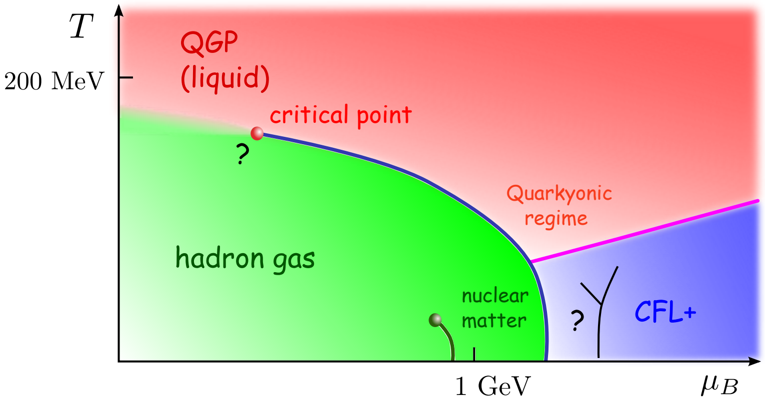
Hydrodynamics for QCD critical point

M. Stephanov

with Y. Yin (MIT), 1712.10305

Critical point between the QGP and hadron gas phases?



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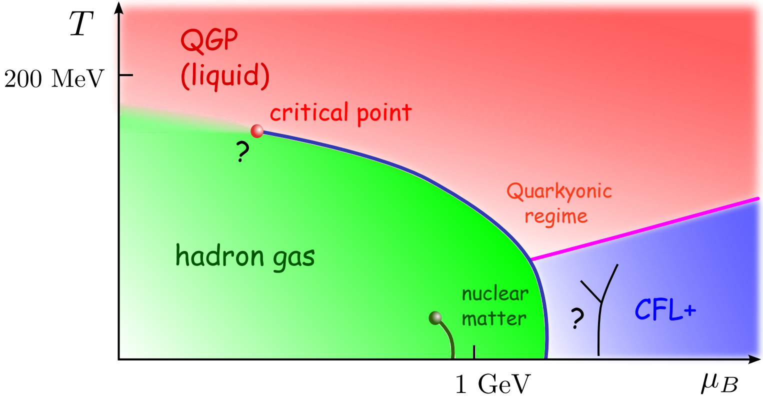


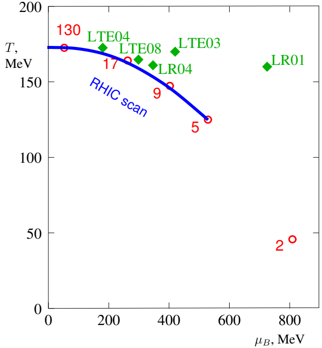
Lattice QCD at $\mu_B \lesssim 2T$ – a crossover.

C.P. is ubiquitous in models (NJL, RM, Holog., Strong coupl. LQCD, . . .)

Essentially two approaches to discovering the QCD critical point.

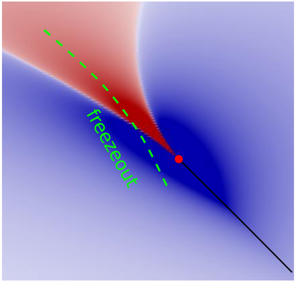
Each with its own challenges.

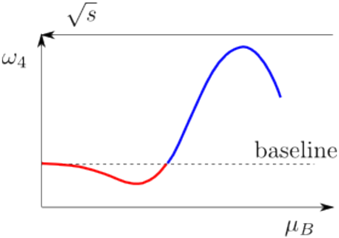
- Lattice simulations. Sign problem.
- Heavy-ion collisions. *Non-equilibrium.*



Beam Energy Scan I: intriguing hints

Equilibrium κ_4 vs T and μ_B:





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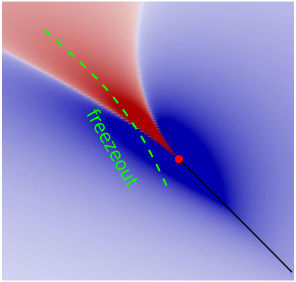

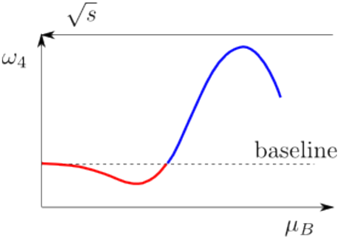

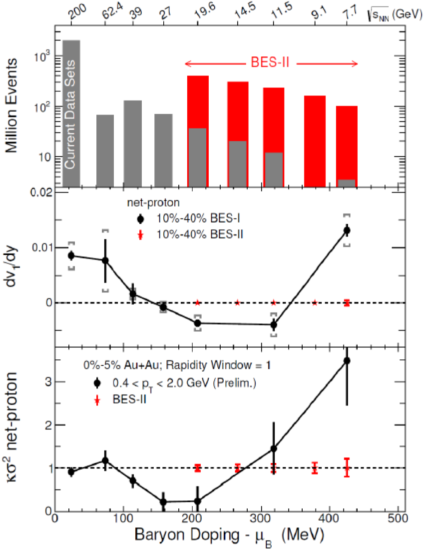

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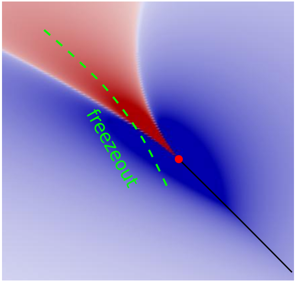

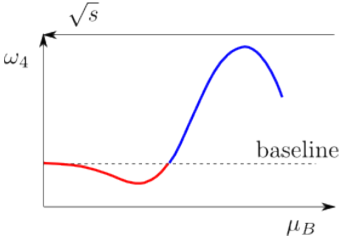

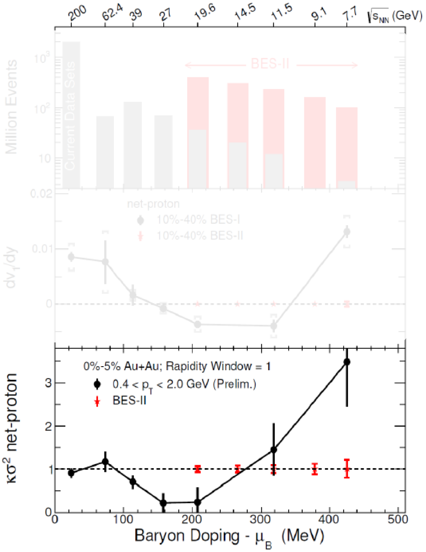

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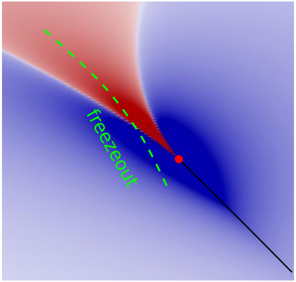

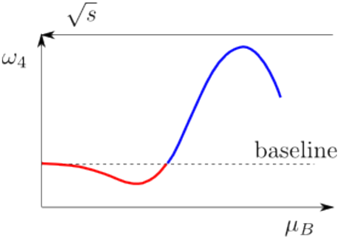

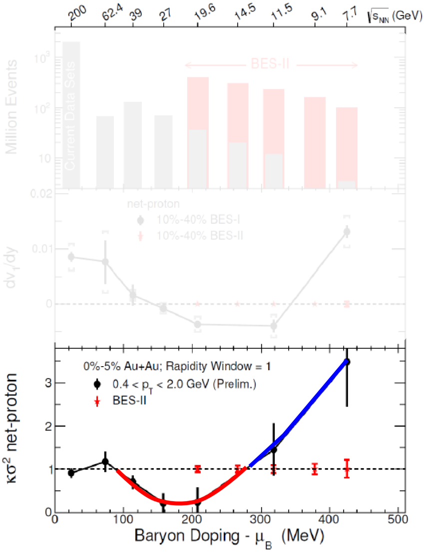

“intriguing hint” (2015 LRPNS)

Non-equilibrium physics is essential near the critical point.

The challenge taken on by BEST COLLABORATION

Magnitude of fluctuation observables and ξ

- Divergent magnitude (κ_2) and non-gaussianity ($\kappa_{3,4}$) are due to divergent correlation length ξ:

$$\kappa_n \sim \xi^{\mathrm{power}(n)}$$

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  Infinite ξ needs infinite time – critical slowing down.
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- Can we describe magnitudes of *critical* fluctuations *directly* from non-equilibrium (hydro)dynamical framework?

Hydrodynamics breaks down at CP

Hydrodynamics is expansion in gradients ($k \sim \nabla$),
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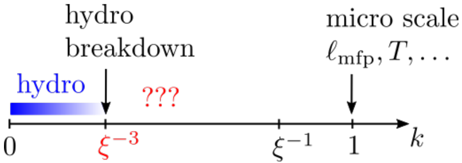
When $k \sim p/\zeta \sim \xi^{-3}$ hydrodynamics breaks down.



Critical slowing down and bulk viscosity

Bulk viscosity is the effect of system taking time to adjust to local equilibrium (Mandel'shtam-Leontovich, Khalatnikov-Landau).

$$p_{\text{hydro}} = p_{\text{equilibrium}} - \zeta\, \boldsymbol{\nabla} \cdot \boldsymbol{v}$$

$\boldsymbol{\nabla} \cdot \boldsymbol{v}$ – expansion rate

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Hydrodynamics breaks down because of *large relaxation time*.

Similar to breakdown of an effective theory (non-locality) due to a low-energy mode which should not have been integrated out.

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(MS-Yin 1704.07396, 1712.10305)

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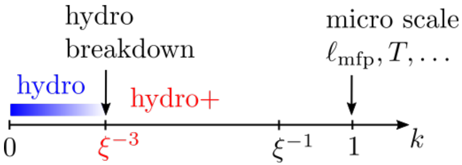
- "Hydro+" extends the range of validity of hydro to parametrically shorter time ($\omega \gg 1/\tau_\phi \sim \xi^{-3}$) and length ($k \gg \xi^{-3}$) scales.



What is the additional slow mode?

- An *equilibrium* thermodynamic state is completely characterized by $\bar{\varepsilon}$, $\bar{n}$, $\ldots$.

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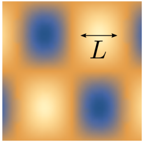
- Hydrodynamics describes *partial-equilibrium states*, i.e., equilibrium is only local, because equilibration time $\sim L^2$.

  Fluctuations in such states are not necessarily in equilibrium.



Nonequilibrium fluctuations

- Measures of fluctuations are *additional* variables needed to characterize the partial-equilibrium state.

  2-point (and n-point) functions of fluctuating hydro variables: $\langle\delta\varepsilon\delta\varepsilon\rangle$, $\langle\delta n\delta n\rangle$, $\langle\delta\varepsilon\delta n\rangle$, $\ldots$. (Or probability functional).

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- Relaxation rates of 2pt functions is of the same order as that of corresponding 1pt functions.

  But effects of fluctuations are *usually* suppressed due to averaging out: $\sqrt{\xi^3/V} \sim (k\xi)^{3/2}$ by CLT.

Critical fluctuations

- Near CP there is *parametric* separation of relaxation time scales.

  The slowest and thus most out-of-equilibrium mode is $s/n \equiv m$.

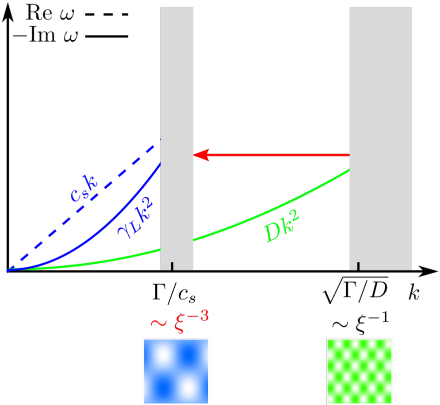
  - Rate of m at scale $k \sim \xi^{-1}$,

    $$\Gamma \sim D\xi^{-2} \sim \xi^{-3},$$

    is of order of that for sound at much smaller $k \sim \xi^{-3}$.

  - The effect of δm fluctuations $(k\xi)^{3/2} = \mathcal{O}(1)$.

- Thus we need $\langle \delta m \delta m \rangle$ as the independent variable(s) in hydro+ equations.



Mode distribution of fluctuations

- The new variable is 2-pt function $\langle \delta m \delta m \rangle$ (Wigner transform):

$$\phi_{\boldsymbol{Q}}(\boldsymbol{x}) = \int_{\Delta \boldsymbol{x}} \langle\, \delta m(\boldsymbol{x} + \Delta\boldsymbol{x}/2)\, \delta m(\boldsymbol{x} - \Delta\boldsymbol{x}/2) \,\rangle \; e^{i\boldsymbol{Q}\cdot\Delta\boldsymbol{x}}$$

  - Dependence on $\boldsymbol{x}$ ($\sim L$) is much slower than on $\Delta\boldsymbol{x}$ ($\sim \xi$).

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- Hydro(+) describes relaxation to eqlbrm, maximizing entropy.

  To ensure the 2nd law of thermodynamics is obeyed we need to know the entropy: $s_{(+)}(\varepsilon, n, \phi_{\boldsymbol{Q}})$, i.e., “EOS+”.

  Derivation: Given an ensemble of state probabilities p_i

$$S = \sum_i p_i \log(1/p_i), \qquad \dots$$

Entropy of fluctuations

- …result resembling 2-PI action: *(1712.10305)*

$$s_{(+)}(\varepsilon, n, \phi_{\boldsymbol{Q}}) = s(\varepsilon, n) + \frac{1}{2}\int_{\boldsymbol{Q}} \left(1 - \frac{\phi_{\boldsymbol{Q}}}{\bar{\phi}_{\boldsymbol{Q}}} + \log\frac{\phi_{\boldsymbol{Q}}}{\bar{\phi}_{\boldsymbol{Q}}}\right)$$

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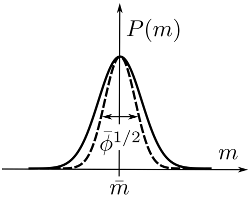
- Two competing effects: e.g., for $\phi > \bar{\phi}$
  - Wider distribution – more microstates – more entropy: $\log(\phi/\bar{\phi})^{1/2}$;

    vs
  - Penalty for larger deviations from peak entropy (at $\delta m = 0$): $-(1/2)\phi/\bar{\phi}$.



Maximimum of $s_{(+)}$ is achieved at $\phi = \bar{\phi}$.

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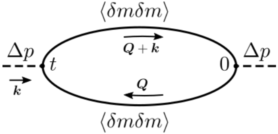
- In equilibrium, Hydro+ = 1-loop.
  Similar to kinetic theory vs HTL.
  Separation of scales: $Q \gg k \sim 1/L$.

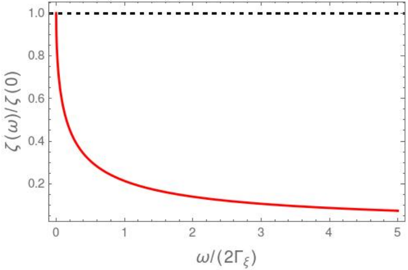


Hydro+ vs Hydro: real-time bulk response

Characteristic Hydro/Hydro+ crossover rate $\Gamma_\xi = D\xi^{-2} \sim \xi^{-3}$.

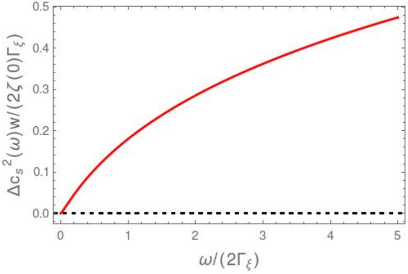
Dissipation during expansion is overestimated in hydro (dashed):

Only modes with $\omega \ll \Gamma_\xi$ experience large ζ.



Stiffness of eos (sound speed) is underestimated in hydro (dashed):

Only modes with $\omega \ll \Gamma_\xi$ are critically soft ($c_s \to 0$ at CP).



Summary

- A fundamental question for Heavy-Ion collision experiments:

  Is there a critical point on the boundary between QGP and hadron gas phases – the endpoint of a first-order transition?

- Intriguing results from experiments (BES-I).

  More to come (BES-II, FAIR/CBM, NICA, J-PARC).

  Quantitative theoretical framework is needed $\Rightarrow$ BEST COLLABORATION.

- In H.I.C., the magnitude of the fluctuation signatures of CP is controlled by dynamical *non-equilibrium effects*.

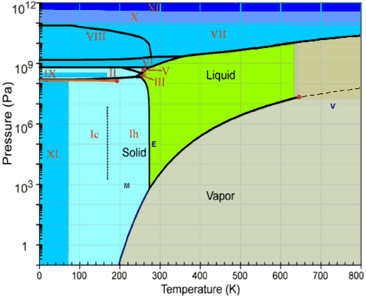
  In turn, critical fluctuations affect hydrodynamics.

  The interplay of critical and dynamical phenomena: Hydro+.

More

Substance[13][14]	Critical temperature	Critical pressure (absolute)
Argon	−122.4 °C (150.8 K)	48.1 atm (4,870 kPa)
Ammonia[15]	132.4 °C (405.5 K)	111.3 atm (11,280 kPa)
Bromine	310.8 °C (584.0 K)	102 atm (10,300 kPa)
Caesium	1,664.85 °C (1,938.00 K)	94 atm (9,500 kPa)
Chlorine	143.8 °C (416.9 K)	76.0 atm (7,700 kPa)
Ethanol	241 °C (514 K)	62.18 atm (6,300 kPa)
Fluorine	−128.85 °C (144.30 K)	51.5 atm (5,220 kPa)
Helium	−267.96 °C (5.19 K)	2.24 atm (227 kPa)
Hydrogen	−239.95 °C (33.20 K)	12.8 atm (1,300 kPa)
Krypton	−63.8 °C (209.3 K)	54.3 atm (5,500 kPa)
CH_4 (methane)	−82.3 °C (190.8 K)	45.79 atm (4,640 kPa)
Neon	−228.75 °C (44.40 K)	27.2 atm (2,760 kPa)
Nitrogen	−146.9 °C (126.2 K)	33.5 atm (3,390 kPa)
Oxygen	−118.6 °C (154.6 K)	49.8 atm (5,050 kPa)
CO_2	31.04 °C (304.19 K)	72.8 atm (7,380 kPa)
N_2O	36.4 °C (309.5 K)	71.5 atm (7,240 kPa)
H_2SO_4	654 °C (927 K)	45.4 atm (4,600 kPa)
Xenon	16.6 °C (289.8 K)	57.6 atm (5,840 kPa)
Lithium	2,950 °C (3,220 K)	652 atm (66,100 kPa)
Mercury	1,476.9 °C (1,750.1 K)	1,720 atm (174,000 kPa)
Sulfur	1,040.85 °C (1,314.00 K)	207 atm (21,000 kPa)
Iron	8,227 °C (8,500 K)	
Gold	6,977 °C (7,250 K)	5,000 atm (510,000 kPa)
Water[2][16]	373.946 °C (647.096 K)	217.7 atm (22.06 MPa)

Critical point is a ubiquitous phenomenon



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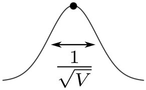

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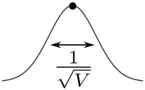
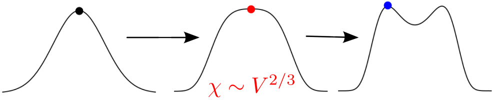


- At the critical point $S(\sigma)$ “flattens”. And $\chi \equiv \langle \delta\sigma^2 \rangle V \to \infty$.



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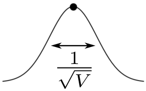

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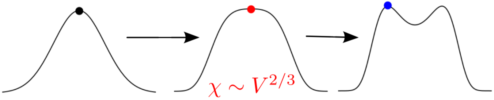
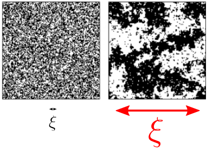




CLT?

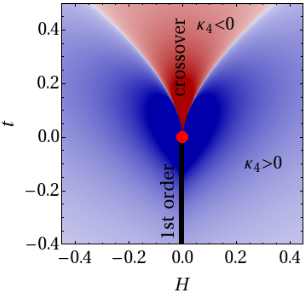
$\boxed{\delta\sigma \text{ is not an average of } \infty \text{ many } \textit{uncorrelated} \text{ contributions: } \xi \to \infty}$

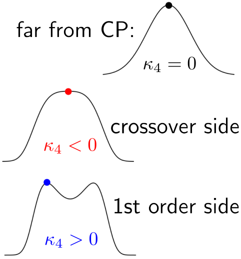
Higher order cumulants

- $n > 2$ cumulants (shape of $P(\sigma)$) depend stronger on ξ.

  E.g., $\langle \sigma^2 \rangle \sim \xi^2$ while $\kappa_4 = \langle \sigma^4 \rangle_c \sim \xi^7$ *[PRL102(2009)032301]*
- For $n > 2$, sign depends on which side of the CP we are.

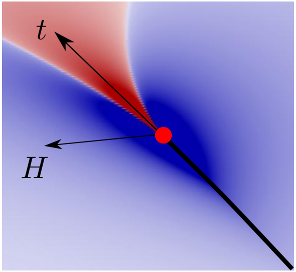
  This dependence is also universal. *[PRL107(2011)052301]*
- Using Ising model variables:





Mapping Ising to QCD phase diagram

T vs μ_B:



- In QCD $(t, H) \to (\mu - \mu_{\rm CP}, T - T_{\rm CP})$
- $\kappa_n(N) = N + \mathcal{O}(\kappa_n(\sigma))$

Critical fluctuations and experimental observables

Observed fluctuations are related to fluctuations of σ.

[MS-Rajagopal-Shuryak PRD60(1999)114028; MS PRL102(2009)032301]

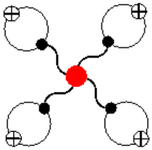
Think of a collective mode described by field σ such that $m = m(\sigma)$:

$$\delta n_{\boldsymbol{p}} = \delta n_{\boldsymbol{p}}^{\text{free}} + \frac{\partial \langle n_{\boldsymbol{p}} \rangle}{\partial \sigma} \times \delta\sigma$$

The cumulants of multiplicity $M \equiv \int_{\boldsymbol{p}} n_{\boldsymbol{p}}$:

- $$\kappa_4[M] = \underbrace{\langle M \rangle}_{\text{baseline}} + \underbrace{\kappa_4[\sigma] \times g^4 \underbrace{\left(\quad \right)^4}_{\sim M^4}}_{\text{this is } \hat{\kappa}_4 (\text{a.k.a.} C_4^{\text{Bzdak-Koch}})} + \ldots,$$

$$= \int_{\boldsymbol{p}} \frac{n_{\boldsymbol{p}}}{\gamma_{\boldsymbol{p}}} \quad \leftarrow \text{acceptance dependent}$$



Why ξ is finite

$\boxed{\text{System expands and is } \textit{out of equilibrium}}$

Kibble-Zurek mechanism:

Critical slowing down means $\tau_{\mathrm{relax}} \sim \xi^z$.
Given $\tau_{\mathrm{relax}} \lesssim \tau$ (expansion time scale):

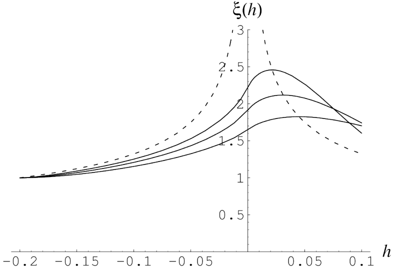
$$\xi \lesssim \tau^{1/z},$$

$z \approx 3$ (universal).

Critical slowing down and ξ

Estimates: $\xi \sim 2 - 3$ fm
(Berdnikov-Rajagopal)

KZ scaling for $\xi(t)$
and cumulants
(Mukherjee-Venugopalan-Yin)



Ingredients of "Hydro+"

- As a warmup consider one extra slow mode.
- Nonequilibrium entropy, or quasistatic EOS:

$$s_{(+)}(\varepsilon, n, \phi)$$

Equilibrium entropy is the maximum of $s_{(+)}$:

$$s(\varepsilon, n) = \max_{\phi} s_{(+)}(\varepsilon, n, \phi)$$

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- The 6th equation (constrained by 2nd law):

$$(u \cdot \partial)\phi = -\gamma_\pi \pi - A_\phi(\partial \cdot u), \qquad \text{where } \pi = -\frac{\partial s_{(+)}}{\partial \phi}$$

ϕ relaxes to equilibrium ($\pi = 0$) at a rate $1/\tau_\phi \equiv \Gamma = \gamma_\pi(\partial\pi/\partial\phi)$.

Linearized Hydro+

- 4 longitudinal modes (sound$\times 2$ + density + ϕ).

  In addition to c_s, D, etc. Hydro+ has two more parameters

  $\Delta c^2 = c^2_{(+)} - c^2_s$ and $\Gamma = 1/\tau_\phi$.

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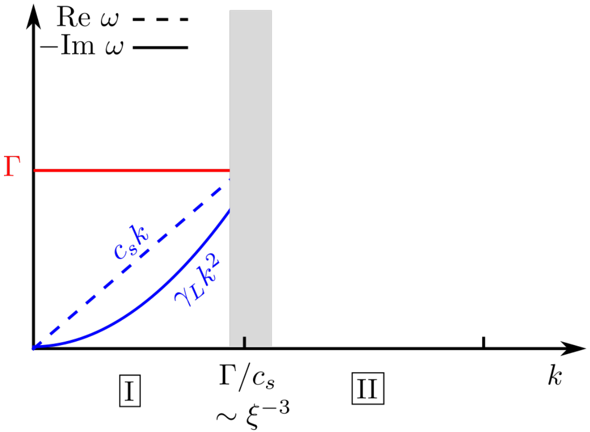
- In Regime I bulk viscosity is divergent as $\Gamma \to 0$:

$$\Delta\zeta = w\Delta c^2/\Gamma$$

  In Regime II bulk viscosity is finite.

Modes



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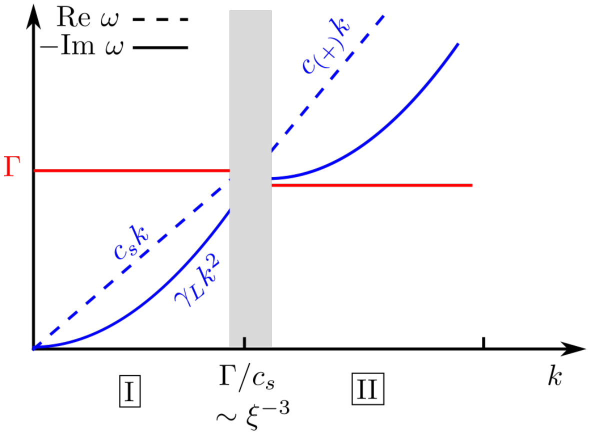

$$\gamma_L \sim \zeta \sim \Gamma^{-1}$$

2-PI entropy

$$S = \sum_i p_i \log \frac{1}{p_i}$$

- Microcanonical: e^{S_0} states in interval $\Delta\Psi$ – $p_i = e^{-S_0}$.
  $S = S_0(\Psi)$. $\Psi = (\varepsilon, n, \ldots)$
- Canonical: $p_i = e^{J\Psi - W[J]}$.
  $S = W[J] - J\langle\Psi\rangle$. "1-PI." $J = (\beta, \beta\mu, \ldots)$.
- Partial equilibrium: $p_i = e^{J\Psi + \frac{1}{2}\Psi K \Psi - W[J,K]}$.
  $S = W[J,K] - J\langle\Psi\rangle - \frac{1}{2}\langle\Psi K \Psi\rangle$. "2-PI"

Hydro+ vs one slow mode vs just hydro