Towards the $N^3$LO evolution of parton distributions

Takahiro Ueda

Seikei University

S. Moch, B. Ruijl, TU, J.A.M. Vermaseren, A. Vogt

Hamburg now ETH Nikhef Liverpool


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Kobe Convention Center
DIS 2018
This talk is about 4-loop splitting functions governing evolution of N^3LO parton distribution functions

DGLAP equation:

\[
\frac{d}{d \ln \mu^2} f_i(x, \mu^2) = \sum_j \left[ P_{ij}(a_s(\mu^2)) \otimes f_j(\mu^2) \right](x)
\]

\[
P_{ij} = a_s P_{ij}^{(0)} + a_s^2 P_{ij}^{(1)} + a_s^3 P_{ij}^{(2)} + a_s^4 P_{ij}^{(3)} + ...
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Do we need this order?
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Do we need this order? Yes!
Precision physics at the LHC

No signals beyond the SM
Breakthrough would come from precision physics(?)

NNLO QCD corrections calculated for many processes

Even $N^3$LO, e.g., inclusive $gg \rightarrow H$

Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Lazopoulos, Mistlberger ’16

$N^3$LO inclusive DIS: Moch, Vermaseren, Vogt ’05; for $F_3$ ’08
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Missing $N^3$LO PDFs

$N^3$LO $gg \rightarrow H$ computed with NNLO PDFs

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Anastasiou et al. ’16

Ideally, $N^3$LO analyses must be performed with PDFs fitted at the $N^3$LO accuracy.

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\( N \)-th Mellin moment of the splitting function \( P_{ab}(x) \)

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\gamma_{ab}(N) = - \int_0^1 dx \, x^{N-1} P_{ab}(x)
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can be computed by (poles of) massless propagator-type integrals with \( N \)-dependence (details later)

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\sum Q \quad N
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Full \( x \)-dependence can be obtained if full \( N \)-dependence is computed

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Fix $N = 2, 4, 6, ...$
→ Just massless propagator-type integrals

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Computable at 4-loops

From values at fixed $N$
we could get approximation
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3-loop: exact vs. approx.
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Analytically performs 4-loop massless propagator-type integrals

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cf. Mincer (3-loop): Gorishnii et al. ’89; Larin, Tkachov, Vermaseren ’91

Example:

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\int d^Dp_1 d^Dp_2 d^Dp_3 d^Dp_4 \frac{(2Q \cdot p_2)^{-n_{12}}(2p_1 \cdot p_4)^{-n_{13}}(2Q \cdot p_3)^{-n_{14}}}{(p_1)^{n_{11}}(p_3)^{n_{12}}}
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Performance: 4-loop QCD \(\beta\)-function < 3 min.

(background field method, \(\xi = 0\), modern 32 core machine)
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Harmonic projection to probe-parton forward scattering

\[ Q^\{\mu_1 \ldots \mu_N\} \frac{1}{N!} \left[ \frac{\partial^N}{\partial P^{\mu_1} \ldots \partial P^{\mu_N}} \right] \]

Pros:
Conceptually simple (computing physical amplitudes). Indeed, 3-loop splitting functions were computed in this way.

Cons:
Quickly becomes hard as \( N \) increases.

Gorishnii, Larin, Tkachev ’83; Gorishnii, Larin ’87
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\[
\frac{Q^{\{\mu_1 \ldots \mu_N\}}}{N!} \frac{\partial^N}{\partial P^{\mu_1} \ldots \partial P^{\mu_N}}
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\(Q^2 \neq 0 \quad P=0\)

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Matrix elements of twist-2 DIS operators

\[ P P P \]

Pros:
Milder increase of complexity as \( N \) increases

Cons:
Complicated Feynman rules, \((L + 2)\)-point vertices

\[ G_\alpha^{\mu_1} D_{\mu_2} \ldots D_{\mu_{N-1}} G_{\mu_N}^\alpha \sim \]

Complicated renormalization: mixing of gauge-invariant operators and gauge-variant operators in the singlet sector

Hamberg, van Neerven ’91
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Strategy

Compute $N$-th moments by Method I (forward scattering) as much as possible (Easy to debug)

Computed up to $N \leq 6$ at 4-loops in general
Confirmed known low-$N$ non-singlet results ($N \leq 4$)
   Velizanin ’11 ’14; Baikov, Chetyrkin, Kühn, Rittinger

Up to $N > 40$ for high-$n_f$ parts (DIS 2017 talk by Josha Davies)
Enough to reconstruct high-$n_f$ parts

Switch to Method II (operator matrix element)
Debug it with low-$N$ results obtained by Method I

New: Computed up to $N \leq 16$ for non-singlet
   Up to $N = 20$ for large-$n_c$ parts
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Reconstructing full-\(N\) expression

Up to \(N = 20\) for non-singlet, large-\(n_c\)

\[\gamma_{NS}^\pm(N): (q_i \pm \bar{q}_i) - (q_k \pm \bar{q}_k)\]

Ansatz: if analogous to lower orders

\[\gamma_{NS}^{(n)}(N) = \sum_{w=0}^{2n+1} c_{0ow} S_w(N) + \sum_{a} \sum_{k=1}^{2n+1-\frac{k}{2}} \sum_{w=0}^{2n+1-k} c_{akw} \frac{S_w(N)}{(N + a)^k}\]

\(\gamma_{NS}\): constrained by ‘self-tuning’ (conjecture, conformal symmetry)

\(\gamma_{NS}^+ = \gamma_{NS}^-\) for large-\(n_c\)

Large-\(N\) and small-\(x\) limits

Small-\(x\) ressumation: known coefficients

\(N \leq 18\) Diophantine eqs. to fix remaining coefficients

Checked by \(N = 19, 20\)
Reconstructing full-$N$ expression

Up to $N = 20$ for non-singlet, large-$n_c$

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Small-$x$ ressumation: known coefficients

$N \leq 18$ Diophantine eqs. to fix remaining coefficients

Checked by $N = 19, 20$
Reconstructing full-$N$ expression

Up to $N = 20$ for non-singlet, large-$n_c$

\[ \gamma_{NS}^{\pm}(N) : (q_i \pm \bar{q}_i) - (q_k \pm \bar{q}_k) \]

Ansatz: if analogous to lower orders

\[ \gamma_{NS}^{(n)}(N) = \sum_{w=0}^{2n+1} c_{OOW} S_w(N) + \sum_{a} \sum_{k=1}^{2n+1} \sum_{w=0}^{2n+1-k} c_{akw} \frac{S_w(N)}{(N + a)^k} \]

\[ \gamma_{NS} : \text{constrained by ‘self-tuning’ (conjecture, conformal symmetry)} \]
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Checked by $N = 19, 20$
Approximation for large-$n_c$ suppressed terms

Remaining large-$n_c$ suppressed terms (non-singlet)
90 resulting trial functions, parameters fixed from the first 8 moments, two representatives chosen that indicate the remaining uncertainty

Checked by the 9th moment, e.g., $P_{N,1}^{(3)} (N = 18)$:
$195.8888792_B \ < \ 195.8888857...$ exact $\ < \ 195.8888968_A$
NS splitting functions: NNLO vs $N^3$LO

\[ P = \alpha_s p^{(0)} + \alpha_s^2 p^{(1)} + \alpha_s^3 p^{(2)} + \alpha_s^4 p^{(3)} + \ldots \]
$q_{NS}^{\pm}$ evolution: NNLO vs $N^3$LO

Logarithmic derivatives w.r.t. the factorization scale

$$\dot{q} \equiv \frac{d \ln q}{d \ln \mu_f^2}$$

Reference point: $x q(x, \mu_0^2) = x^{0.5}(1 - x)^3$ with $\alpha_s(\mu_0^2) = 0.2$
$q_{NS}$ evolution: renormalization scale dependence

\[ \dot{q} \equiv \frac{d \ln q}{d \ln \mu_f^2} \]

$N^3\text{LO}$: stable
scale uncertainty below 1\% for $\mu_f/2 \leq \mu_r \leq 2\mu_f$
Status of singlet case

So far, up to \( N = 8 \) for \( \gamma_{qq}, \gamma_{gq} \) (yesterday),
up to \( N = 6 \) for \( \gamma_{qg}, \gamma_{gg} \) by forward scattering
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Preliminary
Summary and outlook

Precision physics at the LHC requires 4-loop splitting functions

\( \gamma_{NS}^{(3)}(N) \) reconstructed at large-\( n_c \) limit
Remaining large-\( n_c \) suppressed terms approximated
Approximate results for \( P_{NS}^{(3)}(x) \) enough for phenomenology

For more details (valence dist., fragmentation funcs., etc) see JHEP 1710 (2017) 041 [arXiv:1707.08315]

Singlet part: work in progress
Promissing renormalization scale stability
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