## Relativistic hydrodynamics with spin

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based on recent works with B. Friman, A. Jaiswal, R. Ryblewski, and E. Speranza PRC97 (2018) 041901, arXiv:1712.07676 (nucl-th)

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## Introduction \& Motivation

- Non-central heavy-ion collisions create fireballs with large global angular momenta which may generate a spin polarization of the hot and dense matter in a way similar to the Einstein-de Haas and Barnett effects
- Much effort has recently been invested in studies of polarization and spin dynamics of particles produced in high-energy nuclear collisions, both from the experimental and theoretical point of view
L. Adamczyk et al. (STAR), (2017), Nature 548 (2017) 62-65, arXiv: 1701.06657 (nucl-ex) Global $\wedge$ hyperon polarization in nuclear collisions: evidence for the most vortical fluid www.sciencenews.org/article/smashing-gold-ions-creates-most-swirly-fluid-ever



## Global thermodynamic equilibrium (Zubarev, Becattini)

Density operator for any quantum mechanical system

$$
\exp (-E / T) \rightarrow \hat{\rho}(t)=\exp \left[-\int d^{3} \Sigma_{\mu}(x)\left(\hat{T}^{\mu \nu}(x) b_{\nu}(x)-\frac{1}{2} \hat{\jmath}^{\mu, \alpha \beta}(x) \omega_{\alpha \beta}(x)\right)\right]
$$

$\hat{T}^{\mu \nu}(x)$ - energy-momentum tensor, $\hat{J}^{\mu}, \alpha \beta(x)$ - angular-momentum $b_{\nu}(x), \omega_{\alpha \beta}(x)$ - Lagrange multipliers (originally ten independent functions)
$d^{3} \Sigma_{\mu}$ is an element of a space-like, 3-dimensional hypersurface $\Sigma_{\mu}$
we can take, for example, $d^{3} \Sigma_{\mu}=(d V, 0,0,0)$
in global equilibrium $\hat{\rho}(t)$ should be independent of time

$$
\partial_{\mu}\left(\hat{T}^{\mu v}(x) b_{v}(x)-\frac{1}{2} \hat{\jmath}^{\mu, \alpha \beta}(x) \omega_{\alpha \beta}(x)\right)=\hat{T}^{\mu v}(x)\left(\partial_{\mu} b_{v}(x)\right)-\frac{1}{2} \hat{J}^{\mu, \alpha \beta}(x)\left(\partial_{\mu} \omega_{\alpha \beta}(x)\right)=0
$$

for asymmetric energy-momentum tensor:

$$
b_{v}=\text { const. }, \quad \omega_{\alpha \beta}=\text { const. }
$$

for symmetric energy-momentum tensor:

$$
b_{v}=b_{v}^{0}+\omega_{v \gamma}^{0} x^{\gamma} \quad b_{v}^{0}, \omega_{v \gamma}^{0}, \omega_{\alpha \beta}=\text { const. }
$$

## Global thermodynamic equilibrium (Zubarev, Becattini)

asymmetric $\hat{T}^{\mu \nu}(x)$ : splitting angular momentum into its orbital and spin part

$$
\begin{aligned}
\hat{\rho}_{\mathrm{E} \mathcal{G}} & =\exp \left[-\int d^{3} \Sigma_{\mu}(x)\left(\hat{T}^{\mu \nu}(x) b_{v}-\frac{1}{2}\left(x^{\alpha} \hat{T^{\mu \beta}}(x)-x^{\beta} \hat{T}^{\mu \alpha}+\hat{S}^{\mu, \alpha \beta}(x)\right) \omega_{\alpha \beta}\right)\right] \\
& =\exp \left[-\int d^{3} \Sigma_{\mu}(x)\left(\hat{T}^{\mu v}(x)\left(b_{v}+\omega_{v \alpha} x^{\alpha}\right)-\frac{1}{2} \hat{S}^{\mu, \alpha \beta}(x) \omega_{\alpha \beta}\right)\right]
\end{aligned}
$$

Introducing the notation

$$
\beta_{v}=b_{v}+\omega_{v \alpha} x^{\alpha}, \quad \omega_{v \alpha}=-\omega_{\alpha v}
$$

we may write

$$
\rho_{\mathrm{E} G}=\exp \left[-\int d^{3} \Sigma_{\mu}(x)\left(\hat{T}^{\mu \nu}(x) \beta_{v}(x)-\frac{1}{2} \hat{S}^{\mu, \alpha \beta}(x) \omega_{\alpha \beta}\right)\right]
$$

We note that $\beta_{v}$ is the Killing vector satisfying the equation $\partial_{\mu} \beta_{v}+\partial_{\nu} \beta_{\mu}=0$

$$
\text { (thermal vorticity) } \omega_{\mu v}=-\frac{1}{2}\left(\partial_{\mu} \beta_{v}-\partial_{\nu} \beta_{\mu}\right)=\text { const, } \quad \omega_{\mu v}=\omega_{\mu v}
$$

symmetric $\hat{T}^{\mu \nu}(x)$ :

$$
\omega_{\mu v}=-\frac{1}{2}\left(\partial_{\mu} \beta_{v}-\partial_{v} \beta_{\mu}\right)=\text { const }, \quad \omega_{\mu v}=\text { const }, \quad \omega_{\mu v} \neq \omega_{\mu v}
$$

## GENERAL STRATEGY FOR HYDRO WITH SPIN

## PRESENT PHENOMENOLOGY PRESCRIPTION USED TO DESCRIBE THE DATA:

1) Run any type of hydro, perfect or viscous, or transport, or whatsoever, without spin
2) Find $\beta_{\mu}(x)=u_{\mu}(x) / T(x)$ on the freeze-out hypersurface (defined often by the condition $T=$ const)
3) Calculate thermal vorticity $\omega_{\alpha \beta}(x) \neq$ const
4) Identify thermal vorticity with the spin polarization tensor $\omega_{\mu v}$
5) Make predictions about spin polarization

SUCH A METHOD WORKS WELL, DESCRIBES THE DATA, BUT...CAN WE TAKE IT FOR GRANTED?

## GENERAL STRATEGY FOR HYDRO WITH SPIN

## THIS TALK:

1) in local equilibrium thermal vorticity and spin polarization tensor are independent $\beta_{\mu}(x)$ and $\omega_{\mu v}(x)$ continue their independent lives
2) eventually, they may become related if the system reaches global equilibrium
3) freedom in the initial conditions should be realized by independent $\beta_{\mu}\left(t_{0}, \boldsymbol{x}\right)$ and $\omega_{\mu v}\left(t_{0}, \boldsymbol{x}\right)$, there must be a (relativistic) delay in the coupling between vorticity and polarization, like shear stress/flow tensors
4) spin polarization may be an early-stage effect that survives the whole evolution
global equilibrium: $\beta_{\mu}$ is the Killing vector, $\omega_{\mu \nu}=\omega_{\mu \nu}=$ const
extended global equilibrium: $\beta_{\mu}$ is the Killing vector, $\omega_{\mu \nu}=$ const, $\omega_{\mu \nu}=$ const, $\omega_{\mu \nu} \neq \omega_{\mu \nu}$
local equilibrium: $\beta_{\mu}$ is not the Killing vector, $\omega_{\mu v}(x)=\omega_{\mu \nu}(\mathrm{x})$
extended local equilibrium: $\beta_{\mu}$ is not the Killing vector, $\omega_{\mu v}(x) \neq \omega_{\mu v}(x)$

## Local distribution functions

Our starting point is the phase-space distribution functions for spin-1/2 particles generalized from scalar functions to two by two spin density matrices for each value of the space-time position $x$ and momentum p, F. Becattini et al., Annals Phys. 338 (2013) 32

$$
f_{r s}^{+}(x, p)=\frac{1}{2 m} \bar{u}_{r}(p) X^{+} u_{s}(p), \quad f_{r s}^{-}(x, p)=-\frac{1}{2 m} \bar{v}_{s}(p) X^{-} v_{r}(p)
$$

Following the notation used by F. Becattini et al., we introduce the matrices

$$
X^{ \pm}=\exp \left[ \pm \xi(x)-\beta_{\mu}(x) p^{\mu}\right] M^{ \pm}
$$

where

$$
M^{ \pm}=\exp \left[ \pm \frac{1}{2} \omega_{\mu v}(x) \hat{\Sigma}^{\mu v}\right]
$$

Here we use the notation $\beta^{\mu}=u^{\mu} / T$ and $\xi=\mu / T$, with the temperature $T$, chemical potential $\mu$ and four velocity $u^{\mu}$. The latter is normalized to $u^{2}=1$. Moreover, $\omega_{\mu \nu}$ is the spin polarization tensor, while $\hat{\Sigma}^{\mu v}$ is the spin operator expressed in terms of the Dirac gamma matrices, $\hat{\Sigma}^{\mu \nu}=(i / 4)\left[\gamma^{\mu}, \gamma^{\nu}\right]$.

## Spin-polarization tensor

$$
\begin{gathered}
\omega_{\mu v} \equiv k_{\mu} u_{v}-k_{\nu} u_{\mu}+\epsilon_{\mu v \beta \gamma} u^{\beta} \omega^{\gamma}, \quad(k \cdot u=\omega \cdot u=0) \\
k_{\mu}=\omega_{\mu \nu} u^{v}, \quad \omega_{\mu}=\frac{1}{2} \epsilon_{\mu v \alpha \beta} \omega^{v \alpha} u^{\beta} \quad \text { electric- and magnetic-like components }
\end{gathered}
$$ dual spin polarization tensor $\tilde{\omega}_{\mu \nu}=\frac{1}{2} \epsilon_{\mu v \alpha \beta} \omega^{\alpha \beta}$

$$
\frac{1}{2} \omega_{\mu v} \omega^{\mu v}=k \cdot k-\omega \cdot \omega, \quad \frac{1}{2} \tilde{\omega}_{\mu v} \omega^{\mu v}=2 k \cdot \omega
$$

Using the conditions $k \cdot \omega=0$ and $k \cdot k-\omega \cdot \omega \geq 0$ we find the compact form with

$$
M^{ \pm}=\cosh (\zeta) \pm \frac{\sinh (\zeta)}{2 \zeta} \omega_{\mu v} \hat{\Sigma}^{\mu v}, \quad \zeta=\frac{1}{2} \sqrt{k \cdot k-\omega \cdot \omega}
$$

this allows for construction of a consistent thermodynamic and hydrodynamic framework
the conditions above can be relaxed, discussion of the equilibrium with acceleration E. Speranza, F. Becattini, WF, arXiv: 1803.11098

## Charge current

The charge current (S. de Groot, W. van Leeuwen, and C. van Weert)

$$
N^{\mu}=\int \frac{d^{3} p}{2(2 \pi)^{3} E_{p}} p^{\mu}\left[\operatorname{tr}_{4}\left(X^{+}\right)-\operatorname{tr}_{4}\left(X^{-}\right)\right]=n u^{\mu}
$$

where ' $\operatorname{tr}_{4}$ ' denotes the trace over spinor indices and $n$ is the charge density

$$
n=4 \cosh (\zeta) \sinh (\xi) n_{(0)}(T)=\left(e^{\zeta}+e^{-\zeta}\right)\left(e^{\xi}-e^{-\xi}\right) n_{(0)}(T)
$$

Here $n_{(0)}(T)=\langle(u \cdot p)\rangle_{0}$ is the number density of spin 0 , neutral Boltzmann particles, obtained using the thermal average

$$
\langle\cdots\rangle_{0} \equiv \int \frac{d^{3} p}{(2 \pi)^{3} E_{p}}(\cdots) e^{-\beta \cdot p}, \quad E_{p}=\sqrt{m^{2}+\mathbf{p}^{2}}
$$

simple thermodynamic interpretation
four species: particles and antiparticles with spin up and down

## Energy-momentum tensor tensor

The energy-momentum tensor S. de Groot, W. van Leeuwen, and C. van Weert; F. Becattini et al., Annals Phys. 338 (2013) 32

$$
T^{\mu v}=\int \frac{d^{3} p}{2(2 \pi)^{3} E_{p}} p^{\mu} p^{\nu}\left[\operatorname{tr}_{4}\left(X^{+}\right)+\operatorname{tr}_{4}\left(X^{-}\right)\right]=(\varepsilon+P) u^{\mu} u^{v}-P g^{\mu v}
$$

where the energy density and pressure are given by

$$
\varepsilon=4 \cosh (\zeta) \cosh (\xi) \varepsilon_{(0)}(T)
$$

and

$$
P=4 \cosh (\zeta) \cosh (\xi) P_{(0)}(T),
$$

respectively. In analogy to the density $n_{(0)}(T)$, we define the auxiliary quantities $\varepsilon_{(0)}(T)=\left\langle(u \cdot p)^{2}\right\rangle_{0}$ and $P_{(0)}(T)=-(1 / 3)\left\langle\left[p \cdot p-(u \cdot p)^{2}\right]\right\rangle_{0}$.
$T^{\mu \nu}$ is symmetric, expected for classical particles with $\boldsymbol{p}=\boldsymbol{v} E_{p}$, if $T^{\mu \nu}=\Delta p^{\mu} / \Delta \Sigma_{v}$ spin tensor is conserved in this case

## Entropy current

The entropy current is given by an obvious generalization of the Boltzmann expression

$$
S^{\mu}=-\int \frac{d^{3} p}{2(2 \pi)^{3} E_{p}} p^{\mu}\left(\operatorname{tr}_{4}\left[X^{+}\left(\ln X^{+}-1\right)\right]+\operatorname{tr}_{4}\left[X^{-}\left(\ln X^{-}-1\right)\right]\right)
$$

This leads to the following entropy density

$$
s=u_{\mu} S^{\mu}=\frac{\varepsilon+P-\mu n-\Omega w}{T}
$$

where $\Omega$ is defined through the relation $\zeta=\Omega / T$ and

$$
w=4 \sinh (\zeta) \cosh (\xi) n_{(0)}
$$

This suggests that $\Omega$ should be used as a thermodynamic variable of the grand canonical potential, in addition to $T$ and $\mu$. Taking the pressure $P$ to be a function of $T, \mu$ and $\Omega$, we find

$$
s=\left.\frac{\partial P}{\partial T}\right|_{\mu, \Omega}, \quad n=\left.\frac{\partial P}{\partial \mu}\right|_{T, \Omega}, \quad w=\left.\frac{\partial P}{\partial \Omega}\right|_{T, \mu} .
$$

## Basic conservation laws

The conservation of energy and momentum requires that $\partial_{\mu} T^{\mu \nu}=0$ This equation can be split into two parts, one longitudinal and the other transverse with respect to $u^{\mu}$ :

$$
\begin{aligned}
\partial_{\mu}\left[(\varepsilon+P) u^{\mu}\right] & =u^{\mu} \partial_{\mu} P \equiv \frac{d P}{d \tau} \\
(\varepsilon+P) \frac{d u^{\mu}}{d \tau} & =\left(g^{\mu \alpha}-u^{\mu} u^{\alpha}\right) \partial_{\alpha} P
\end{aligned}
$$

Evaluating the derivative on the left-hand side of the first equation we find

$$
T \partial_{\mu}\left(s u^{\mu}\right)+\mu \partial_{\mu}\left(n u^{\mu}\right)+\Omega \partial_{\mu}\left(w u^{\mu}\right)=0 .
$$

The middle term vanishes due to charge conservation,

$$
\partial_{\mu}\left(n u^{\mu}\right)=0 .
$$

Thus, in order to have entropy conserved in our system (for the perfect-fluid description we are aiming at), we demand that

$$
\partial_{\mu}\left(w u^{\mu}\right)=0 .
$$

Consequently, we self-consistently arrive at the conservation of entropy, $\partial_{\mu}\left(s u^{\mu}\right)=0$ Equations above form dynamic background for the spin dynamics.

## Spin dynamics

Since we use a symmetric form of the energy-momentum tensor $T^{\mu v}$, the spin tensor $S^{\lambda, \mu v}$ satisfies the conservation law,

$$
\partial_{\lambda} S^{\lambda, \mu v}=0 .
$$

For $S^{\lambda, \mu v}$ we use a phenomenological form

$$
S^{\lambda, \mu v}=\int \frac{d^{3} p}{2(2 \pi)^{3} E_{p}} p^{\lambda} \operatorname{tr}_{4}\left[\left(X^{+}-X^{-}\right) \hat{\Sigma}^{\mu v}\right]=\frac{w u^{\lambda}}{4 \zeta} \omega^{\mu v}
$$

Using the conservation law for the spin density and introducing the rescaled spin tensor $\bar{\omega}^{\mu \nu}=\omega^{\mu \nu} /(2 \zeta)$, we obtain

$$
u^{\lambda} \partial_{\lambda} \bar{\omega}^{\mu v}=\frac{d \bar{\omega}^{\mu v}}{d \tau}=0
$$

with the normalization condition $\bar{\omega}_{\mu \nu} \bar{\omega}^{\mu \nu}=2$.

## TRANSPORT OF THE SPIN POLARIZATION DIRECTION ALONG THE FLUID STREAM LINES

## CHANGE OF THE POLARIZATION MAGNITUDE DESCRIBED BY THE BACKGROUND EQUATIONS

## Global equilibrium with rotation - stationary vortex 1

The hydrodynamic flow $u^{\mu}=\gamma(1, \boldsymbol{v})$ with the components ( $\Omega$ is a constant)

$$
u^{0}=\gamma, \quad u^{1}=-\gamma \tilde{\Omega} y, \quad u^{2}=\gamma \tilde{\Omega} x, \quad u^{3}=0,
$$

$\gamma=1 / \sqrt{1-\tilde{\Omega}^{2} r^{2}}, r$ - distance from the vortex centre in the transverse plane


$$
T=T_{0} \gamma, \quad \mu=\mu_{0} \gamma, \quad \Omega=\Omega_{0} \gamma,
$$

with $T_{0}, \mu_{0}$ and $\Omega_{0}$ being constants. One possibility is that the vortex represents an unpolarized fluid with $\omega_{\mu \nu}=0$ and thus, with $\Omega_{0}=0$.

## Global equilibrium with rotation - stationary vortex 2

Another possibility is that the particles in the fluid are polarized and $\Omega_{0} \neq 0$. In the latter case we expect that the spin tensor has the structure

$$
\omega_{\mu \nu}=\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & \tilde{\Omega} / T_{0} & 0 \\
0 & -\tilde{\Omega} / T_{0} & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right),
$$

where the parameter $T_{0}$ has been introduced to keep $\omega_{\mu \nu}$ dimensionless. This form yields $k_{\mu}=\tilde{\Omega}^{2}\left(\gamma / T_{0}\right)(0, x, y, 0)$ and $\omega_{\mu}=\tilde{\Omega}\left(\gamma / T_{0}\right)(0,0,0,1)$. As a consequence, we find $\zeta=\tilde{\Omega} /\left(2 T_{0}\right)$, which, for consistency with the hydrodynamic background equations, implies

$$
\tilde{\Omega}=2 \Omega_{0} .
$$

In this case the spin polarization tensor agrees with the thermal vorticity, namely

$$
\omega_{\mu v}=-\frac{1}{2}\left(\partial_{\mu} \beta_{v}-\partial_{v} \beta_{\mu}\right)=\omega_{\mu v}
$$

as emphasised in the works by Becattini and collaborators.

## Expanding vortex

What can happen if the external boundary is removed? Expansion into external vacuum.


Stream lines and temperature (color gradient), $T_{0}=200 \mathrm{MeV}, m=1 \mathrm{GeV}$

## Quasi-realistic model for low-energy collisions 1

## Initial gaussian temperature profile

$$
T_{\mathrm{i}}=T_{0} \exp \left(-\frac{x^{2}}{2 x_{0}^{2}}-\frac{y^{2}}{2 y_{0}^{2}}-\frac{z^{2}}{2 z_{0}^{2}}\right)
$$

$x_{0}=1$ (beam direction, one can possibly use the Landau model)
$y_{0}=2.6$ and $z_{0}=2$ (from GLISSANDO version of the Glauber Model, Au+Au, 20-30\%)
Initial flow profile

$$
\tilde{\Omega} \rightarrow \frac{1}{r} \tanh \frac{r}{r_{0}}, \quad v_{x}=-\frac{y}{r} \tanh \frac{r}{r_{0}}, \quad v_{y}=\frac{x}{r} \tanh \frac{r}{r_{0}}
$$

the parameter $r_{0}$ controls the magnitude of the initial angular velocity, in this talk $r_{0}=1.0$

## Quasi-realistic model for low-energy collisions 2



Figure: Initial conditions for the quasi-realistic model

## Quasi-realistic model for low-energy collisions 3



## Conclusions and Summary

We have introduced a hydrodynamic framework, which includes the evolution of the spin density in a consistent fashion. Equations that determine the dynamics of the system follow solely from conservation laws - minimal extension of the well established perfect-fluid picture.

Our framework can be used to determine the space-time dynamics of fluid variables, now including also the spin tensor, from initial conditions defined on an initial space-like hypersurface. This property makes them useful for practical applications in studies of polarization evolution in high-energy nuclear collisions and also in other physics systems exhibiting fluid-like, collective dynamics connected with non-trivial polarization phenomena.

The possibility to study the dynamics of systems in local thermodynamic equilibrium represents an important advance compared to studies, where global equilibrium was assumed.

Next steps: spin-orbit interactions, asymmetric $T_{\mu v}$, dissipation, ...

## Torrieri/Tinti:

$$
\zeta u^{\lambda} \partial_{\lambda} \bar{\omega}^{\mu \nu}=\zeta \frac{d \bar{\omega}^{\mu \nu}}{d \tau}=-\frac{\omega_{\mu v}-\omega_{\mu v}}{\tau_{\text {rel }}}
$$

consistent incorporation of the relaxation of the spin polarisation towards thermal vorticity

