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# Non-Equilibrium Quantum Transport of Chiral Fluids from Kinetic Theory

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Yoshimasa Hidaka, Shi Pu, DY, arXiv:1612.04630, PRD 95 (2017) no.9, 091901  
Yoshimasa Hidaka, Shi Pu, DY, arXiv:1710.00278, PRD 97 (2018) no.1, 016004  
Yoshimasa Hidaka, DY, arXiv:1801.08253

# Chiral kinetic theory

- Chiral kinetic theory (CKT): kinetic theory + chiral anomaly
- A useful tool to study anomalous transport “in and out of equilibrium” at “weak coupling”.
- Different derivations :
- ❖ The semi-classical (non-QFT) approach : classical action + Berry phase  $\mathcal{O}(\hbar)$

D. T. Son and N. Yamamoto, 12

M. Stephanov and Y. Yin, 12

$$\text{Berry curvature : } \Omega_p = \frac{\mathbf{p}}{2|\mathbf{p}|^3} \quad (\text{quantum correction})$$

➤ EOM :  $\begin{aligned} \dot{\mathbf{x}} &= \hat{\mathbf{p}} + \hbar \dot{\mathbf{p}} \times \Omega_p, \\ \dot{\mathbf{p}} &= \mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}. \end{aligned}$   $\Rightarrow \frac{d}{dt} f(x, \mathbf{p}) = (\partial_t + \dot{\mathbf{x}} \cdot \nabla_{\mathbf{x}} + \dot{\mathbf{p}} \cdot \nabla_{\mathbf{p}}) f = 0$

(on-shell) (collisionless)

- ❖ Not manifestly Lorentz covariance :
- $f$  is not a Lorentz scalar (frame dependent) J.-Y. Chen, et.al. 14
- Modified Lorentz transformation for  $f$  : side jumps J.-Y. Chen, D. T. Son, and M. Stephanov, 15
- ❖ QFT derivation : Wigner-function approach steady state & large  $\mu$  :  
J.-W. Chen, S. Pu, Q. Wang, and X.-N. Wang, 13
- Collisions? D. T. Son and N. Yamamoto, 13
- ❖ World-line formalism N. Mueller, R. Venugopalan, 16

# Outline

- CKT from QFT (Wigner-function approach,  $\mathcal{O}(\hbar)$ ) :
  - Side-jump terms : modified Lorentz (frame) transformation
  - Covariant CKT with background fields & collisions
    - Hidaka, Pu, DY, 16
- Applications : non(near)-equilibrium transport for chiral fluids
  - Wigner functions & anomalous transport in global/local equilibrium
  - Non-linear (2<sup>nd</sup>-order) responses for anomalous transport
    - Hidaka, Pu, DY, 17
    - Hidaka, DY, 18

(I will focus on R-handed fermions.)

# Wigner functions (WF)

- less (greater) propagators :

$$S^>(x, y) = \langle \psi(x) \mathcal{P} \mathcal{U}^\dagger(A_\mu, x, y) \psi^\dagger(y) \rangle$$

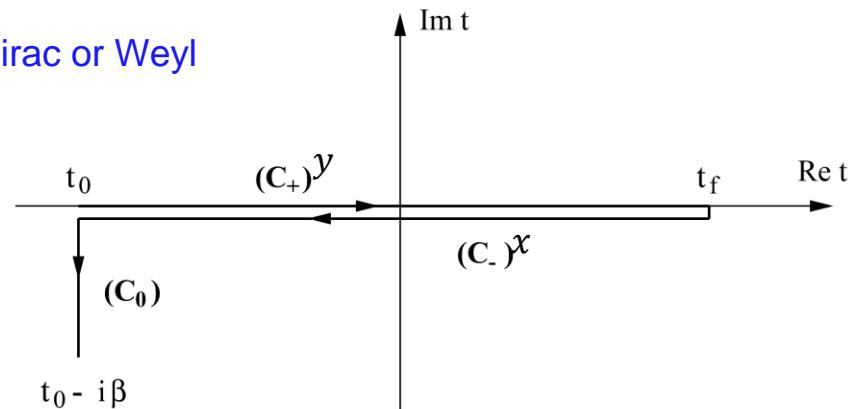
$$S^<(x, y) = \langle \psi^\dagger(y) \mathcal{P} \mathcal{U}(A_\mu, x, y) \psi(x) \rangle$$

gauge link

$$\downarrow$$

$$X = \frac{x+y}{2}, \quad Y = x - y$$

Dirac or Weyl



review : J. Blaizot, E. Iancu, Phys.Rept. 359 (2002) 355-528

Wigner functions :

$$\dot{S}^{<(>)}(q, X) = \int d^4Y e^{\frac{i q \cdot Y}{\hbar}} S^{<(>)} \left( X + \frac{Y}{2}, X - \frac{Y}{2} \right)$$

- Wigner functions are always covariant :  $J^\mu = \int \frac{d^4q}{(2\pi)^4} \text{tr} \left( \sigma^\mu \dot{S}^< \right)$
- Kadanoff-Baym-like equations up to  $\mathcal{O}(\hbar)$  : ( $q \gg \partial$  : weak fields)

$$\sigma^\mu \left( q_\mu + \frac{i\hbar}{2} \Delta_\mu \right) \dot{S}^< = \frac{i\hbar}{2} \left( \Sigma^< \dot{S}^> - \Sigma^> \dot{S}^< \right), \quad \Delta_\mu = \partial_\mu + F_{\nu\mu} \partial / \partial q_\nu$$

systematically include collisions  
(R-handed fermions)

$$\left( q_\mu - \frac{i\hbar}{2} \Delta_\mu \right) \dot{S}^< \sigma^\mu = -\frac{i\hbar}{2} \left( \dot{S}^> \Sigma^< - \dot{S}^< \Sigma^> \right).$$

# Quantum corrections for WF

- WF up to  $\mathcal{O}(\hbar)$  : Hidaka, Pu, DY, 16

key eq.

$$\dot{S}^{<\mu}(q, X) = 2\pi\bar{\epsilon}(q \cdot n) \left( q^\mu \delta(q^2) f_q^{(n)} + \hbar \delta(q^2) S_{(n)}^{\mu\nu} \mathcal{D}_\nu f_q^{(n)} + \hbar \epsilon^{\mu\nu\alpha\beta} q_\nu F_{\alpha\beta} \frac{\partial \delta(q^2)}{2\partial q^2} f_q^{(n)} \right)$$

$$\mathcal{D}_\beta f_q^{(n)} = \Delta_\beta f_q^{(n)} - \mathcal{C}_\beta, \quad C_\beta[f] = \Sigma_\beta^< \bar{f} - \Sigma_\beta^> f$$

$$\text{spin tensor : } S_{(n)}^{\mu\nu} = \frac{\epsilon^{\mu\nu\alpha\beta}}{2(q \cdot n)} q_\alpha n_\beta$$

CME in equilibrium

side-jump term :

magnetization current

CVE or non-equilibrium effects

$S_{(n)}^{\mu\nu} \partial_\nu f_q^{(n)}$  : break translational inv.

- Frame vector  $n^\mu$  : choice of the spin basis  $\sigma^0 = I$  ( $n^\mu = (1, 0)$ )  $\implies n_\mu \sigma^\mu = I$
- L.T. of phase-space coordinates is equivalent to inverse L.T. of frames.

$$q'^\mu = \Lambda^\mu_\nu q^\nu, \quad X'^\mu = \Lambda^\mu_\nu X^\nu. \leftrightarrow n'^\mu = (\Lambda^{-1})^\mu_\nu n^\nu$$

- The full WF has to be frame independent  $\implies f_q^{(n)}$  is frame dependent

$$(\Lambda^{-1})_\mu^\nu \dot{S}'_\nu^{<} (q', X') - \dot{S}_\mu^{<} (q, X) = 0$$

- The modified frame transformation :

(agrees with J.-Y. Chen, et.al. 14)

$$f_q^{(n')} = f_q^{(n)} + \frac{\hbar \epsilon^{\nu\mu\alpha\beta} q_\alpha n'_\beta n_\mu}{2(q \cdot n)(q \cdot n')} \mathcal{D}_\nu f_q^{(n)}$$

# Covariant CKT with collisions

- A general form of CKT (for  $n^\mu = n^\mu(X)$ ) :

$$\Delta_\mu \dot{S}^{<\mu} = \Sigma_\mu^{<} \dot{S}^{>\mu} - \Sigma_\mu^{>} \dot{S}^{<\mu}$$

Hidaka, Pu, DY, 17

( previous expression of CKT in e.g. Son & Yamamoto, 12 :  $n^\mu = (1, \mathbf{0})$  & onshell)

Hidaka, Pu, DY, 16

- Energy-momentum tensor and charged four current :

$$T^{\mu\nu} = \int \frac{d^4q}{(2\pi)^4} [q^\mu \dot{S}^{<\nu} + q^\nu \dot{S}^{<\mu}], \quad J^\mu = 2 \int \frac{d^4q}{(2\pi)^4} \dot{S}^{<\mu}.$$

# Global/local equilibrium

- Global equilibrium ( $T, \mu = \text{const. } E = 0$ ) such that  $\mathcal{C}[f] = 0$  :

equilibrium  $f$  for arbitrary  $n^\mu$  :

J.-Y. Chen, D. T. Son, and M. A. Stephanov, 15

$$f_q^{\text{eq}(n)} = (e^g + 1)^{-1}, \quad g = \left( \beta q \cdot u - \bar{\mu} + \frac{\hbar S_{(n)}^{\mu\nu}}{2} \partial_\mu (\beta u_\nu) \right)$$

spin-vorticity coupling  
 $u^\mu$ : fluid velocity

- Local equilibrium ( $T, \mu \neq \text{const.}$ ) :  $f$  can be nontrivially defined in  $n^\mu = u^\mu$ .

$$\Rightarrow f_q^{\text{eq}(u)} = \left( \exp \left[ \beta(q \cdot u - \mu) + \frac{\hbar q \cdot \omega}{2q \cdot u} \right] + 1 \right)^{-1}, \quad \omega^\mu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu (\partial_\alpha u_\beta). \quad \text{Hidaka, Pu, DY, 17}$$

- Near local equilibrium :

$$T^{\mu\nu} = u^\mu u^\nu \epsilon - p \Theta^{\mu\nu} + \Pi_{\text{non}}^{\mu\nu} + \Pi_{\text{dis}}^{\mu\nu}, \quad J^\mu = N_0 u^\mu + v_{\text{non}}^\mu + v_{\text{dis}}^\mu, \quad \Theta^{\mu\nu} = \eta^{\mu\nu} - u^\mu u^\nu$$

- Equilibrium anomalous transport :  $F_{\alpha\beta} = -\epsilon_{\mu\nu\alpha\beta} B^\mu u^\nu + u_\beta E_\alpha - u_\alpha E_\beta$

$$v_{\text{non}}^\mu = \hbar \sigma_B B^\mu + \hbar \sigma_\omega \omega^\mu \quad \Pi_{\text{non}}^{\mu\nu} = \hbar \xi_\omega (\omega^\mu u^\nu + \omega^\nu u^\mu) + \hbar \xi_B (B^\mu u^\nu + B^\nu u^\mu)$$

$$\sigma_\omega = \frac{T^2}{12} \left( 1 + \frac{3\bar{\mu}^2}{\pi^2} \right), \quad \sigma_B = \frac{\mu}{4\pi^2}, \quad \xi_\omega = \frac{T^3}{6} \left( \bar{\mu} + \frac{\bar{\mu}^3}{\pi^2} \right), \quad \xi_B = \frac{T^2}{24} \left( 1 + \frac{3\bar{\mu}^2}{\pi^2} \right).$$

(agree with different approaches,  
e.g. Son & Surowka, 09.  
K. Landsteiner, et.al. Lect. Notes,  
13.)

CVE

CME

Hidaka, Pu, DY, 17

# RT approximation & matching conditions

- 2<sup>nd</sup>-order anomalous responses : Hidaka, Pu, DY, 17 Hidaka, DY, 18
  - Solve CKT for  $f_q^{(u)} - f_q^{\text{eq}} = \delta f_q = \delta f_q^{(c)} + \hbar \delta f_q^{(Q)}$
  - RTA :  $\square(q, X) f_q^{(u)} = \mathcal{C}_{\text{full}}$ ,
- $$\mathcal{C}_{\text{full}} = q^\mu \mathcal{C}_\mu + \hbar \frac{S_{(u)}^{\mu\nu} E_\mu}{(q \cdot u)} \mathcal{C}_\nu + \hbar (\partial_\rho S_{(u)}^{\rho\mu}) \mathcal{C}_\mu \xrightarrow{\text{RTA}} \mathcal{C}_{\text{full}} = -\tau_R^{-1} (q \cdot u) \delta f_q$$
- Constrains from hydrodynamic EOM :  $\partial_\mu T^{\mu\nu} = F^{\nu\rho} J_\rho, \quad \partial_\mu J^\mu = \frac{\hbar}{4\pi^2} (\mathbf{E} \cdot \mathbf{B})$   
(to satisfy \*charge & energy conservation)  $\xrightarrow{} DT, D\mu, Du^\mu. \quad D = u \cdot \partial$
  - Matching conditions of RTA from CKT :

$$\partial_\mu J^\mu = -\frac{\hbar}{4\pi^2} E_\mu B^\mu - \frac{u_\mu \delta J^\mu}{\tau_R}, \quad \partial_\mu T^{\mu\nu} = F^{\nu\mu} J_\mu - \frac{u_\mu \delta T^{\mu\nu}}{\tau_R}.$$

\*charge & energy conservation  $\xrightarrow{} u_\mu \delta J^\mu = u_\mu \delta T^{\mu\nu} = 0$

No non-equilibrium corrections on charge density & energy-density current (same as standard KT)

# Non-equilibrium charged currents

- 2<sup>nd</sup>-order corrections on currents :  $V_{\perp}^{\mu} \equiv P_{\nu}^{\mu} V^{\nu}$     $P^{\mu\nu} \equiv \eta^{\mu\nu} - u^{\mu} u^{\nu}$

$J_{Q\perp}^{\mu}$	$\mathcal{O}(\partial)$	$\mathcal{O}(\partial^2)$
$\mathcal{O}(\hbar)$	$\sigma_B B^{\mu} + \sigma_{\omega} \omega^{\mu}$	$\tau_R [\epsilon^{\mu\nu\alpha\beta} u_{\nu} (\hat{\gamma}_E \partial_{\alpha} E_{\beta} + \hat{\gamma}_{\mu} E_{\alpha} \partial_{\beta} \mu + \hat{\gamma}_T E_{\alpha} \partial_{\beta} T + \hat{\gamma}_{T\mu} (\partial_{\alpha} T) (\partial_{\beta} \mu)) + \delta \hat{\sigma}_{BL} \theta B^{\mu} + \delta \hat{\sigma}_{BH} \pi^{\mu\nu} B_{\nu} + \delta \hat{\sigma}_{\omega L} \theta \omega^{\mu} + \delta \hat{\sigma}_{\omega H} \pi^{\mu\nu} \omega_{\nu}]$

$$\theta \equiv \partial \cdot u$$

$$\pi^{\mu\nu} \equiv P_{\rho}^{\mu} P_{\sigma}^{\nu} (\partial^{\rho} u^{\sigma} + \partial^{\sigma} u^{\rho} - 2\eta^{\rho\sigma} \theta / 3) / 2$$

key findings:

Hidaka, Pu, DY, 17  
Hidaka, DY, 18

Viscous corrections  
for CME/CVE

- 2<sup>nd</sup>-order quantum transport coefficients :  $\mathcal{P}, \mathcal{T}$  – odd

- The origin of viscous corrections : time-dep. B

$$\partial_{\nu} \tilde{F}^{\mu\nu} = 0$$

→ 
$$u \cdot \partial B^{\rho} + B^{\rho} \partial \cdot u - B \cdot \partial u^{\rho} + u^{\rho} B^{\mu} u \cdot \partial u_{\mu} + \epsilon^{\rho\mu\alpha\beta} (u_{\beta} \partial_{\mu} E_{\alpha} + u_{\mu} E_{\alpha} u \cdot \partial u_{\beta}) = 0$$

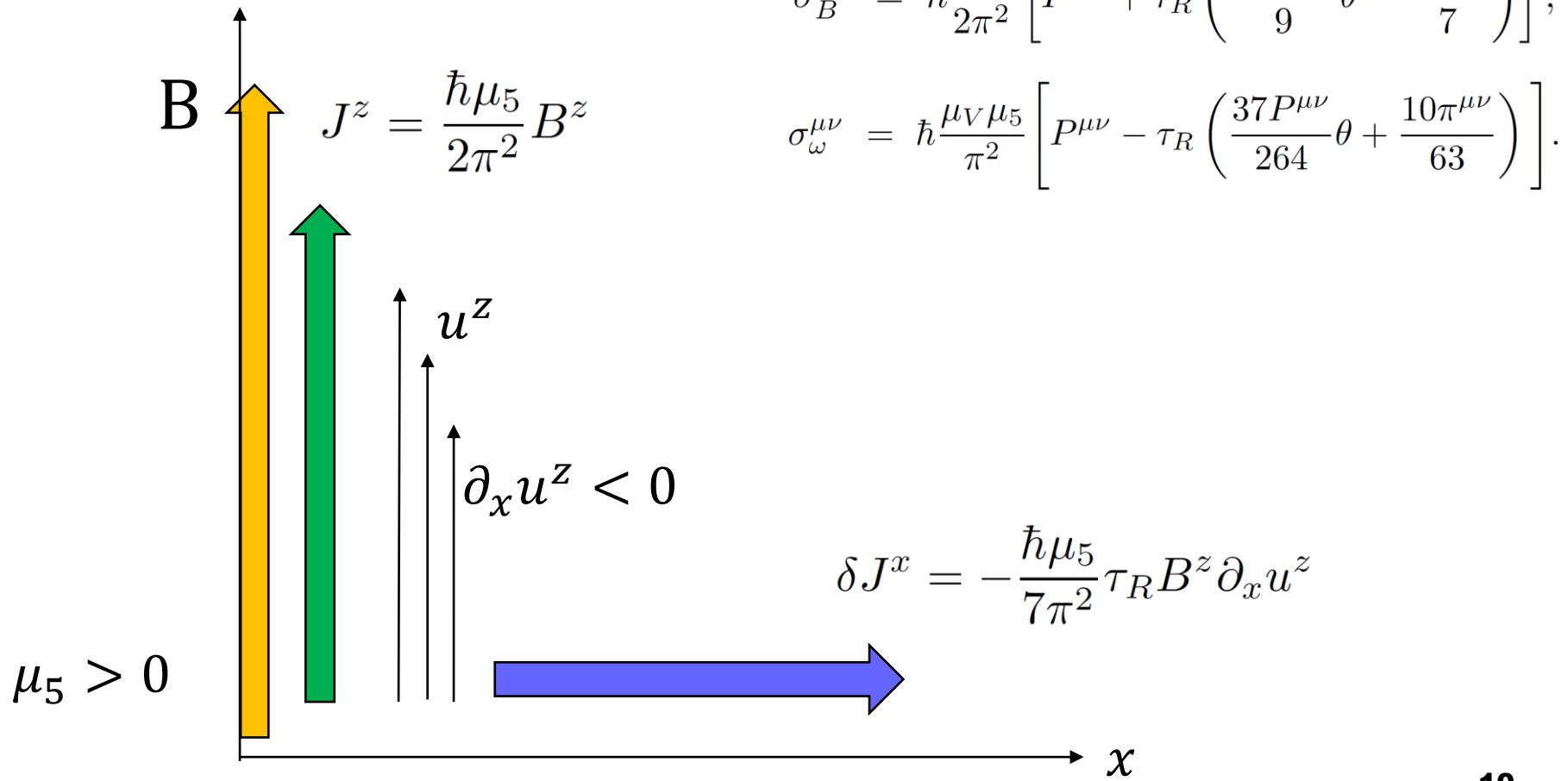
- Recall that AC conductivity of CME is int.-dep.

D. Kharzeev & H. Warringa, 09  
D. Satow & H. U. Yee, 14.  
D. Kharzeev, et.al., 17,

# High-temperature limit

- Viscous corrections on CME/CVE ( $\bar{\mu} \ll 1$ ) :  $J_{Q\perp}^\mu = \sigma_B^{\mu\nu} B_\nu + \sigma_\omega^{\mu\nu} \omega_\nu,$

Hidaka, DY, 18



# Conclusions & outlook

- We have presented a QFT-based derivation for CKT for Weyl fermions.
- In WF formalism, Lorentz covariance is manifested and both background fields and collisions are involved.
- Novel 2<sup>nd</sup> –order non-equilibrium anomalous transport including the viscous corrections on CME/CVE is found.
  
- Phenomenological applications : HIC, Weyl semimetals, etc.
- Realistic collisions beyond RTA
- Ongoing studies :
- Angular momenta (polarization) of chiral fluids.
- Theoretical issues : how to go beyond the weak-field limit?
- Landau-level (LL) WF : subjected to constant B (no Lorentz sym.).
- Perturbative sol. up to  $\mathcal{O}(\hbar^2)$  : complements the LL WF.

relevant work :

J.-H. Gao, Z.-T. Liang, Q. Wang, and X.-N. Wang, 18

Thank you!

# Non-equilibrium anomalous transport

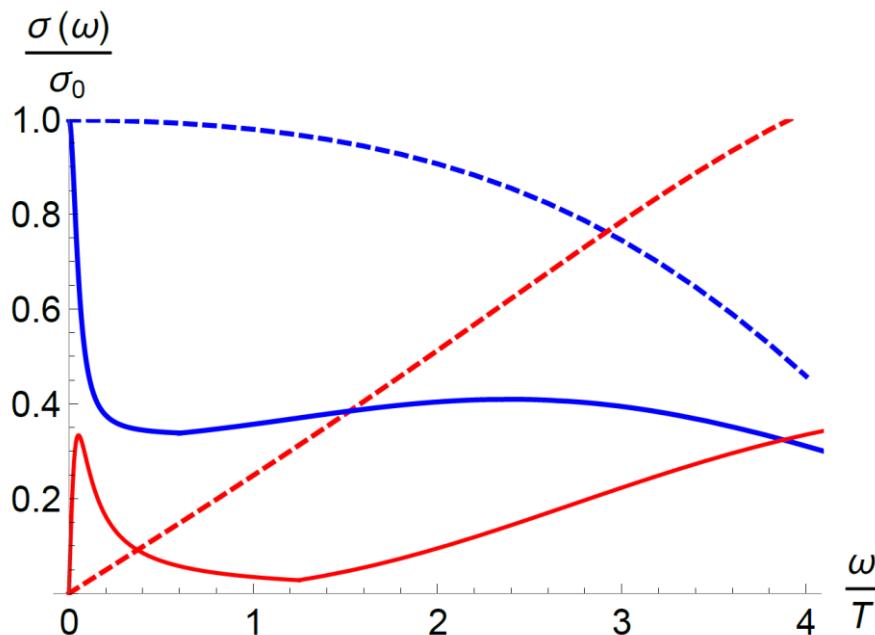
- CME in thermal equilibrium (BF) : independent of interactions
- Non-equilibrium cases : interactions could be involved
- e.g. AC conductivity of CME :

D. Kharzeev & H. Warringa, 09

D. Satow & H. U. Yee, 14.

D. Kharzeev, et.al., 17,

(time-dep. B field)



blue : real part

red : imaginary part

solid : kinetic theory

dashed : AdS/CFT

$$\sigma(\omega) = \sigma_0 \left( 1 - \frac{2}{3} \frac{\omega}{\omega + i\tau_R^{-1}} \right)$$

$$\tau_R^{-1} \approx 1.3 \alpha_s^2 \log(1/\alpha_s) T$$

(2 – flavor QCD)

# Solving WF perturbatively

- To derive anomalous(quantum) corrections : [Hidaka, Pu, DY, 16](#)
- work in Weyl bases ( $\grave{S}^< = \bar{\sigma}^\mu \grave{S}_\mu^<$ , for R-handed fermions).
- solve for the perturbative solution :  $\grave{S}_\mu^< = 2\pi q_\mu \delta(q^2) f + \hbar \delta S_\mu^<$
- usually choosing a basis for  $\sigma^0 = I$ .
- a more general choice :  $n_\mu \sigma^\mu = I$  with  $n^\mu$  being a frame vector.
- The frame vector can be regarded as the zeroth component of a vierbein.

Local coordinates :  $\sigma^a = (I, \sigma^i)$      $\longrightarrow$     Global spacetime :  $\sigma^\mu = e_a^\mu \sigma^a$

$$\bar{\sigma}^a = (I, -\sigma^i)$$

(local spin direction)

$$n^\mu \equiv e_0^\mu$$

flat :  $e_a^\mu(X) = \delta_a^\mu$

$$n^2 = 1$$

$n^\mu = (1, \mathbf{0})$

- Global spacetime coordinate transf.  $\longleftrightarrow$  Frame transf.  $n^\mu \rightarrow n'^\mu$
  - KB eq. :  $\mathcal{D} \cdot \grave{S}^< = 0, \quad q \cdot \grave{S}^< = 0,$   $\mathcal{D}_\beta f_q^{(n)} = \Delta_\beta f_q^{(n)} - \mathcal{C}_\beta,$
- give WF 
$$\begin{cases} 2\pi\delta(q^2)(q \cdot n\mathcal{D}_\mu - q_\mu n \cdot \mathcal{D})f = -2\epsilon_{\alpha\mu\nu\beta}n^\alpha q^\nu \delta\grave{S}^<\beta, & \Delta_\mu = \partial_\mu + F_{\nu\mu}\partial/\partial q_\nu \\ 2\pi\epsilon_{\alpha\mu\nu\beta}\delta(q^2)n^\alpha q^\nu \mathcal{D}^\beta f = 2\left(q \cdot n\delta\grave{S}_\mu^< - q_\mu n \cdot \delta\grave{S}^<\right). & \mathcal{C}_\beta[f] = \Sigma_\beta^<\bar{f} - \Sigma_\beta^>f \end{cases}$$

# Origin of side-jumps

- An alternative way to derive WF (no background fields & no collisions) with a “scalar  $f$ ”. [Hidaka, Pu, DY, 16](#)

- Nontrivial phase for massless particles with helicity: Helicity dep. phase

[S. Weinberg, QFT, Vol. I](#)

$$\text{L. T. : } v_+(\Lambda p) = e^{i\Phi(p, \Lambda)} U(\Lambda) v_+(p)$$

- Second quantization :  $\psi(x) = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2|\mathbf{p}|}} e^{-ip \cdot x} v_+(p) a_{\mathbf{p}}$  (neglect anti-fermions)

$$\hat{S}^<(x, y) = \langle \psi^\dagger(y) \psi(x) \rangle = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2|\mathbf{p}|}} \int \frac{d^3 p'}{(2\pi)^3 \sqrt{2|\mathbf{p}'|}} v_+(p) v_+^\dagger(p') \langle a_{\mathbf{p}'}^\dagger a_{\mathbf{p}} \rangle e^{i(p' - p) \cdot X - \frac{i}{2}(p' + p) \cdot Y}$$

→ Helicity dep. phase

- Under the L.T. :  $N(p', p) \equiv \langle a_{\mathbf{p}'}^\dagger a_{\mathbf{p}} \rangle \rightarrow e^{-i(\Phi(\Lambda, p) - \Phi(\Lambda, p'))} \langle a_{\mathbf{p}'}^\dagger a_{\mathbf{p}} \rangle$
- Could we define a scalar distribution function?
- ❖ Introduce a phase field :  $\phi(p) \rightarrow \phi'(\Lambda p) = \phi(p) - \Phi(p, \Lambda)$
- ❖ Reparametrize the wave function and annihilation operator :

$$v_+(p) \rightarrow e^{i\phi(p)} v_+(p)$$

$$a_{\mathbf{p}} \rightarrow e^{-i\phi(p)} a_{\mathbf{p}}$$

# Manifestation of Lorentz symmetry

- From  $a_{\mathbf{p}} \rightarrow e^{-i\phi(p)} a_{\mathbf{p}}$ , we may define a scalar distribution function :

scalar
non-scalar
Hidaka, Pu, DY, 16

$$\check{N}(p', p) \equiv e^{-i(\phi(p) - \phi(p'))} N(p', p) \quad \Rightarrow \quad \check{f}(q, X) \equiv \int \frac{d^3 \bar{p}}{(2\pi)^3} \check{N}\left(q - \frac{\bar{p}}{2}, q + \frac{\bar{p}}{2}\right) e^{-i\bar{p} \cdot X}$$

$$N(p', p) \equiv \langle a_{\mathbf{p}'}^\dagger a_{\mathbf{p}} \rangle$$

- The derivation of WF implicitly involves the contribution from anti-fermions.

Key eq. :  $\text{Im} \left[ c_\pm^\dagger(q) \sigma^k \frac{\partial}{\partial q_\beta} c_\pm(q) \right] = \mp a_\pm^\beta v^k - \frac{1}{2|\mathbf{q}|} \epsilon^{kj\beta} v_j$  from  $c_+(p)c_+^\dagger(p) + c_-(p)c_-^\dagger(p) = I$

$$\Rightarrow \dot{S}_\mu^<(q, X) = (2\pi)\theta(q^0)\delta(q^2) \left( q_\mu \left( 1 - \hbar(\partial_q^\nu \phi - a^\nu) \partial_\nu \right) + \hbar \delta_{\mu i} \epsilon_{ijk} \frac{q_j}{2|\mathbf{q}|} \partial_k \right) \check{f}(q, X),$$

covariant
scalar

- Compare to the previous expression :

$$\dot{S}^{<\mu} = 2\pi\theta(q^0)\delta(q^2) \left( q^\mu + \hbar \delta^{\mu i} \epsilon^{ijk} \frac{q_j}{2|\mathbf{q}|} \partial_k \right) f(q, X)$$

“the origin of side-jumps”

$$\Rightarrow f(q, X) = \check{f}(q_\mu, X^\mu - \hbar \partial_q^\mu \phi(q) + \hbar a^\mu)$$

non-scalar

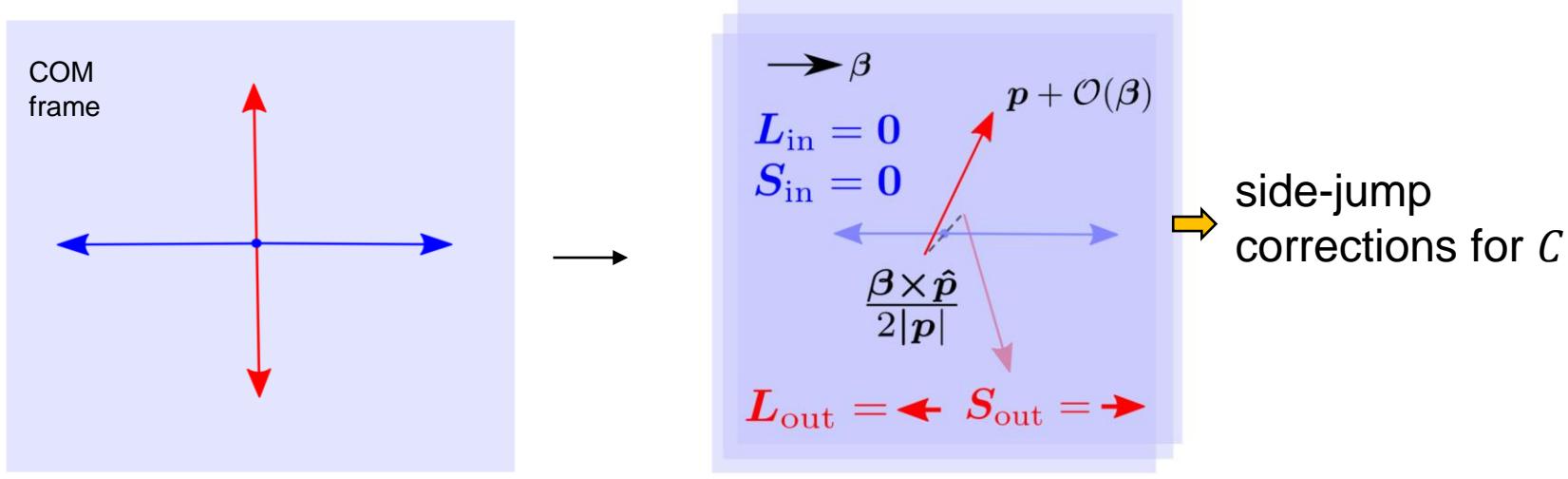
- Choices of phase field corresponds to the gauge degrees of freedom for the Berry connection.
- The perturbative solution could be uniquely determined by Lorentz symmetry.

# A no-jump frame

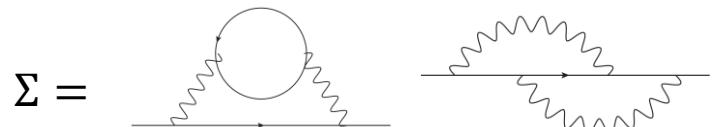
- Conservation of the angular momentum : COM frame = no-jump frame

J.-Y. Chen, D. T. Son, and M. A. Stephanov, 15

- A phenomenological argument for side jumps in collisions :



E.g. 2-2 Coulomb scattering ( $n_c^\mu = (q^\mu + q'^\mu)/\sqrt{s}$ ) :



$$q^\mu C_\mu [f^{(n_c)}] = \frac{1}{4} \int_{\mathbf{q}', \mathbf{k}, \mathbf{k}'} |\mathcal{M}|^2 \left[ \bar{f}^{(n_c)}(q) \bar{f}^{(n_c)}(q') f^{(n_c)}(k) f^{(n_c)}(k') - f^{(n_c)}(q) f^{(n_c)}(q') \bar{f}^{(n_c)}(k) \bar{f}^{(n_c)}(k') \right]$$

# The no-jump frame in 2-2 scattering

- Introducing a frame :  $\tilde{S}_\mu^< = 2\pi\delta(q^2) \left[ q_\mu f - \frac{\hbar}{2q \cdot u} \epsilon_{\mu\nu\alpha\beta} u^\nu q^\alpha \left( \partial^\beta f + \Sigma^{>\beta} f - \Sigma^{<\beta} \bar{f} \right) \right]$
- Conservation of the angular momentum : COM frame = no-jump frame

J.-Y. Chen, D. T. Son, and M. A. Stephanov, Phys. Rev. Lett. 115, 021601 (2015)

- Choosing the COM frame :  $u_c^\mu = (q+q')^\mu/\sqrt{s}$

$$\Sigma_\mu^< = \int_{q',k,k'} \mathcal{P}(q',k,k') \tilde{S}_\mu^>(q') (\tilde{S}^<(k) \cdot \tilde{S}^<(k')),$$

$$\mathcal{P}(q',k,k') = 4e^4 \left( \frac{1}{(q-k)^2} + \frac{1}{(q-k')^2} \right)^2,$$

$$\int_{\mathbf{q}',\mathbf{k},\mathbf{k}'} = \int \frac{d^3\mathbf{q}' d^3\mathbf{k} d^3\mathbf{k}'}{(2\pi)^5} \frac{\delta^{(4)}(q+q'-k-k')}{8E_{q'} E_k E_{k'}}.$$

$$\Rightarrow \tilde{S}^>\mu \Sigma_\mu^< = 2\pi\delta(q^2) \int_{\mathbf{q}',\mathbf{k},\mathbf{k}'} \mathcal{P}(q',k,k') (k \cdot k') (q \cdot q') \times \bar{f}^{(u_c)}(q) \bar{f}^{(u_c)}(q') f^{(u_c)}(k) f^{(u_c)}(k'),$$

- No “**explicit**”  $\mathcal{O}(\hbar)$  corrections in  $C_\mu$  :  $\partial_\mu \tilde{S}^{\mu<} = 2\pi\delta(q^2) q^\mu C_\mu [f^{(u_c)}]$ ,
- The final expression is concise but not pragmatic.
- $u_c$  is momentum-dependent : hard to write down  $f^{(u_c)}$  with different  $q'$

$$q^\mu C_\mu [f^{(u_c)}] = \frac{1}{4} \int_{\mathbf{q}',\mathbf{k},\mathbf{k}'} |\mathcal{M}|^2 \times [\bar{f}^{(u_c)}(q) \bar{f}^{(u_c)}(q') f^{(u_c)}(k) f^{(u_c)}(k') - f^{(u_c)}(q) f^{(u_c)}(q') \bar{f}^{(u_c)}(k) \bar{f}^{(u_c)}(k')]$$

no side-jumps

# Anomalous Hydrodynamics

- Anomalous hydro :  $(T, \bar{\mu} = \frac{\mu}{T}, u^\mu)$ : free parameters)

$$\text{E \& B : } n^\nu F_{\mu\nu} = E_\mu,$$

constraints :  $\partial_\mu T^{\mu\nu} = F^{\nu\rho} J_\rho, \quad \partial_\mu J^\mu = \frac{\hbar}{4\pi^2} (\mathbf{E} \cdot \mathbf{B})$

$$\frac{1}{2} \epsilon^{\mu\nu\alpha\beta} n_\nu F_{\alpha\beta} = B^\mu$$



in the local  
rest frame  
 $u^\mu \approx (1, \mathbf{0})$

$$\frac{\partial_0 T}{T} = \hbar \mathcal{E} \cdot (\tilde{T}_B \mathbf{B} + \tilde{T}_\omega \boldsymbol{\omega} T), \quad \partial_0 \bar{\mu} = \hbar \mathcal{E} \cdot (\tilde{\mu}_B \mathbf{B} + \tilde{\mu}_\omega \boldsymbol{\omega} T), \quad \mathcal{E}_\nu = E_\nu + T \partial_\nu \bar{\mu}$$

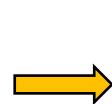
$$\partial_0 \mathbf{u} = \partial_0 \mathbf{u}^{(0)} + \hbar \partial_0 \delta \mathbf{u},$$

Hidaka, Pu, DY, 17

$$\partial_0 \mathbf{u}^{(0)} = -\frac{\nabla T}{T} + \frac{N_0 \mathcal{E}}{4p}, \quad \hbar \partial_0 \delta \mathbf{u} = \hbar \left( \tilde{U}_E \nabla \times \mathbf{E} + \tilde{U}_T \frac{\mathbf{E} \times \nabla T}{T} + \tilde{U}_{\bar{\mu}} \mathbf{E} \times \nabla \bar{\mu} \right)$$

- Free to choose a different frame in hydrodynamics :

e.g. Landau frame :  $\tilde{u}^\mu = u^\mu + \hbar \frac{\xi_\omega \omega^\mu + \xi_B B^\mu}{\epsilon + p},$



$$T_{\text{leq}}^{\mu\nu} = \tilde{u}^\mu \tilde{u}^\nu \epsilon - p \tilde{\Theta}^{\mu\nu},$$

$$J_{\text{leq}}^\mu = N_0 \tilde{u}^\mu + \hbar \tilde{\sigma}_\omega \omega^\mu + \hbar \tilde{\sigma}_B B^\mu,$$

$$\partial_0 \tilde{\mathbf{u}} = -\frac{\nabla T}{T} + \frac{N_0 \mathcal{E}}{4p} - \frac{\hbar \sigma_\omega}{8p} \boldsymbol{\omega} \times \mathbf{B}$$

(yields Chiral Alfvén waves)

# Non-equilibrium distribution functions

- Solving CKT for non-equilibrium fluctuations of  $f$  :  $f_q^{(u)} - f_q^{\text{eq}} = \delta f_q = \delta f_q^{(c)} + \hbar \delta f_q^{(Q)}$

$$\left[ q \cdot \Delta + \hbar \frac{S_{(u)}^{\mu\nu} E_\mu}{(q \cdot u)} \Delta_\nu + \hbar S_{(u)}^{\mu\nu} (\partial_\mu F_{\rho\nu}) \partial_q^\rho + \hbar (\hat{\Pi}_1^\mu + \hat{\Pi}_2^\mu) \Delta_\mu \right] f_q^{(u)} = \mathcal{C}_{\text{full}}, \quad \partial_\mu S_{(u)}^{\mu\nu} = \hat{\Pi}_1^\nu + \hat{\Pi}_2^\nu.$$

classical

quantum

- $\mathcal{C}_{\text{full}}$  :  $\mathcal{C}_{\text{full}} = q^\mu \mathcal{C}_\mu + \hbar \frac{S_{(u)}^{\mu\nu} E_\mu}{(q \cdot u)} \mathcal{C}_\nu + \hbar (\partial_\rho S_{(u)}^{\rho\mu}) \mathcal{C}_\mu \xrightarrow{\text{RTA}} \mathcal{C}_{\text{full}} = -\frac{1}{\tau_R} \left( q \cdot u + \hbar \frac{q^\mu \mathcal{A}_\mu}{(q \cdot u)^2} \right) \delta f_q$

$$\partial_\rho S_{(u)}^{\rho\mu} = \frac{1}{2} \left[ \omega^\mu - \frac{(q \cdot \omega) u^\mu}{q \cdot u} - \frac{(q \cdot \omega) q^\mu}{(q \cdot u)^2} \right] + \frac{\epsilon^{\mu\nu\alpha\beta}}{2q \cdot u} \left( q_\alpha \kappa_{\beta\nu} - \frac{u_\nu q_\alpha q^\rho}{q \cdot u} (\sigma_{\beta\rho} + \kappa_{\beta\rho}) \right)$$

- The perturbative solution :

Hidaka, Pu, DY, 17

$$\delta f_q = -\frac{\tau_R}{q \cdot u} \left( 1 - \frac{\hbar q \cdot \mathcal{A}}{q \cdot u} \right) \left[ q \cdot \Delta + \hbar \frac{S_{(u)}^{\mu\nu} E_\mu}{(q \cdot u)} \Delta_\nu + \hbar S_{(u)}^{\mu\nu} (\partial_\mu F_{\rho\nu}) \partial_q^\rho + \hbar (\hat{\Pi}_1^\mu + \hat{\Pi}_2^\mu) \Delta_\mu \right] f_q^{\text{eq}}$$

- Different contributions for  $\hbar \delta f_q^{(Q)} = \delta f_q^K + \delta f_q^H + \delta f_q^C$

from CKT      from hydro EOM      from collisions

(neglect nonlinear classical responses)

- The non-equilibrium four current :

$$\delta J_Q^\mu = 2\hbar \int \frac{d^4 q}{(2\pi)^3} \bar{\epsilon}(q \cdot u) \delta(q^2) \left[ q^\mu \delta f_q^{(Q)} - \frac{1}{2} \left( \epsilon^{\mu\nu\alpha\beta} \frac{u_\nu q_\alpha}{q \cdot u} \Delta_\beta + \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \frac{\partial_{q\nu}}{2} \right) \delta f_q^{(c)} \right]$$

# Anomalous Hall currents

- Non-equilibrium (quantum) charge currents : Hidaka, Pu, DY, 17

$$\delta \mathbf{J}_{QR/L} = \mp \frac{\hbar \tau_R}{84\pi^2} \left[ \mu_{R/L} \nabla \times \mathbf{E} + \frac{7\mu_{R/L}}{2T} \mathbf{E} \times \nabla T - \frac{1}{2} \mathbf{E} \times \nabla \mu_{R/L} + \frac{12\mu_{R/L}}{T} (\nabla \mu_{R/L}) \times (\nabla T) \right], \quad \bar{\mu}_{R/L} \ll 1$$

emerge from hydro.

$$\delta \mathbf{J}_{QR/L} = \pm \frac{\hbar \tau_R}{4\pi^2} \left[ \frac{\pi^2 T^2}{3\mu_{R/L}} \nabla \times \mathbf{E} + \frac{2\mu_{R/L}}{3T} \mathbf{E} \times \nabla T - \frac{2}{3} \mathbf{E} \times \nabla \mu_{R/L} - \frac{2\mu_{R/L}}{3T} (\nabla \mu_{R/L}) \times (\nabla T) \right], \quad \bar{\mu}_{R/L} \gg 1$$

- Some observations :
  - $B$  and  $\omega$  do not contribute to nonlinear corrections in the inviscid case.
  - The transport coefficients are parity-odd (charged currents from chiral imbalance).
- A caveat : part of anomalous Hall currents vanishes without hydro in the “naïve” RT ( $\tau_R = \text{const.}$ ) approximation.

$$\tau_R(T, \mu) : \delta \mathbf{J}_{\tau_R} = \frac{\hbar}{12\pi^2} \left( (\mu \partial_T \tau_R - I_1 \partial_{\bar{\mu}} \tau_R) (\nabla \mu) \times (\nabla T) + \bar{\mu} (\bar{\mu} \partial_{\bar{\mu}} \tau_R - T \partial_T \tau_R) \mathbf{E} \times \nabla T - \bar{\mu} (\partial_{\bar{\mu}} \tau_R) \mathbf{E} \times \nabla \mu \right)$$

→ exist without hydro.  
(could exist in Weyl semimetals?)

# Matching Conditions & Entropy Production

- From CKT :

$$\partial_\mu J^\mu = \frac{\hbar}{4\pi^2}(\mathbf{E} \cdot \mathbf{B}) + 2 \int_q \left[ \delta(q^2)q^\mu + \hbar\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\frac{\partial_{q\nu}\delta(q^2)}{4} \right] \mathcal{C}_\mu$$

$$\partial_\mu T^{\mu\nu} = F^{\nu\mu}J_\mu + 2 \int_q \delta(q^2) \left[ q^\nu q^\mu + \frac{\hbar\epsilon^{\sigma\mu\alpha\beta}}{4} \left( \eta_\sigma^\nu (q_\beta \partial_\alpha + F_{\alpha\beta}) + q^\nu F_{\alpha\beta} \partial_{q\sigma} \right) \right] \mathcal{C}_\mu$$

- Matching conditions for the classical RT approximation :  $u_\mu \delta J^\mu = 0$ ,  
 $u_\mu \delta T^{\mu\nu} = 0$
- Entropy currents :  $s^\mu = \frac{1}{T} \left( p u^\mu + T^{\mu\nu} u_\nu - \mu J^\mu \right) + \hbar(D_B B^\mu + D_\omega \omega^\mu)$   
 $T^{\mu\nu} = u^\mu u^\nu \epsilon - p P^{\mu\nu} + \Pi_{\text{dis}}^{\mu\nu} + \Pi_{\text{non}}^{\mu\nu}$
- 2<sup>nd</sup> law is protected by classical parts :

$$\partial \cdot s = \frac{1}{T} \left[ \Pi_{\text{dis}}^{\mu\nu} \partial_\mu u_\nu - (E_\mu + T \partial_\mu \bar{\mu}) \delta J^\mu \right]$$

Hidaka, DY, 18