Phenomenology of models of Generalized Parton Distributions built from Light-front wave-functions

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Outline

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Definition of GPDs

- Quark GPD (twist-2, spin-0 hadron): **(Müller et al., 1994; Radyushkin, 1996; Ji, 1997)**

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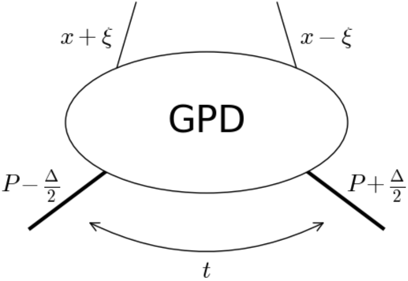

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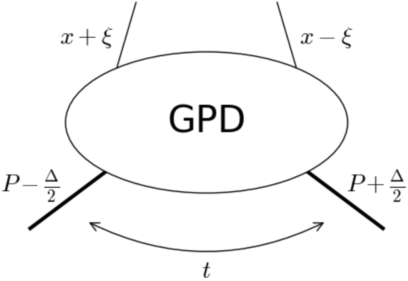

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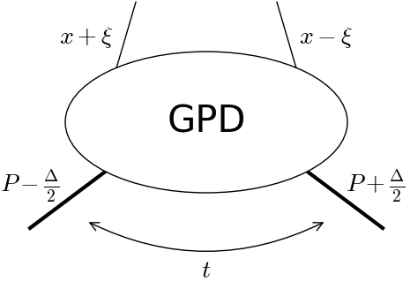

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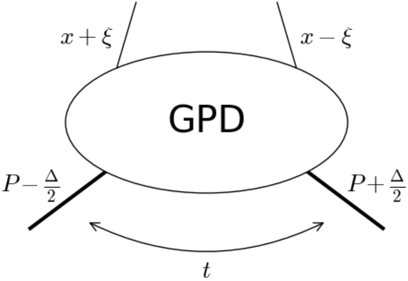

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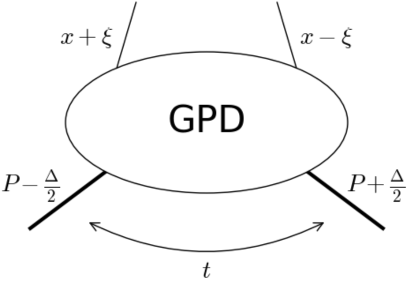

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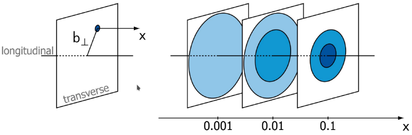

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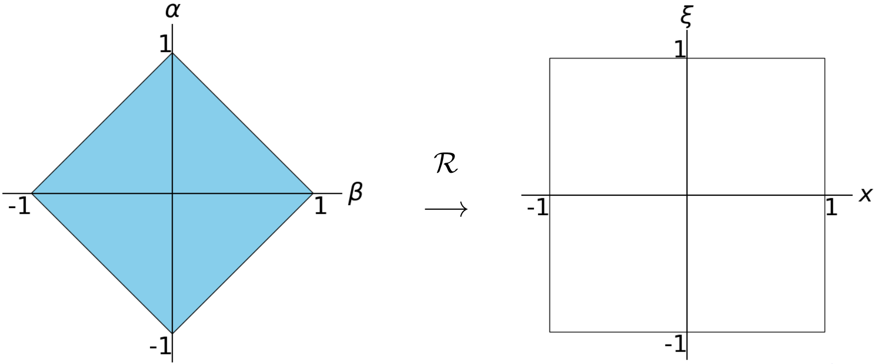

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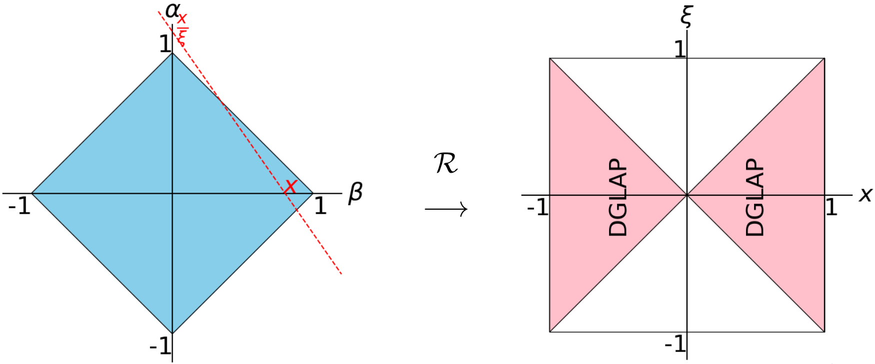

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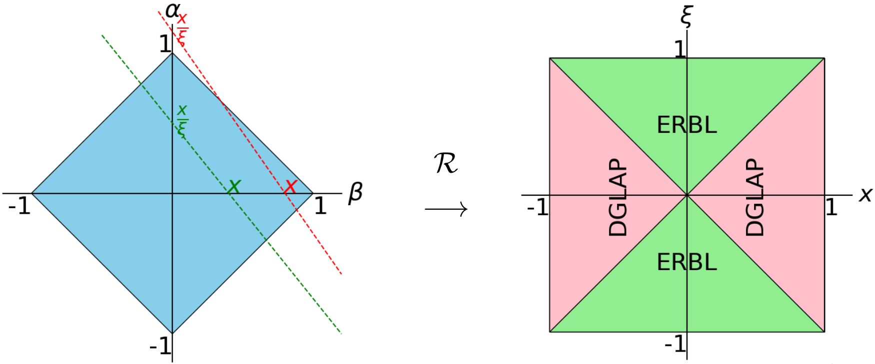

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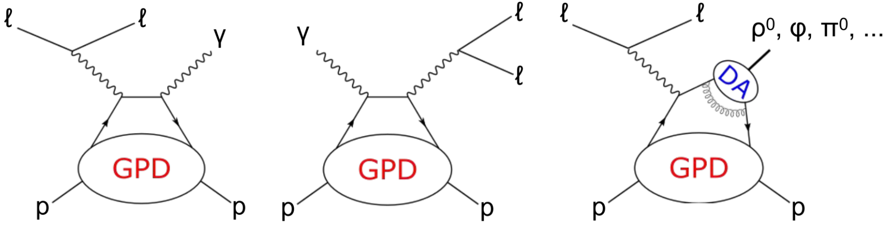

DVCS TCS DVMP

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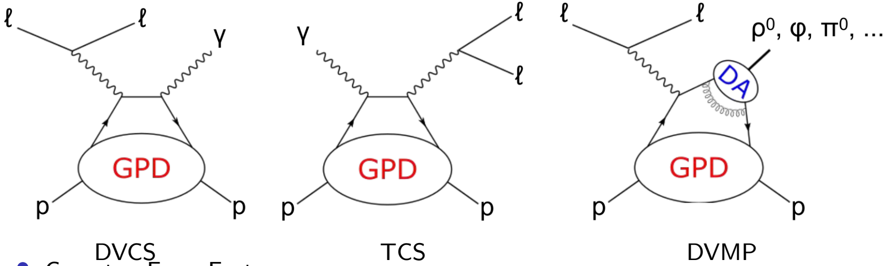

- Compton Form Factors: **(Belitsky et al., 2002)**

$$\mathcal{F}\left(\xi, t, Q^2\right) = \int_{-1}^{1} \mathrm{d}x\, C\left(x, \xi, \alpha_S\left(\mu_F\right), \frac{Q}{\mu_F}\right) F\left(x, \xi, t, \mu_F\right). \quad (12)$$

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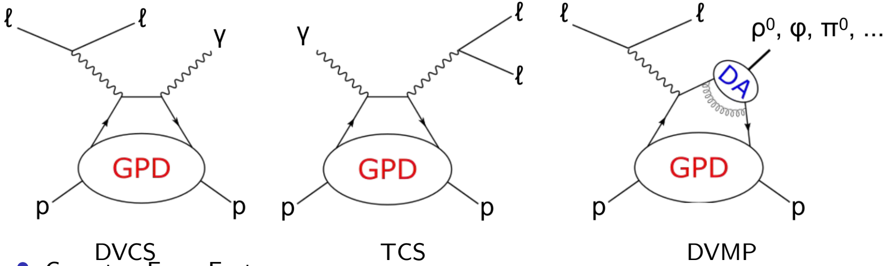

- Compton Form Factors: **(Belitsky et al., 2002)**

$$\mathcal{F}\left(\xi, t, Q^2\right) = \int_{-1}^{1} \mathrm{d}x \, C\left(x, \xi, \alpha_S\left(\mu_F\right), \frac{Q}{\mu_F}\right) F\left(x, \xi, t, \mu_F\right). \tag{12}$$

- Observables are convolutions of:

Accessing GPDs

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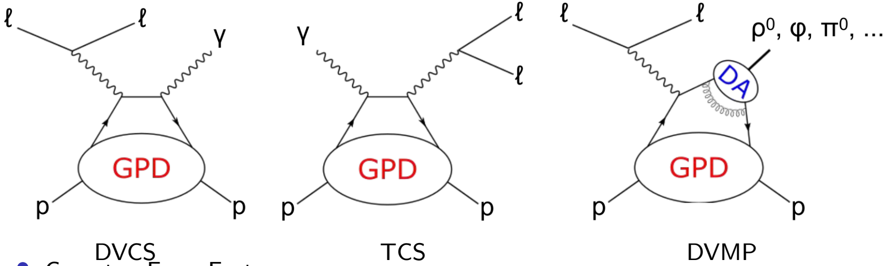

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- Observables are convolutions of:
  - a soft part, i.e. the GPD, with long distance interactions encoded (non-perturbative QCD).

Accessing GPDs

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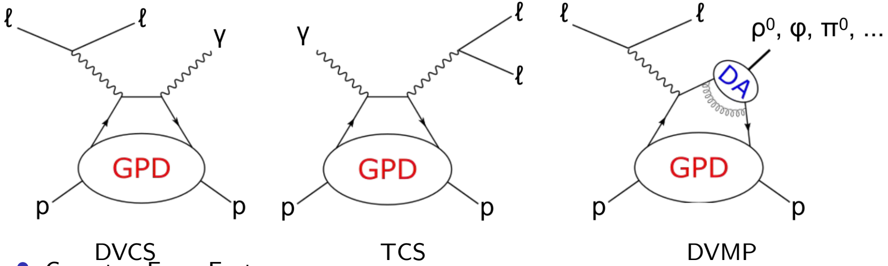

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- Observables are convolutions of:
  - a soft part, i.e. the GPD, with long distance interactions encoded (non-perturbative QCD).
  - a hard-scattering kernel, calculated with perturbative QCD (short distance interactions).

PARTONS software

(Berthou et al., 2016)

- Framework to study GPDs.

PARTONS software

(Berthou et al., 2016)

- Framework to study GPDs.
  - Goal: support the effort of the GPD community.

PARTONS software

(Berthou et al., 2016)

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  - For both theorists and experimentalists.

PARTONS software

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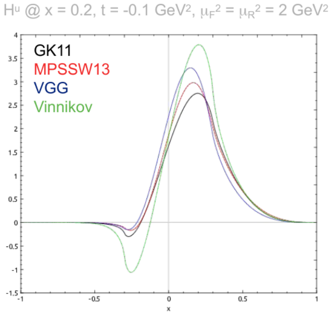
PARTONS



PARTONS software

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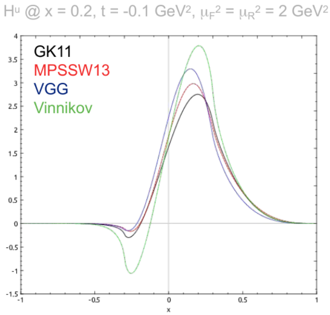

PARTONS software

(Berthou et al., 2016)

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  - Goal: support the effort of the GPD community.
  - For both theorists and experimentalists.
- A certain number of built-in physics developments.
  - Easy addition of your own models.
- C++ library with XML interface for automated tasks.

```
// Get service
GPDService* pGPDService =
        Partons::getInstance()->getServiceObjectRegistry()->getGPD
        Service();

// Create GPD module with the ModuleObjectFactory
GPDModule* pGPDModel
        =Partons::getInstance()->getModuleObjectFactory()->newGPDM
        odule(MMS13Model::classId);

// Create a GPDKinematic(x, xi, t, MuF, MuR) object
GPDKinematic gpdKinematic(-0.99, 0.99, 0., 1., 1.);

GPDResult gpdResult = pGPDService->computeGPDModel(gpdKinematic,
        pGPDModel);
```

PARTONS software

(Berthou et al., 2016)

- Framework to study GPDs.
  - Goal: support the effort of the GPD community.
  - For both theorists and experimentalists.
- A certain number of built-in physics developments.
  - Easy addition of your own models.
- C++ library with XML interface for automated tasks.

```
<?xml version="1.0" encoding="UTF-8" standalone="yes" ?>

<scenario date="2016-03-25"
    description="This is a test scenario to use the GPD
      service">

    <task service="GPDService" method="computeGPDModel">
        <kinematic type=GPDKinematic"">
            <param name="x" value="0.1" />
            <param name="xi" value="0.00050025" />
            <param name="t" value="-0.3" />
            <param name="MuF2" value="8" />
            <param name="MuR2" value="8" />
        </kinematic>

        <computation_configuration>
            <module type="GPDModule">
                <param name="className" value="GK11Model" />
            </module>
        </computation_configuration>
    </task>

</scenario>
```

PARTONS software

(Berthou et al., 2016)

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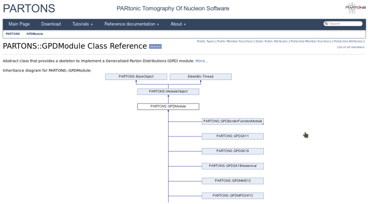
- Open-source: `http://partons.cea.fr`



PARTONS software

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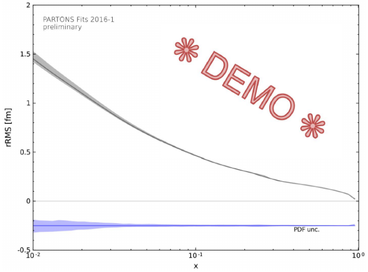
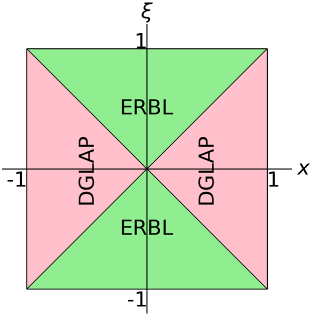
- A certain number of built-in physics developments.
  - Easy addition of your own models.
- C++ library with XML interface for automated tasks.
- Open-source: `http://partons.cea.fr`
- Global fit of JLab data (ongoing work by Pawel Snajder).



Outline

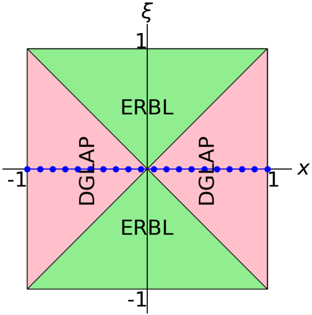
Covariant extension

- For the spatial tomography of hadrons, we need:



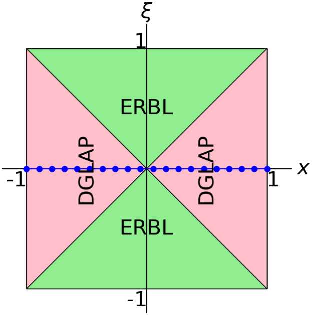
Covariant extension

- For the spatial tomography of hadrons, we need:
  - GPD at $\xi = 0$.



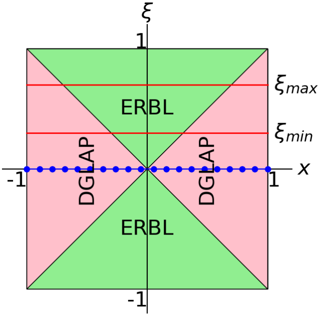
Covariant extension

- For the spatial tomography of hadrons, we need:
  - GPD at $\xi = 0$.
- Experimental access through exclusive processes:



Covariant extension

- For the spatial tomography of hadrons, we need:
  - GPD at $\xi = 0$.
- Experimental access through exclusive processes:
  - Integrals over x of GPD at $\xi > 0$.



Covariant extension

- For the spatial tomography of hadrons, we need:
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- Experimental access through exclusive processes:
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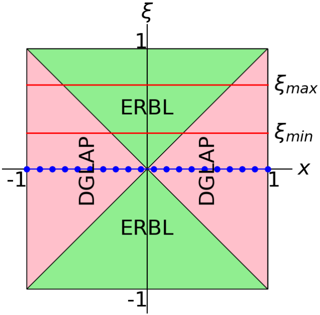
  - Extrapolation?



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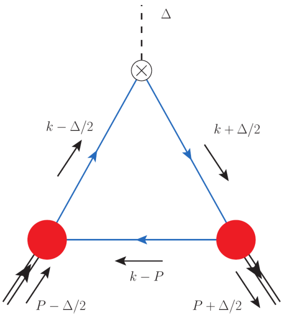
  - Extrapolation?
- Covariant approach?



Covariant extension

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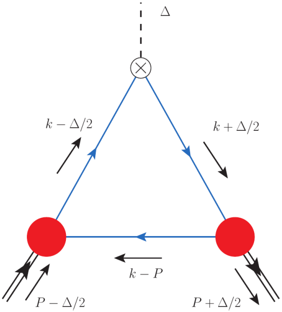
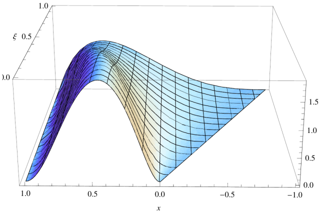
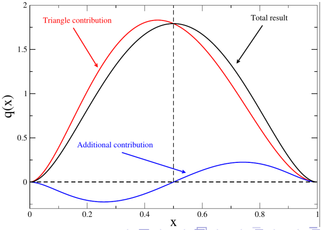
- Covariant approach?
  - Positivity?







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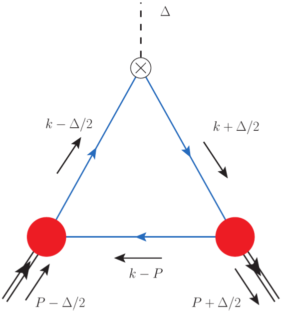
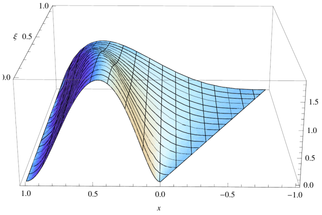
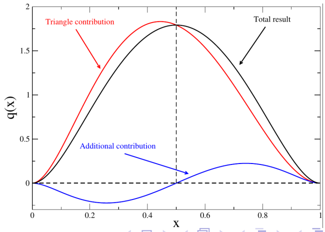
- Covariant approach?
  - Positivity?
  - Loss of symmetries...







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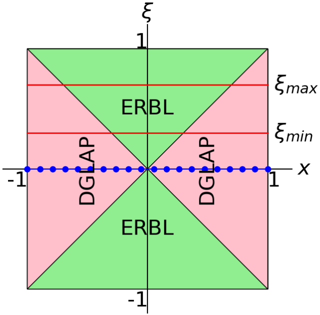

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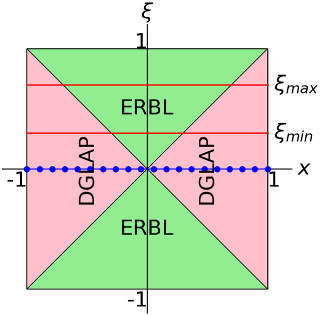
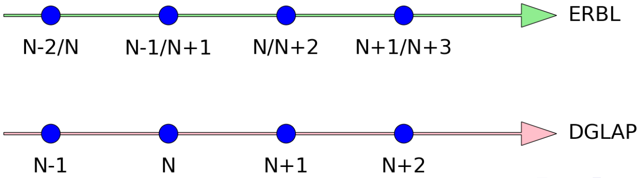
  - Derive all parton distributions (not just GPDs)!





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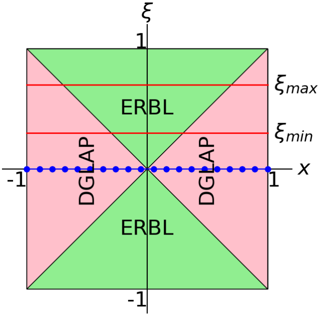

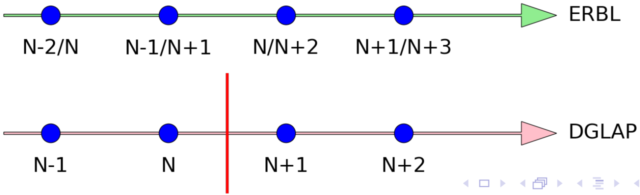

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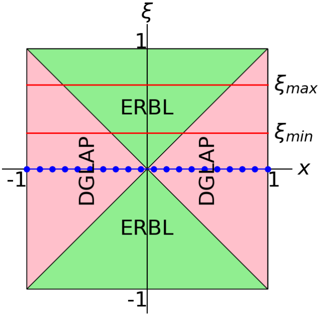

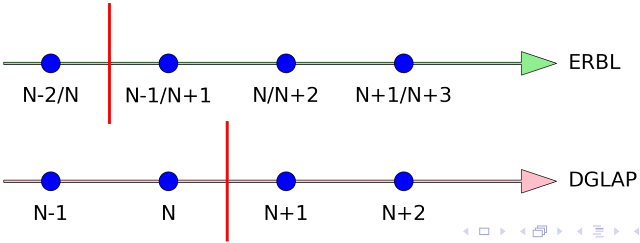

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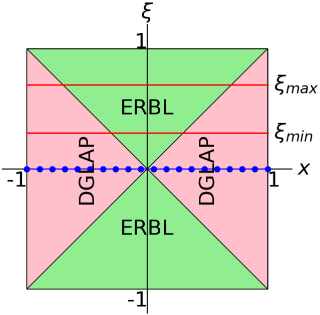

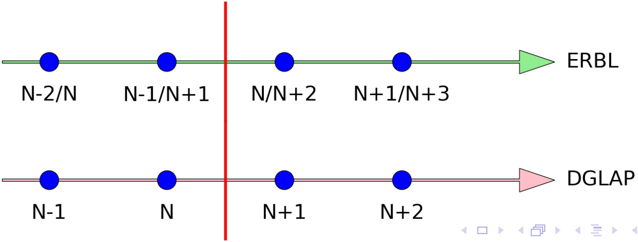

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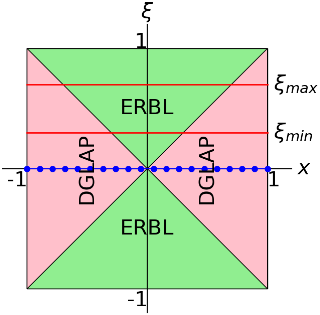

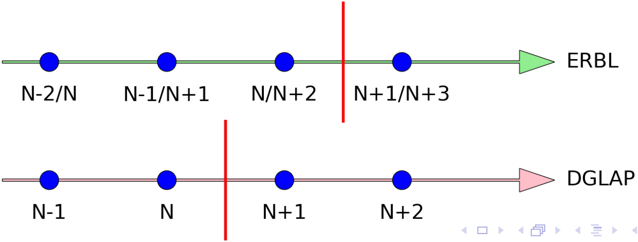

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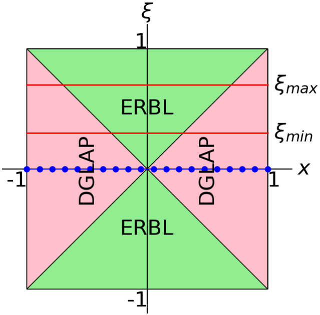
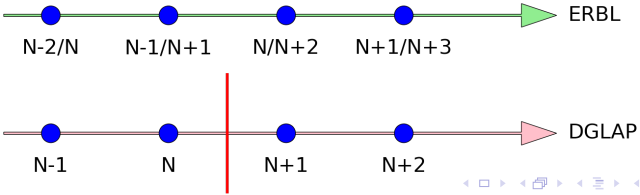
- **Use Lorentz invariance to extend from DGLAP!**
  (Hwang and Mueller, 2008; Chouika et al., 2017)





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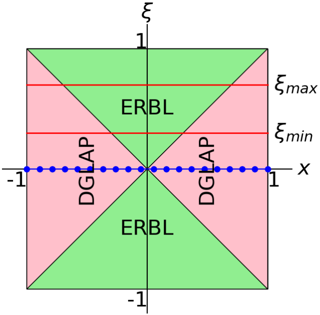

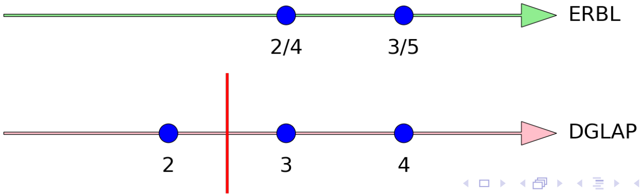

Inversion

Problem

Find $h(\beta, \alpha)$ on square $\{|\alpha| + |\beta| \leq 1\}$ such that

$$H(x,\xi)|_{\mathrm{DGLAP}} \propto \int \mathrm{d}\beta\, \mathrm{d}\alpha\, h(\beta,\alpha)\, \delta(x - \beta - \alpha\xi)\ .$$

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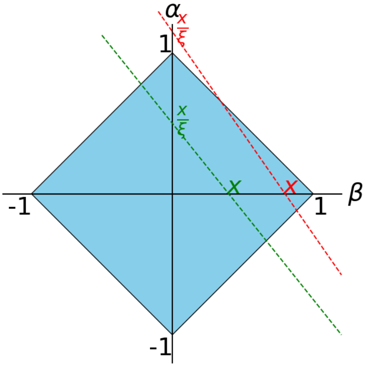

$\mathcal{R}$

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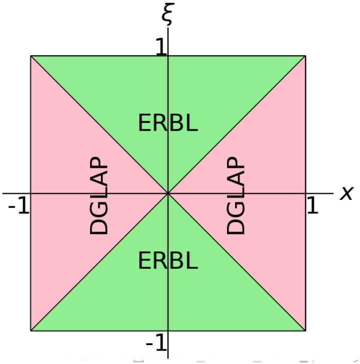
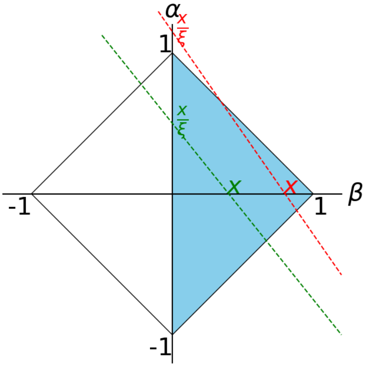


Inversion

- Quark GPD: $H(x,\xi) = 0$ for $-1 < x < -|\xi| \Longrightarrow h(\beta,\alpha) = 0$ for $\beta < 0$.



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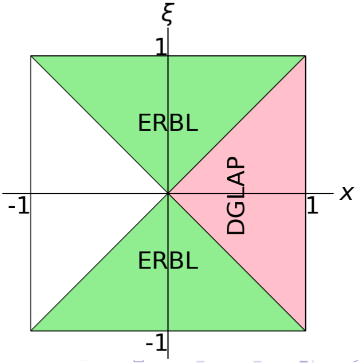
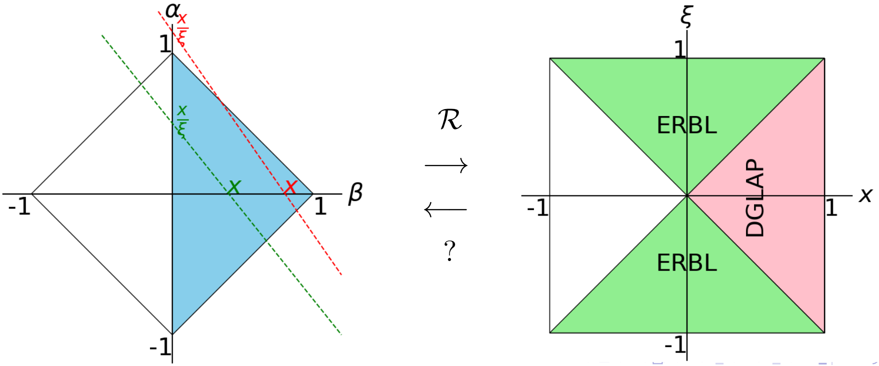


Inversion

- Quark GPD: $H(x,\xi) = 0$ for $-1 < x < -|\xi| \Longrightarrow h(\beta,\alpha) = 0$ for $\beta < 0$.
- Domains $\beta < 0$ and $\beta > 0$ are uncorrelated in the DGLAP region.



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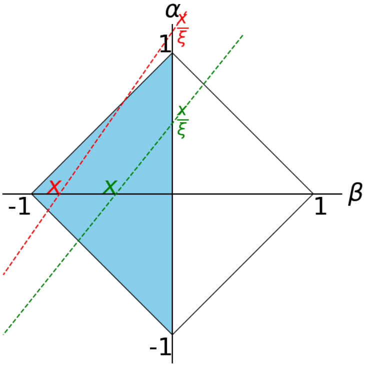

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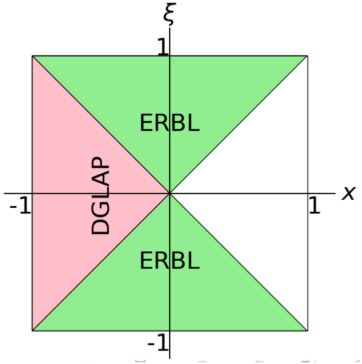

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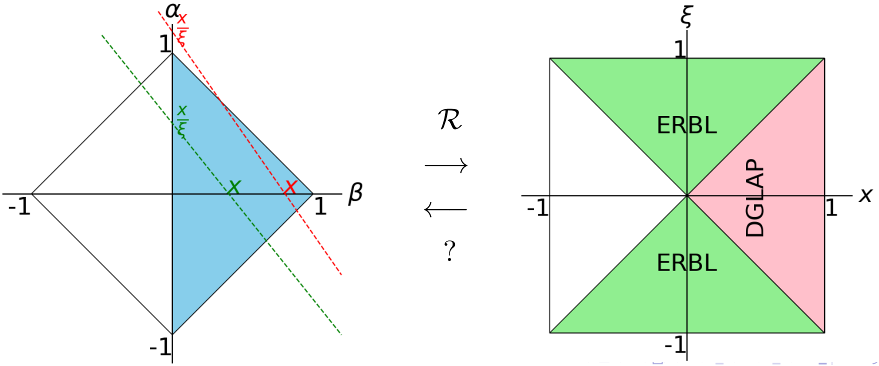
- Domains $\beta < 0$ and $\beta > 0$ are uncorrelated in the DGLAP region.
- Divide and conquer:
  - Better numerical stability.
  - Lesser complexity: $O(N^p + N^p) \ll O((N+N)^p)$.



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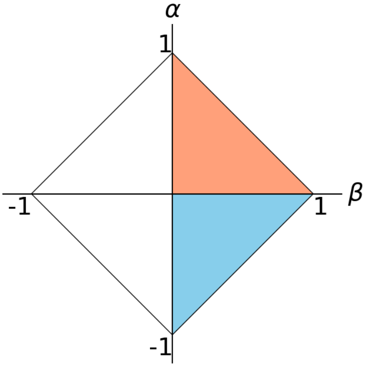
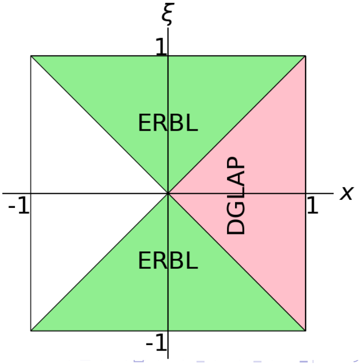
- α-parity of the DD: $h(\beta,-\alpha) = h(\beta,\alpha)$.



$$\overset{\mathcal{R}}{\longrightarrow}$$
$$\underset{?}{\longleftarrow}$$



Some examples: Dyson-Schwinger model

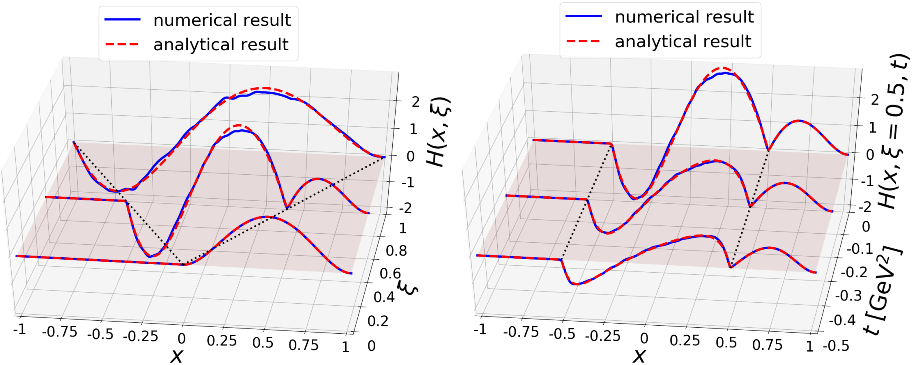


Figure: Extension of GPDs for the pion DSE model of Refs. **(Mezrag, 2015; Mezrag et al., 2016)**. Comparison to the analytical result. Left: Plot for fixed ξ values 0, 0.5 and 1, at $t = 0$ GeV^2. Right: Plot for fixed t values 0, -0.25 and -0.5 GeV^2, at $\xi = 0.5$. See **(Chouika et al., 2018)**.

Some examples: Spectator di-quark model

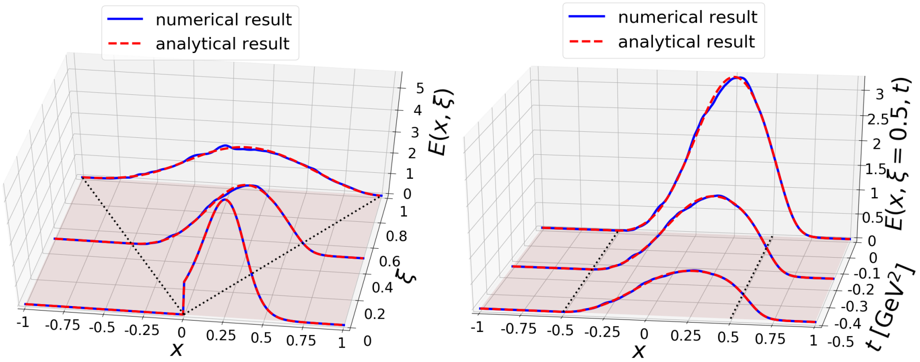
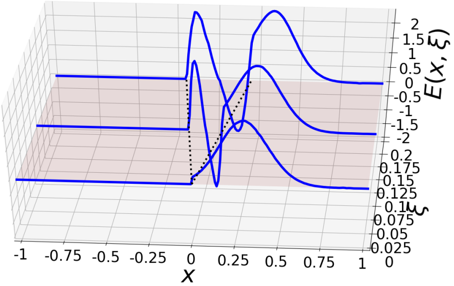


Figure: Extension of GPD E for the nucleon model of Ref. **(Hwang and Mueller, 2008)**. Comparison to the analytical result of the authors. Left: Plot for fixed ξ values 0, 0.5 and 1, at $t = 0$ GeV^2. Right: Plot for fixed t values 0, -0.25 and -0.5 GeV^2, at $\xi = 0.5$.

Phenomenology of quark models: application to χQSM



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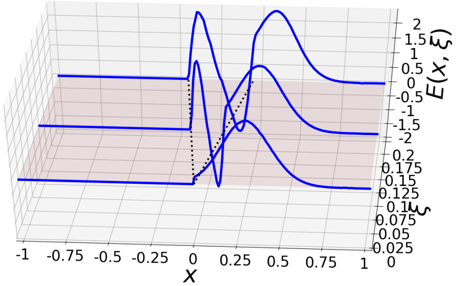

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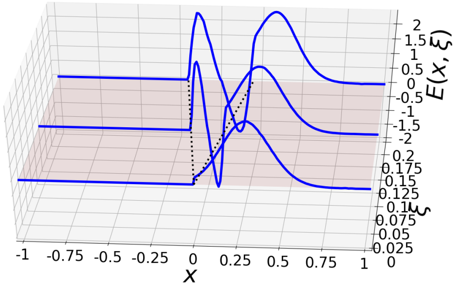

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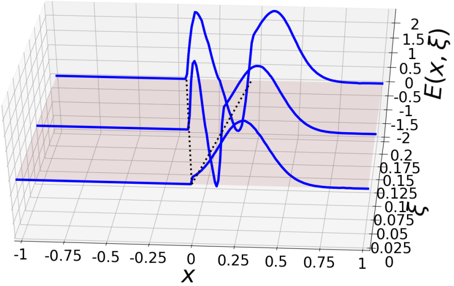

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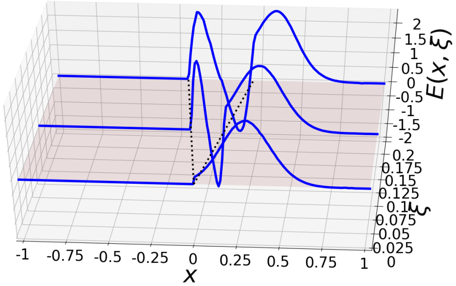

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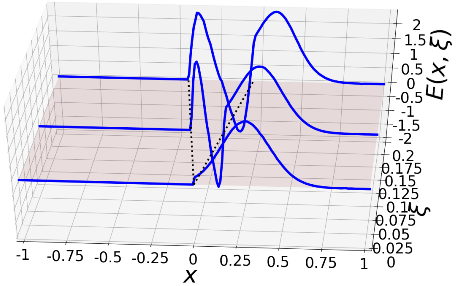

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- **Work in progress** with **PARTONS**...

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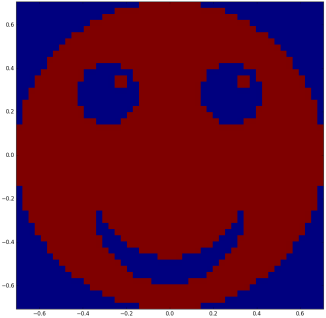

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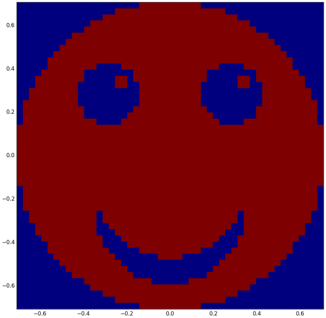

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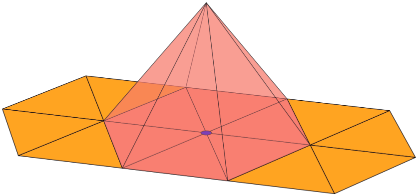
Discretization

- Expansion of the DD into basis functions $\{v_j\}$:

$$h(\beta, \alpha) = \sum_j h_j \, v_j(\beta, \alpha) \, , \tag{15}$$

  - Piece-wise constant, **piece-wise linear**, etc.
  - n columns of the matrix.
- Sampling:
  - Random couples $(x, \xi) \longrightarrow m \geq n$ lines of the matrix.
- Linear problem: $A\,X = B$ where $B_i = H(x_i, \xi_i)$ and $A_{ij} = \mathcal{R} v_j(x_i, \xi_i)$.
- Regularization necessary: discrete ill-posed problem.
  - Trade-off between noise and convergence.

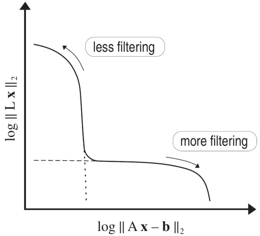


Ill-posed problems and Regularization

- Ill-posed problems?
  - For example the inversion of a Fredholm equation of the first kind:

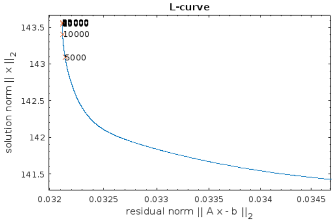
$$\int K(x,y)\, f(y)\, \mathrm{d}y = g(x)\,. \qquad (16)$$

  - The inverse is not continuous: an arbitrarily small variation Δg of the rhs can lead to an arbitrarily large variation Δf of the solution.
- The corresponding discrete problem needs to be regularized.
  - E.g Tikhonov regularization: $\min\left\{\|AX - B\|^2 + \epsilon\,\|X\|^2\right\}$.



Theoretical "L-curve": curve parameterized by the regularization factor.

(fig. taken from Ref. (Hansen, 2007))



L-curve with the iteration number as regularization factor.

D-term considerations

- Polynomiality property:

$$\int_{-1}^{1} \mathrm{d}x\, x^m H(x,\xi,t) = \sum_{\substack{k=0 \\ k \text{ even}}}^{m+1} c_k^{(m)}(t)\xi^k \, . \tag{17}$$

- Recast polynomiality property for $H - D$:

$$\int_{-1}^{1} \mathrm{d}x\, x^m \left(H(x,\xi,t) - D\left(\frac{x}{\xi}, t\right)\right) = \sum_{\substack{k=0 \\ k \text{ even}}}^{m} c_k^{(m)}(t)\xi^k \, , \tag{18}$$

where $D\left(\frac{x}{\xi}, t\right)$ is the so-called D-term with support on $-\xi < x < \xi$.

- $H - D$ is a Radon Transform:

$$H(x,\xi,t) - D\left(\frac{x}{\xi}, t\right) = \int_\Omega \mathrm{d}\beta\, \mathrm{d}\alpha\, h_{\mathrm{PW}}(\beta,\alpha)\, \delta(x - \beta - \alpha\xi) \, . \tag{19}$$

  - **The DGLAP region gives no information on the D-term.**

- With other DD representations, we can generate intrinsic D-terms, e.g. Pobylitsa representation:

$$H(x,\xi,t) = (1-x)\int_\Omega \mathrm{d}\beta\, \mathrm{d}\alpha\, h_{\mathrm{P}}(\beta,\alpha)\, \delta(x - \beta - \alpha\xi) \, . \tag{20}$$

  - **Still freedom of extra D-term.**

Some examples (Regge behavior)

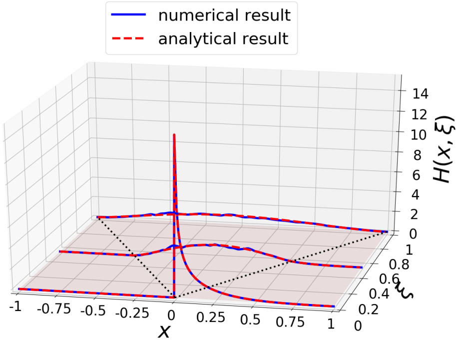


Figure: Extension of GPD for a nucleon toy model with Regge behavior. Plot for fixed ξ values 0, 0.5 and 1.

- Integrable singularity for the GPD at $x \sim 0$: $H(x,\xi) \propto \frac{1}{\sqrt{x}}$.
- Equivalent to an integrable singularity for the DD at $\beta \sim 0$: $h(\beta,\alpha) \propto \frac{1}{\sqrt{\beta}}$.
  - We solve for $\sqrt{\beta}\, h(\beta,\alpha)$ instead of $h(\beta,\alpha)$!