neq a,b} |M(k_i)|^2 \right) |\tilde{M}(k_a, k_b)|^2 + \\
\sum_{a>b} \sum_{c>d} \frac{1}{(n-4)!} \left( \prod_{i\neq a,b,c,d} |M(k_i)|^2 \right) |\tilde{M}(k_a, k_b)|^2 |\tilde{M}(k_c, k_d)|^2 + \ldots \\
\left. + \sum_{a>b>c} \frac{1}{(n-3)!} \left( \prod_{i\neq a,b,c} |M(k_i)|^2 \right) |\tilde{M}(k_a, k_b, k_c)|^2 + \ldots \right\} + \ldots
\]
Squared amplitude decomposition

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\[
|M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 = |M_B(\tilde{p}_1, \tilde{p}_2)|^2 \left\{ \frac{1}{n!} \prod_{i=1}^{n} |M(k_i)|^2 \right\}^{LL} + \\
\left[ \sum_{a>b} \frac{1}{(n-2)!} \left( \prod_{i=1, i\neq a, b}^{n} |M(k_i)|^2 \right) |\tilde{M}(k_a, k_b)|^2 \right] + \\
\sum_{a>b} \sum_{c>d} \frac{1}{(n-4)!^2} \left( \prod_{i=1, i\neq a, b, c, d}^{n} |M(k_i)|^2 \right) |\tilde{M}(k_a, k_b)|^2 |\tilde{M}(k_c, k_d)|^2 + \cdots \\
+ \left[ \sum_{a>b>c} \frac{1}{(n-3)!} \left( \prod_{i=1, i\neq a, b, c}^{n} |M(k_i)|^2 \right) |\tilde{M}(k_a, k_b, k_c)|^2 + \cdots \right] + \cdots \right\},
\]
**Squared amplitude decomposition**

- Write all-order cross section as 
  \[ V(\{\tilde{p}\}, k_1, \ldots, k_n) = |\tilde{k}_{t1} + \cdots + \tilde{k}_{tn}| \]

\[
\Sigma(v) = \int d\Phi_B \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} (dk_i) |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 \Theta(v - V(\{\tilde{p}\}, k_1, \ldots, k_n))
\]

- Recast all-order squared ME for \( n \) real emissions as iteration of **correlated blocks**
- Scaling of the observable in the presence of radiation **must** preserve the above hierarchy

\[ |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 = |M_B(\tilde{p}_1, \tilde{p}_2)|^2 \left\{ \frac{1}{n!} \prod_{i=1}^{n} |M(k_i)|^2 \right\}_{LL} + \]

\[
\sum_{a>b} \frac{1}{(n-2)!} \left( \prod_{i=1}^{n} |M(k_i)|^2 \right) |\tilde{M}(k_a, k_b)|^2 \right\}_{NLL} + 
\]

\[
\sum_{a>b} \sum_{c>d} \frac{1}{(n-4)!} \left( \prod_{i=1}^{n} |M(k_i)|^2 \right) |\tilde{M}(k_a, k_b)|^2 |\tilde{M}(k_c, k_d)|^2 + \ldots \right\}_{NLL} + 
\]

\[
+ \left\{ \sum_{a>b>c} \frac{1}{(n-3)!} \left( \prod_{i=1}^{n} |M(k_i)|^2 \right) |\tilde{M}(k_a, k_b, k_c)|^2 + \ldots \right\} + \ldots \right\}_{NLL}.
\]
Squared amplitude decomposition

- Write all-order cross section as \( V(\{\tilde{p}\}, k_1, \ldots, k_n) = |\vec{k}_{t1} + \cdots + \vec{k}_{tn}| \)

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\]

- Recast all-order squared ME for \( n \) real emissions as iteration of correlated blocks

- Scaling of the observable in the presence of radiation must preserve the above hierarchy

\[ |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 = |M_B(\tilde{p}_1, \tilde{p}_2)|^2 \left\{ \begin{array}{c}
\frac{1}{n!} \prod_{i=1}^{n} |M(k_i)|^2 \\
\text{LL}
\end{array} \right\} + \]

\[ \sum_{a>b} \frac{1}{(n-2)!} \left( \prod_{i=1, i \neq a,b}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b) \right|^2 + \]

\[ \sum_{a>b} \sum_{c>d} \frac{1}{(n-4)!} \left( \prod_{i=1, i \neq a,b,c,d}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b) \left| \tilde{M}(k_c, k_d) \right|^2 + \ldots \right) + \]

\[ \sum_{a>b} \sum_{c>d} \frac{1}{(n-3)!} \left( \prod_{i=1, i \neq a,b,c}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b, k_c) \right|^2 + \ldots + \ldots \right) + \ldots \}

- \begin{array}{c}
\text{LL} \\
\text{NLL} \\
\text{NLL} \\
\text{NNLL}
\end{array}
Squared amplitude decomposition

- Write all-order cross section as \( V(\{\tilde{p}\}, k_1, \ldots, k_n) = |\tilde{k}_{t1} + \cdots + \tilde{k}_{tn}| \)

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\Sigma(v) = \int d\Phi_B \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 \Theta(v - V(\{\tilde{p}\}, k_1, \ldots, k_n))
\]

- Recast all-order squared ME for \( n \) real emissions as iteration of correlated blocks

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\textit{e.g.} soft radiation (analogous considerations for hard-collinear)

\[
|M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 = |M_B(\tilde{p}_1, \tilde{p}_2)|^2 \left\{ \frac{1}{n!} \prod_{i=1}^{n} |M(k_i)|^2 \right\}_{\text{LL}} + \sum_{a>b} \frac{1}{(n-2)!} \left( \prod_{i=1}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b) \right|^2_{\text{NLL}}
\]

\[
+ \sum_{a>b} \sum_{c,d\neq a,b} \frac{1}{(n-4)!} \left( \prod_{i=1}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b, k_c) \right|^2_{\text{NLL}} + \ldots
\]

\[
+ \sum_{a>b>c} \frac{1}{(n-3)!} \left( \prod_{i=1}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b, k_c, k_d) \right|^2_{\text{NNLL}} + \ldots + \ldots
\]

In addition to this counting, requiring that the observable is recursively IRC safe allows one to construct a (simpler) all-order subtraction scheme.
Monte Carlo formulation

- One great simplification: choice of the resolution variable such that correlated blocks entering at $N^k$LL in the unresolved radiation only contribute at $N^{k+1}$LL in the resolved case.

- i.e. we can expand out the cutoff dependence and retain in the Radiator only the terms necessary to cancel the singularities in the resolved radiation.

\[
R(\epsilon k_{t1}) = R(k_{t1}) + R'(k_{t1}) \ln \frac{1}{\epsilon} + \frac{1}{2} R''(k_{t1}) \ln^2 \frac{1}{\epsilon} + \ldots
\]

\[
R'(k_{ti}) = R'(k_{t1}) + R''(k_{t1}) \ln \frac{k_{t1}}{k_{ti}} + \ldots
\]

Expansion is safe since in the resolved radiation $k_{t1}/k_{ti} \sim 1$

e.g. at NLL

\[
\int \frac{dk_{t1}}{k_{t1}} \partial_L \left( -e^{-R(k_{t1})} L(k_{t1}) \right) \prod_{i=2}^{n+1} \frac{1}{n!} \int_{c_{k_{t1}}}^{k_{t1}} \frac{dk_{ti}}{k_{ti}} R'(k_{t1})
\]
Monte Carlo formulation

- One great simplification: choice of the resolution variable such that correlated blocks entering at $N^k\text{LL}$ in the unresolved radiation only contribute at $N^{k+1}\text{LL}$ in the resolved case

- i.e. we can expand out the cutoff dependence and retain in the Radiator only the terms necessary to cancel the singularities in the resolved radiation

$$R(\epsilon k_{t1}) = R(k_{t1}) + R'(k_{t1}) \ln \frac{1}{\epsilon} + \frac{1}{2} R''(k_{t1}) \ln^2 \frac{1}{\epsilon} + \ldots$$

$$R'(k_{ti}) = R'(k_{t1}) + R''(k_{t1}) \ln \frac{k_{t1}}{k_{ti}} + \ldots$$

- Corrections beyond NLL are obtained as follows
  
  - Add subleading effects in the Sudakov radiator and constants
  
  - Correct a fixed number of the NLL resolved emissions:
    
    - only one at NNLL
    - two at $N^3\text{LL}$
    - ...

Expansion is safe since in the resolved radiation $k_{t1}/k_{ti} \sim 1$

-e.g. at NNLL see:
Banfi, PM, Salam, Zanderighi '12
Banfi, McAslan, PM, Zanderighi '14-'16
Numerical implementation: **RadISH**

- Since the transverse momenta of the *resolved* reals are of the same order, we can expand the whole integrand about \( k_{ti} \sim k_{t1} \) up to the desired logarithmic accuracy.
- This expansion allows us to compute higher-order corrections to the NLL *resolved* reals by simply including one correction at a time.

**e.g.** expansion up to NLL

\[
\frac{d\Sigma(v)}{d\Phi_B} = \int \frac{d k_{t1}}{k_{t1}} \frac{d \phi_1}{2\pi} \partial_L \left( -e^{-R(k_{t1})} \mathcal{L}_{N^3LL}(k_{t1}) \right) \int dZ([R', k_i]) \Theta(v - V([\bar{p}], k_1, \ldots, k_{n+1}))
\]

\[
\mathcal{L}_{N^3LL}(k_{t1}) = \sum_{c,c'} \frac{d|M_B|}{d\Phi_B} \sum_{i,j} \int \frac{dz_1}{x_1} \int \frac{dz_2}{x_2} f_i \left( k_{t1}, \frac{x_1}{z_1} \right) f_j \left( k_{t1}, \frac{x_2}{z_2} \right)
\]

\[
\left\{ \delta_{c_1} \delta_{c_1'} \delta(1-z_1) \delta(1-z_2) \left( 1 + \frac{\alpha_s(\mu_R)}{2\pi} H^{(1)}(\mu_R) + \frac{\alpha_s^2(\mu_R)}{(2\pi)^2} H^{(2)}(\mu_R) \right) \right.
\]

\[
+ \frac{\alpha_s(\mu_R)}{2\pi} \frac{1}{1 - 2\alpha_s(\mu_R) \beta_0 L} \left( 1 - \alpha_s(\mu_R) \frac{\beta_1}{\beta_0} \ln \frac{1 - 2\alpha_s(\mu_R) \beta_0 L}{1 - 2\alpha_s(\mu_R) \beta_0 L} \right)
\]

\[
\times \left( C^{(1)}_{c_1}(z_1) \delta(1-z_2) \delta_{c_1'} + \{ z_1 \leftrightarrow z_2; c, i \leftrightarrow c', j \} \right)
\]

\[
+ \frac{\alpha_s^2(\mu_R)}{(2\pi)^2} \frac{1}{(1 - 2\alpha_s(\mu_R) \beta_0 L)^2} \left( C^{(2)}_{c_1}(z_1) - 2\pi \beta_0 C^{(1)}_{c_1}(z_1) \ln \frac{M^2}{\mu^2_R} \right) \delta(1-z_2) \delta_{c_1'}
\]

\[
+ \{ z_1 \leftrightarrow z_2; c, i \leftrightarrow c', j \} + \frac{\alpha_s^2(\mu_R)}{(2\pi)^2} \frac{1}{(1 - 2\alpha_s(\mu_R) \beta_0 L)^2} \left( C^{(1)}_{c_1}(z_1) C^{(1)}_{c_1'}(z_2) + G^{(1)}_{c_1}(z_1) G^{(1)}_{c_1'}(z_2) \right)
\]

\[
+ \frac{\alpha_s^2(\mu_R)}{(2\pi)^2} H^{(1)}(\mu_R) \frac{1}{1 - 2\alpha_s(\mu_R) \beta_0 L} \left( C^{(1)}_{c_1}(z_1) \delta(1-z_2) \delta_{c_1'} + \{ z_1 \leftrightarrow z_2; c, i \leftrightarrow c', j \} \right)
\]

- Coefficient functions and hard-virtual corrections absorbed into effective parton luminosities.
Numerical implementation: RadISH

- Since the transverse momenta of the resolved reals are of the same order, we can expand the whole integrand about $k_{ti} \sim k_{t1}$ up to the desired logarithmic accuracy.
- This expansion allows us to compute higher-order corrections to the NLL resolved reals by simply including one correction at a time.

\[ \frac{d\Sigma(v)}{d\Phi_B} = \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} \partial_L \left( -e^{-R(k_{t1})} L_{N_{LL}}(k_{t1}) \right) \int dZ[[R', k_i]] \Theta(v - V(\{p}, k_1, \ldots, k_{n+1})) \]

\[ k_{ti}/k_{t1} = \zeta_i = O(1) \]

- Coefficient functions and hard-virtual corrections absorbed into effective parton luminosities.
- The ensemble of NLL real emissions $dZ$ is generated as a parton shower. Fast numerical evaluation with Monte-Carlo methods.

\[ \int dZ[[R', k_i]]G(\{p}, \{k_i\}) = \epsilon^{R'(k_{t1})} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_{\epsilon}^{1} d\zeta_i \int_0^{2\pi} \frac{d\phi_i}{2\pi} R'(k_{t1})G(\{p}, k_1, \ldots, k_{n+1}) \]
Numerical implementation: RadISH

- Since the transverse momenta of the resolved reals are of the same order, we can expand the whole integrand about $k_{ti} \sim k_{t1}$ up to the desired logarithmic accuracy.

- This expansion allows us to compute higher-order corrections to the NLL resolved reals by simply including one correction at a time.

  e.g. expansion up to NNLL

  \[
  \frac{d\Sigma(v)}{d\Phi_B} = \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} \partial_L \left( -e^{-R(k_{t1})} L_{N^3\text{LL}}(k_{t1}) \right) \int dZ[\{R', k_i\}] \Theta(v - V(\{p\}, k_1, \ldots, k_{n+1})) \\
  + \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(k_{t1})} \int dZ[\{R', k_i\}] \int_0^1 \frac{d\zeta_s}{\zeta_s} \frac{d\phi_s}{2\pi} \left\{ \left( R'(k_{t1}) L_{N^{\text{NNLL}}}(k_{t1}) - \partial_L L_{N^{\text{NNLL}}}(k_{t1}) \right) \\
  \times \left( R''(k_{t1}) \ln \frac{1}{\zeta_s} + \frac{1}{2} R''(k_{t1}) \ln^2 \frac{1}{\zeta_s} \right) - R'(k_{t1}) \left( \partial_L L_{N^{\text{NNLL}}}(k_{t1}) - 2 \frac{\beta_0}{\pi} \alpha_s^2(k_{t1}) \hat{P}^{(0)} \otimes L_{\text{NNLL}}(k_{t1}) \ln \frac{1}{\zeta_s} \right) \right. \\
  \left. + \frac{\alpha_s^2(k_{t1})}{\pi^2} \hat{P}^{(0)} \otimes L_{\text{NNLL}}(k_{t1}) \right\} \Theta(v - V(\{p\}, k_1, \ldots, k_{n+1}, k_s)) - \Theta(v - V(\{p\}, k_1, \ldots, k_{n+1})) \right\}
  \]

  - Coefficient functions and hard-virtual corrections absorbed into effective parton luminosities.

  - The ensemble of NLL real emissions $dZ$ is generated as a parton shower. Fast numerical evaluation with Monte-Carlo methods.
Numerical implementation: **RadISH**

- Since the transverse momenta of the *resolved* reals are of the same order, we can expand the whole integrand about \( k_{ti} \sim k_{t1} \) up to the desired logarithmic accuracy.
- This expansion allows us to compute higher-order corrections to the NLL *resolved* reals by simply including one correction at a time.

\[
\frac{d\Sigma(v)}{d\Phi_B} = \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} \partial_L \left( -e^{-R(k_{t1})} \mathcal{L}_{N^3LL}(k_{t1}) \right) \int dZ[[R', k_i]] \Theta(v - V([\bar{p}], k_1, \ldots, k_{n+1}))
\]

\[
+ \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(k_{t1})} \int dZ[[R', k_i]] \int_0^1 \frac{d\zeta_s}{\zeta_s} \frac{d\phi_s}{2\pi} \left\{ \left( R'(k_{t1}) \mathcal{L}_{NNLL}(k_{t1}) - \partial_L \mathcal{L}_{NNLL}(k_{t1}) \right) \right.
\]

\[
\times \left( R''(k_{t1}) \ln \frac{1}{\zeta_s} + \frac{1}{2} R''(k_{t1}) \ln^2 \frac{1}{\zeta_s} \right) - R'(k_{t1}) \left( \partial_L \mathcal{L}_{NNLL}(k_{t1}) - \frac{2\beta_0}{\pi} \alpha_s^2(k_{t1}) \hat{p}^{(0)} \otimes \mathcal{L}_{NNLL}(k_{t1}) \ln \frac{1}{\zeta_s} \right)
\]

\[
+ \left. \frac{\alpha_s^2(k_{t1})}{\pi^2} \hat{p}^{(0)} \otimes \mathcal{L}_{NNLL}(k_{t1}) \right\}
\]

- Coefficient functions and hard-virtual corrections absorbed into effective parton luminosities.
- The ensemble of NLL real emissions \( dZ \) is generated as a parton shower. Fast numerical evaluation with Monte-Carlo methods.

\[
\Theta(v - V([\bar{p}], k_1, \ldots, k_{n+1}, k_s)) - \Theta(v - V([\bar{p}], k_1, \ldots, k_{n+1}, k_s) -
\Theta(v - V([\bar{p}], k_1, \ldots, k_{n+1}, k_{s2})) + \Theta(v - V([\bar{p}], k_1, \ldots, k_{n+1}))) + \mathcal{O} \left( \alpha_s^n \ln^{2n-6} \frac{1}{v} \right)
\]
Equivalence to CSS formula

- Hard-collinear emissions off initial-state legs require some care in the treatment of kinematics. Final result reads

\[
\frac{d\Sigma(v)}{dp_t d\Phi_B} = \int_{c_1} \frac{dN_1}{2\pi i} \int_{c_2} \frac{dN_2}{2\pi i} x_1^{-N_1} x_2^{-N_2} \sum_{c_1, c_2} \frac{d|M_B|^2_{c_1, c_2}}{d\Phi_B} f_{N_1}(\mu_0) \frac{d\Sigma_{c_1, c_2}^{N_1, N_2}(v)}{dp_t} f_{N_2}(\mu_0)
\]

\[
\hat{\Sigma}_{c_1, c_2}^{N_1, N_2}(v) = \left[ \mathcal{C}_{c_1}^{T}(\alpha_s(\mu_0)) H(\mu_R) \mathcal{C}_{c_2}^{T}(\alpha_s(\mu_0)) \right] \int_0^M \frac{dk_{t_1}}{k_{t_1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \left( \Gamma_{N_1}(\alpha_s(k_{t_1})) + \int_0^{2\pi} \frac{d\phi_1}{2\pi} \Gamma_{N_1}(\alpha_s(k_{t_1})) \right)
\]

\[
\times e^{-R(\epsilon k_{t_1})} \exp \left\{ -\sum_{\ell_1=1}^{2} \left( \mathcal{R}_{\ell_1}(k_{t_1}) + \frac{\alpha_s(k_{t_1})}{\pi} \Gamma_{N_1}(\alpha_s(k_{t_1})) + \int_0^{2\pi} \frac{d\phi_1}{2\pi} \Gamma_{N_1}(\alpha_s(k_{t_1})) \right) \right\}
\]

\[
\times \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_{\epsilon}^{\zeta_i} d\zeta_i \int_0^{2\pi} \frac{d\phi_i}{2\pi} \sum_{\ell_1=1}^{2} \left( \mathcal{R}_{\ell_1}(k_{t_1}) + \frac{\alpha_s(k_{t_1})}{\pi} \Gamma_{N_1}(\alpha_s(k_{t_1})) + \int_0^{2\pi} \frac{d\phi_1}{2\pi} \Gamma_{N_1}(\alpha_s(k_{t_1})) \right)
\]

\[
\times \Theta(v - V(\{\hat{p}, k_1, \ldots, k_{n+1}\})),
\]

- Formulation equivalent to b-space result, up to a scheme change. Using the delta representation for the distribution one finds

\[
\frac{d\Sigma(v)}{dp_t d\Phi_B} = \int_{c_1} \frac{dN_1}{2\pi i} \int_{c_2} \frac{dN_2}{2\pi i} x_1^{-N_1} x_2^{-N_2} \sum_{c_1, c_2} \frac{d|M_B|^2_{c_1, c_2}}{d\Phi_B} f_{N_1}(\mu_0) \frac{d\Sigma_{c_1, c_2}^{N_1, N_2}(v)}{dp_t} f_{N_2}(\mu_0) = \]

\[
\delta^{(2)}(\hat{p}_t - (\vec{k}_{t_1} + \ldots + \vec{k}_{t_n})) = \int \frac{d^2\vec{p}_t}{4\pi^2} e^{-i\vec{p}_t \cdot \vec{k}_t} \prod_{i=1}^{n} e^{i\vec{p}_t \cdot \vec{k}_{t_i}}
\]

\[
(1 - J_0(bk_i)) \simeq \Theta(k_t - \frac{b_0}{b}) + \frac{\zeta_3}{12 \partial \ln(M b / b_0)^3} \Theta(k_t - \frac{b_0}{b}) + \ldots
\]

\[
= \sum_{c_1, c_2} \frac{d|M_B|^2_{c_1, c_2}}{d\Phi_B} \int b \frac{dp_t}{k_t} J_0(p_t b) \mathcal{C}_{c_1}^{T}(\alpha_s(b_0 / b)) H(M) \mathcal{C}_{c_2}^{T}(\alpha_s(b_0 / b)) f(b_0 / b)
\]

\[
\times \exp \left\{ -\sum_{\ell=1}^{2} \int_0^M \frac{dk_{t_1}}{k_{t_1}} \mathcal{R}_{\ell}(k_{t_1}) (1 - J_0(bk_{t_1})) \right\}.
\]