Analytical solutions for higher order hydrodynamics in Bjorken and Gubser flows

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Outline

- Systematic approach of deriving an out-of-equilibrium ‘hydrodynamic’ theory from Kinetic theory
- Solutions of third-order ‘hydrodynamics’ for Bjorken expansion
- Approximations using different series solutions
- Higher-order hydrodynamics in Gubser flow: Analytical Results
- Gradient expansion and slow-roll series for third-order hydro
- Emergent attractor behavior in hydrodynamics
Viscous hydro from Kinetic theory

- $T^{\mu\nu}(x)$ in terms of phase-space distribution function:

$$T^{\mu\nu}(x) = \int dp \ p^\mu p^\nu f(x, p) = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu} + \pi^{\mu\nu}.$$  

- Write $f = f_{eq} + \delta f$; For Boltzmann statistics, $f_{eq} \equiv \text{Exp}\left[\frac{-p^\mu u_\mu}{T}\right].$

- Deviation from local equilibration $\Rightarrow$ Dissipation,

$$\pi^{\mu\nu} = \int dp \ p^{\langle \mu} p^{\nu \rangle} f(x, p),$$

$$A^{\langle \mu\nu \rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta} A_{\alpha\beta}.$$
Perturbative solution of Boltzmann equation

- Boltzmann Eq. in relaxation-time approximation,

\[ p^\mu \partial_\mu f = -(u \cdot p) \frac{\delta f}{\tau_\pi}, \]

\( \tau_\pi \) is the relaxation time.

- Solve perturbatively assuming small relaxation time \( \tau_\pi \).

- To first- and second-order in derivatives,

\[ \delta f^{(1)} = -\frac{\tau_\pi}{u \cdot p} p^\mu \partial_\mu f_{eq}, \]

\[ \delta f^{(2)} = \frac{\tau_\pi}{u \cdot p} p^\mu p^\nu \partial_\mu \left( \frac{\tau_\pi}{u \cdot p} \partial_\nu f_{eq} \right). \]
Equations of shear stress tensor

▶ Up to first-order in derivatives

\[
\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \int dp \ p^{\alpha} \ p^{\beta} \ \delta f^{(1)} = 2\tau_{\pi} \beta_{\pi} \sigma^{\mu\nu}
\]

\[
\beta_{\pi} = 4P / 5
\]

▶ Resummation at second-order:

\[
\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_{\pi}} = -\int dp \ p^{\mu} \ p^{\nu} \ p^{\gamma} \nabla_{\gamma} f
\]

▶ Second-order equation for \( \pi^{\mu\nu} \)

\[
\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_{\pi}} = 2\beta_{\pi} \sigma^{\mu\nu} + 2\pi^{\langle\mu\omega^{\nu}\rangle}_{\gamma} - \frac{10}{7} \pi^{\langle\mu\sigma^{\nu}\rangle}_{\gamma} - \frac{4}{3} \pi^{\mu\nu} \theta,
\]

where \( \omega^{\mu\nu} \equiv (\nabla^{\mu} u^{\nu} - \nabla^{\nu} u^{\mu}) / 2 \) is the vorticity tensor.
To third-order in derivatives,

\[ \dot{\pi}^{\langle \mu \nu \rangle} = -\frac{\pi^{\mu \nu}}{\tau_\pi} + 2\beta_\pi \sigma^{\mu \nu} + 2\pi^{\gamma \omega}^{\langle \mu \omega \nu \rangle} - \frac{10}{7} \pi^{\gamma \sigma}^{\langle \mu \sigma \nu \rangle} - \frac{4}{3} \pi^{\mu \nu} \theta \]

\[ + \frac{25}{7\beta_\pi} \pi^\rho^{\langle \mu \omega \nu \rangle}^{\gamma} \pi^\rho_{\gamma} - \frac{1}{3\beta_\pi} \pi^{\gamma \rho}^{\langle \mu \rho \nu \rangle} \theta - \frac{38}{245\beta_\pi} \pi^{\mu \nu} \pi^\rho_{\gamma} \sigma^\rho_{\gamma} \]

\[ - \frac{22}{49\beta_\pi} \pi^\rho^{\langle \mu \pi \nu \rangle}^{\gamma} \sigma^\rho_{\gamma} - \frac{24}{35} \nabla^{\langle \mu \rangle} \left( \pi^\nu_{\gamma} \dot{u}_{\gamma} \tau_\pi \right) \]

\[ + \frac{4}{35} \nabla^{\langle \mu \rangle} \left( \tau_\pi \nabla^{\gamma} \pi^\nu_{\gamma} \right) - \frac{2}{7} \nabla^{\gamma} \left( \tau_\pi \nabla^{\langle \mu \pi \nu \rangle} \right) \]

\[ + \frac{12}{7} \nabla^{\gamma} \left( \tau_\pi \dot{u}_{\gamma}^{\langle \mu \pi \nu \rangle} \right) - \frac{1}{7} \nabla^{\gamma} \left( \tau_\pi \nabla^{\gamma} \pi^{\langle \mu \nu \rangle} \right) \]

\[ + \frac{6}{7} \nabla^{\gamma} \left( \tau_\pi \dot{u}_{\gamma}^{\langle \mu \nu \rangle} \pi^{\langle \mu \nu \rangle} \right) - \frac{2}{7} \tau_\pi \omega^\rho^{\langle \mu \omega \nu \rangle} \pi^\rho_{\gamma} \]

\[ - \frac{2}{7} \tau_\pi \pi^\rho_{\gamma}^{\langle \mu \omega \nu \rangle} \omega^\rho_{\gamma} - \frac{10}{63} \tau_\pi \pi^{\mu \nu} \theta^2 + \frac{26}{21} \tau_\pi \pi^\gamma_{\gamma}^{\langle \mu \omega \nu \rangle} \theta \]
Bjorken flow

- For boost-invariant longitudinal expansion, $v^z = \frac{z}{t}$, $v^x = v^y = 0$.

- Milne coordinate system: proper time $\tau = \sqrt{t^2 - z^2}$ and space-time rapidity $\eta_s = \tanh^{-1}(z/t)$.

\[
\frac{d\epsilon}{d\tau} = -\frac{1}{\tau} \left( \frac{4}{3} \epsilon - \pi \right),
\]
\[
\frac{d\pi}{d\tau} = -\frac{\pi}{\tau} + \frac{1}{\tau} \left( \frac{4}{3} \beta_\pi - (\lambda + \frac{4}{3})\pi - \chi \frac{\pi^2}{\beta_\pi} \right),
\]

The transport coefficients are,

\[
\beta_\pi = \frac{4P}{5}, \quad \lambda = \frac{10}{21}, \quad \chi = \frac{72}{245}.
\]
Proper time evolution of pressure anisotropy

$T_0 = 300 \text{ MeV, } \tau_0 = 0.25 \text{ fm/c}$

- $\eta/s = 1/4\pi$
- $\eta/s = 3/4\pi$
- $\eta/s = 10/4\pi$

- Exact RTA
- CE higher-order
- Israel-Stewart
Assume a constant relaxation time and define $\bar{\pi} = \pi / (\epsilon + P)$.


Using a rescaled variable $\hat{\tau} = \tau / \tau_{\pi}$,

\[
\frac{d\bar{\pi}}{d\hat{\tau}} = -\bar{\pi} + \frac{1}{\hat{\tau}} \left( a - \lambda \bar{\pi} - \left( \frac{4}{3} + 5\chi \right) \bar{\pi}^2 \right)
\]

Convert to second-order linear differential equation via

\[
\frac{1}{y} \frac{dy}{d\hat{\tau}} = \left( \frac{4}{3} + 5\chi \right) \frac{\bar{\pi}}{\hat{\tau}},
\]

\[
\frac{d^2 y}{d\hat{\tau}^2} + \left( 1 + \frac{1 + \lambda}{\hat{\tau}} \right) \frac{dy}{d\hat{\tau}} - \frac{4(a + \chi)}{3} \frac{y}{\hat{\tau}^2} = 0
\]

where $a = 4/15$. 
General solution of this linear ODE is in terms of Whittaker functions $M_{k,m}(\hat{\tau})$ and $W_{k,m}(\hat{\tau})$:

$$y(\hat{\tau}) = Ae^{-\hat{\tau}/2} \hat{\tau}^k \left[M_{k,m}(\hat{\tau}) + \alpha W_{k,m}(\hat{\tau})\right],$$

where $A$ and $\alpha$ are constants.

$$k = -\frac{\lambda+1}{2}, \quad m = \frac{1}{2} \sqrt{\frac{16}{3} (a + \chi)} + \lambda^2.$$

Solution for $\bar{\pi}$ is

$$\bar{\pi}(\hat{\tau}) = \frac{\pi}{\epsilon + P} = \frac{[6k + 6m + 3] M_{k+1,m}(\hat{\tau}) - 6\alpha W_{k+1,m}(\hat{\tau})}{8(1 + \frac{\chi}{a}) [M_{k,m}(\hat{\tau}) + \alpha W_{k,m}(\hat{\tau})]}.$$
Emergent attractor behavior

- Want to look at late time behavior of

\[
\bar{\pi}(\hat{\tau}) = \frac{\pi}{\epsilon + P} = \frac{[6k + 6m + 3] M_{k+1,m}(\hat{\tau}) - 6\alpha W_{k+1,m}(\hat{\tau})}{8(1 + \frac{\chi}{\alpha}) [M_{k,m}(\hat{\tau}) + \alpha W_{k,m}(\hat{\tau})]}
\]

- Want to approximate late time approach to universal curve

- Consider the object

\[
\frac{\partial \bar{\pi}}{\partial \alpha}
\]

and take \( \hat{\tau} \to \infty \)

- Using the analytical solution of \( \bar{\pi} \) we get

\[
\frac{\partial \bar{\pi}}{\partial \alpha} \sim \frac{e^{-\hat{\tau}}}{\hat{\tau}}
\]
The attractor solution

Proper time evolution of normalised shear

\[ \frac{\tau}{(\varepsilon + P)} \]

Third-order
Second-order

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Write $\bar{\pi}$ as a gradient-series:

$$\bar{\pi} = a \sum \frac{c_n}{\hat{\tau}^n}$$

Substituting in

$$\frac{d\bar{\pi}}{d\hat{\tau}} + \bar{\pi} + \frac{\lambda \bar{\pi}}{\hat{\tau}} - \frac{a}{\hat{\tau}} + \left(\frac{4}{3} + 5\chi\right) \frac{\bar{\pi}^2}{\hat{\tau}} = 0$$

Recursion relation for the coefficients for $n \geq 1$ :

$$c_{n+1} = n c_n - \lambda c_n + 5 a (a + \chi) \sum_{m=0}^{n} c_m c_{n-m}$$

$c_1 = 1$, $c_2 = 1 - \lambda$, , $c_3 = (1 - \lambda)(2 - \lambda + 10a(a + \chi))$, ...
Gradient expansion as a perturbation series

- Label each term by the number of gradients:

\[
\epsilon \frac{d \bar{\pi}}{d \hat{\tau}} + \bar{\pi} + \epsilon \frac{\lambda \bar{\pi}}{\hat{\tau}} - \frac{a}{\hat{\tau}} + \epsilon^2 \left( \frac{4}{3} + 5 \chi \right) \frac{\bar{\pi}^2}{\hat{\tau}} = 0
\]

- Consider a perturbative expansion

\[
\bar{\pi} = \sum_n \bar{\pi}_n(\hat{\tau}) \epsilon^n
\]

- The series:

  order \( \epsilon^0 \) : \( \bar{\pi}_0 = \frac{a}{\hat{\tau}} \),

  order \( \epsilon \) : \( \bar{\pi}_1 = \frac{a(1 - \lambda)}{\hat{\tau}^2} \),

  order \( \epsilon^2 \) : \( \bar{\pi}_2 = (1 - \lambda)(2 - \lambda + 10a(a + \chi))/\hat{\tau}^3 \)

- Same as the gradient expansion
The gradient approximation

The gradient series in IS and third-order theories

\[ \pi/(\varepsilon + P) \]
Assume that time-evolution of $\bar{\pi}$ is ‘slow’; no restriction on gradients:

$$
\epsilon \frac{d\bar{\pi}}{d\hat{\tau}} = -\bar{\pi} + \frac{1}{\tau} \left( a - \lambda \bar{\pi} - \left( \frac{4}{3} + 5\chi \right) \bar{\pi}^2 \right)
$$

Assume perturbative series solution:

$$
\bar{\pi} = \sum \pi_n \epsilon^n
$$

$$
\bar{\pi}_0 = \frac{1}{2(4/3 + 5\chi)} \left( \sqrt{(4/3 + 5\chi)4a + (\lambda + \hat{\tau})^2} - \lambda - \tau \right)
$$

The general relation:

$$
\bar{\pi}_n = \frac{-1}{\sqrt{(4/3 + 5\chi)4a + (\lambda + \hat{\tau})^2}} \left[ \hat{\tau} \frac{d\bar{\pi}}{d\hat{\tau}} + \left( \frac{4}{3} + 5\chi \right) \sum_{m=1}^{n-1} \bar{\pi}_{n-m} \bar{\pi}_m \right]
$$
The slow roll solution in IS and third-order theories
Gubser Flow

- $v^z = z/t, \quad u^r(x) \neq 0, \quad u^\phi(x) = 0$.

- Suitably described in de Sitter coordinates $(\rho, \theta, \phi, \eta)$, in which $u^\mu = (1, 0, 0, 0)$,

$$
\rho = - \sinh^{-1} \left( \frac{1 - q^2 \tau^2 + q^2 r^2}{2q\tau} \right), \quad \theta = \tan^{-1} \left( \frac{2qr}{1 + q^2 \tau^2 - q^2 r^2} \right),
$$

$1/q \approx$ transverse size.

- Weyl rescaled unitless quantities,

$$
\epsilon(\tau, r) = \frac{\hat{\epsilon}(\rho)}{\tau^4}, \quad \pi_{\mu\nu}(\tau, r) = \frac{1}{\tau^2} \frac{\partial \hat{X}^\alpha}{\partial x^\mu} \frac{\partial \hat{X}^\beta}{\partial x^\nu} \hat{\pi}_{\alpha\beta}(\rho).
$$

► Chapman-Enskog Method:

\[
\begin{align*}
\frac{d\hat{\epsilon}}{d\rho} &= - \left( \frac{8}{3} \hat{\epsilon} - \hat{\pi} \right) \tanh \rho, \\
\frac{d\hat{\pi}}{d\rho} &= - \frac{\hat{\pi}}{\hat{\tau}_\pi} + \tanh \rho \left( \frac{4}{3} \beta_\pi - \lambda \hat{\pi} - \chi \frac{\hat{\pi}^2}{\beta_\pi} \right).
\end{align*}
\]

See talk by Ulrich Heinz today at 18:10

► Assume constant relaxation time \( \hat{\tau}_\pi \).

► Decoupled equation for normalised shear \( \hat{\pi} \equiv \hat{\pi}/(\hat{\epsilon} + \hat{P}) \):

\[
\frac{d\hat{\pi}}{d\rho} = - \frac{\hat{\pi}}{\hat{\tau}_\pi} + \tanh \rho \left[ a + \lambda \hat{\pi} \left( \frac{4}{3} + 5\chi \right) \hat{\pi}^2 \right].
\]
If the last non-linear term is ignored, we expect a solution of the form \( \hat{\pi}(\rho) \approx e^{-f(\rho)}g(\rho) \).

\[
f'(\rho) = -\left( \frac{1}{\hat{\tau}} - a \tanh \rho \right),
\]

\[
g(\rho) = a \int e^{\rho/\hat{\tau}} \cosh^\lambda \rho \tanh \rho \, d\rho + C
\]

\[
\hat{\pi}(\rho) = C e^{-\rho/\hat{\tau}} \cosh^\lambda \rho - \frac{a}{2\lambda_2} \, _2F_1[1, \lambda_1, 1 + \lambda_2, -e^{2\rho}]
\]

\[
+ \frac{e^{2\rho} a}{2(1 + \lambda_2)} \, _2F_1[1, 1 + \lambda_1, 2 + \lambda_2, -e^{2\rho}],
\]

where \( \lambda_1 = (1/\hat{\tau} - \lambda)/2 \) and \( \lambda_2 = (1/\hat{\tau} + \lambda)/2 \).
The full solution

- Ricatti type equation:

\[
\frac{d\hat{\pi}}{d\rho} = -\frac{\hat{\pi}}{\hat{\tau}_\pi} + \tanh \rho \left( a + \lambda \hat{\pi} - \gamma \hat{\pi}^2 \right)
\]

- Convert to second-order linear differential form using

\[
\frac{1}{y} \frac{dy}{d\rho} = \gamma \tanh \rho \frac{\hat{\pi}}{\hat{\tau}_\pi},
\]

followed by a variable transform, \( x = \tanh \rho \)

- The full equation,

\[
\frac{d^2y}{dx^2} + \frac{1}{1 - x^2} \left[ \frac{1}{\hat{\tau}_\pi} - \frac{1}{x} - (1 + \lambda)x \right] \frac{dy}{dx} - \frac{a\gamma x^2}{(1 - x^2)^2} y = 0.
\]
Analytical solution Gubser 3rd order

Analytical solution for $\hat{\pi}$:

$$
\hat{\pi} = \left[ c(tanh(\rho) + 1)^{-\frac{a_1}{4}} \left( (1 + (\lambda + a_1)\hat{\pi})(tanh(\rho) - 1) - 4\hat{\pi}b_4(tanh(\rho) + 1) \right) HeunG(s) 
+ (tanh(\rho) + 1)^{\frac{a_1}{4}} \left( (1 + (\lambda - a_1)\hat{\pi})(tanh(\rho) - 1) - 4\hat{\pi}b_4(tanh(\rho) + 1) \right) HeunG(r) 
- 4\hat{\pi}(tanh(\rho) - 1) \left( c(tanh(\rho) + 1)^{1-\frac{a_1}{4}} HeunGP(s) + (tanh(\rho) + 1)^{1+\frac{a_1}{4}} HeunGP(r) \right) \right] / 
\left[ 4\phi tanh(\rho)\hat{\pi} \left( c(tanh(\rho) + 1)^{-\frac{a_1}{4}} HeunG(s) + (tanh(\rho) + 1)^{\frac{a_1}{4}} HeunG(r) \right) \right]
$$

Where HeunGP is HeunGPrime and

$$
r := [2, \frac{b_3 - b_2}{24 a_3}, \frac{a_1 + a_3}{4}, \frac{a_1 a_3 b_1}{4 a_3}, \frac{a_1}{2} + 1, -1, tanh(\rho) + 1]
$$

$$
s := [2, \frac{b_3 + b_2}{24 a_3}, -\frac{a_1 + a_3}{4}, -\frac{a_1 a_3 b_1}{4 a_3}, \frac{a_1}{2} + 1, -1, tanh(\rho) + 1]
$$
The solutions for various initial conditions in two theories
Gradient Expansion for Gubser flow: Third-order hydro

- Gradient expansion of:

\[
\frac{d\hat{\pi}^\rho}{d\rho} = -\frac{\hat{\pi}}{\hat{\tau}_\pi} + \tanh(\rho \left(a + \lambda\hat{\pi} - \gamma\hat{\pi}^2\right))
\]

- Contrary to Bjorken, θ ≡ 2 tanh ρ is not a good expansion parameter [G.S. Denicol, Jorge Noronha arXiv:1804.0477]

- Label each term by the order of gradients appearing,

\[
\epsilon \frac{d\hat{\pi}}{d\rho} = -\frac{\hat{\pi}}{\hat{\tau}_\pi} + \tanh(\rho \left(a + \epsilon\lambda\hat{\pi} - \epsilon^2\gamma\hat{\pi}^2\right))
\]

- Perturbative series solution

\[
\hat{\pi} = \sum \hat{\pi}_n \epsilon^n
\]
The gradient series in Gubser flow

- The Navier-Stokes term at order $\epsilon^0$:
  \[ \hat{\pi}_0 = a \hat{\tau}_\pi \tanh \rho \]

- order $\epsilon^1$
  \[ \hat{\pi}_1 = a \hat{\tau}_\pi^2 \left[ \tanh^2 \rho (1 + \lambda) - 1 \right] \]

- Similarly,
  \[ \hat{\pi}_2 = -\hat{\tau}_\pi \frac{d\hat{\pi}_1}{d\rho} + \hat{\tau}_\pi \tanh \rho \left( \lambda \hat{\pi}_1 - \gamma \hat{\pi}_0^2 \right) \]

- In general for $n \geq 2$
  \[ \hat{\pi}_n = -\hat{\tau}_\pi \left[ \frac{d\hat{\pi}_{n-1}}{d\rho} - \lambda \tanh(\rho) \hat{\pi}_{n-1} + \gamma \tanh \rho \sum_{m=0}^{n-2} \hat{\pi}_{n-2-m} \hat{\pi}_m \right] \]
The gradient approximation

The gradient series in IS and third-order theories for Gubser flow
As in Bjorken one can express
\[
\epsilon \frac{d\hat{\pi}}{d\rho} = - \frac{\hat{\pi}}{\hat{\tau}_\pi} + \tanh \rho \left( a + \lambda \hat{\pi} - \gamma \hat{\pi}^2 \right).
\]

Assume a perturbative solution \( \hat{\pi} = \sum \hat{\pi}_n \epsilon^n \)

The zeroeth order solution:
\[
\hat{\pi}_0 = \frac{\coth \rho}{2\gamma} \left[ -\frac{1}{\tau_\pi} + \lambda \tanh \rho + \sqrt{4a\gamma \tanh^2 \rho + \left( \frac{1}{\tau_\pi} - \lambda \tanh \rho \right)^2} \right]
\]

General recursive relation
\[
\hat{\pi}_n = -\frac{\tau_\pi}{\sqrt{(1 - \lambda \tau_\pi \tanh \rho)^2 + 4a\gamma \tau_\pi^2 \tanh^2 \rho}} \left[ \frac{d\hat{\pi}_{n-1}}{d\rho} \right.
\]
\[
+ \gamma \tanh \rho \sum_{m=1}^{n-1} \hat{\pi}_{n-m} \hat{\pi}_m \left. \right]
\]
Slow-roll approximation

The slow roll solution in two theories
Conclusions

- Derived analytical solutions for higher-order evolution equations in Bjorken and Gubser flows.
- Obtained attractor behavior for time evolution of normalised shear.
- Quantified the effects of various transport coefficients on determining the attractor.
- Future work: Investigate the presence of attractors in realistic flow profiles with lesser symmetries.