Constraining the QCD critical endpoint with lattice simulations?

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The question

- There exists now lattice calculations of χ_6^B and χ_8^B for fine lattices.
- Such fluctuations were used to try to constrain the critical point before. Two recent examples:
  - D'Elia et al, 1611.08285, Phys.Rev. D95 (2017) no.9, 094503
  - Bazavov et al, 1701.04325, Phys.Rev. D95 (2017) no.5, 054504
- Can we really learn something about the QCD critical point from these lattice simulations?

Spoilers: Not yet. Our results on fine lattices, up to χ_6^B and χ_8^B (with large errors) show no sign of criticality.

Theoretical Background: Lee-Yang zeros

Look at the partition function as a complex function of its parameters and look for zeros:

$$\mathcal{Z}\left(\mu^2, \beta\right) = 0$$

Two choices:

- Fix a real μ and look for zeros in the complex β plane (Fischer zeros)
- Fix a real β ($\sim$ temperature) and look for zeros in the complex μ^2 plane

In the case of a 1st order transition it is easy to show that:

$$\mathrm{Im}\,\beta_{\mathrm{LY}} \text{ or } \mathrm{Im}\,\mu^2_{\mathrm{LY}} \sim \frac{1}{\mathrm{V}}$$

If we have a nonzero infinite volume limit of $\mathrm{Im}\,\beta_{\mathrm{LY}}$ or $\mathrm{Im}\,\mu^2_{\mathrm{LY}}$ we have a crossover. In this case $\mathrm{Im}\,\beta_{\mathrm{LY}}$ can be tought of as a measure of the strength of the crossover.

Connection to the Taylor expansion

In finite V:

$$\mathcal{Z}(\mu,\beta) = \sum_{-3N_s^3}^{N=3N_s^3} Z_N(\beta) e^{N\mu/T}$$

- polynomial in $e^{\mu/T} \equiv e^{\hat{\mu}} \implies$ has $6N_s^3$ complex roots
- roots of $\mathcal{Z} \implies$ singularity in $p \propto \ln \mathcal{Z}$

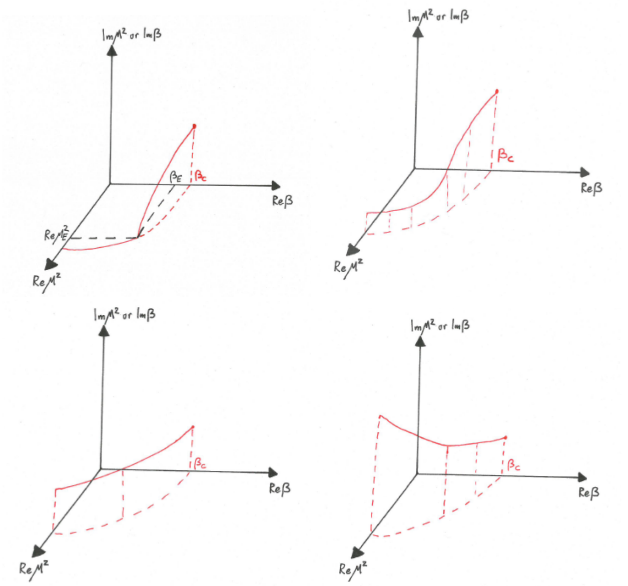
The Taylor expansion of the pressure in any finite volume:

$$p(\mu,\beta) = p_0(\beta) + p_2(\beta)\hat{\mu}^2 + p_4(\beta)\hat{\mu}^4 + \cdots \propto \log \mathcal{Z}(\mu,\beta)$$

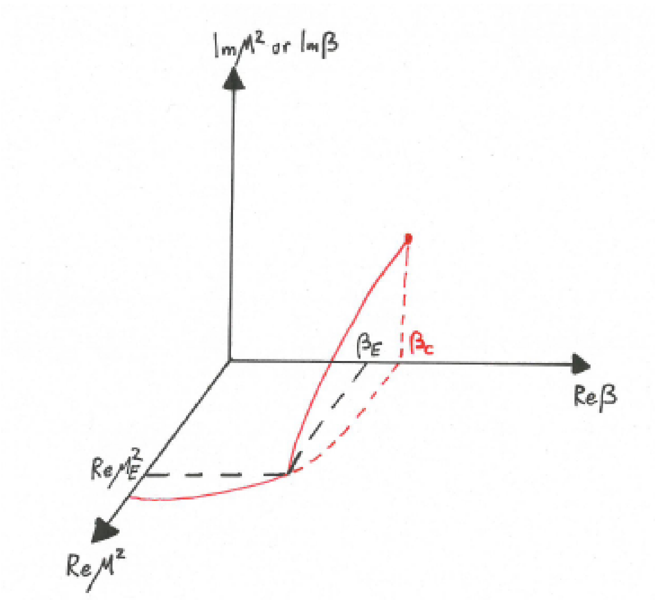
will have a convergence radius equal to the modulus of the Lee Yang zero nearest to $\mu = 0$.

- $\exists N : \forall n > N : p_{2n} \geq 0 \implies \mu_{LY}^2 \in \mathbb{R}, \mu_{LY}^2 > 0$
- $\mathrm{Im}\, \mu_{LY}^2 \neq 0 \implies \exists n : p_{2n} < 0$

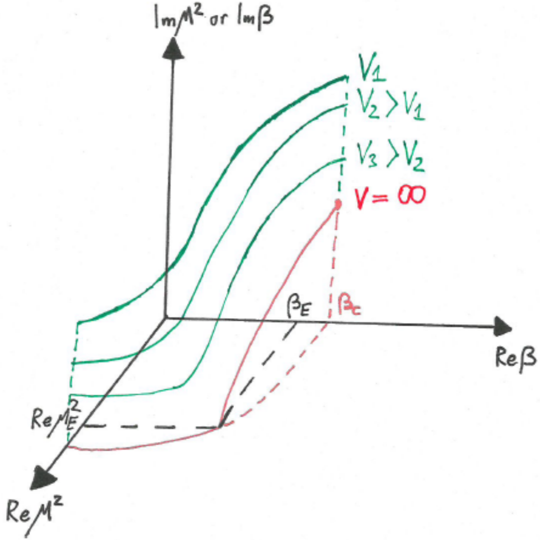
Some possible scenarios



A scenario with a critical point



A scenario with a critical point



The problem

- Known algorithms for calculating the Lee-Yang zeros always involve a sign and an overlap problem, both exponential in the volume
- They also get prohibitively expensive for small volumes, but lattices closer to the continuum limit
- For fine lattices, we are basically limited to calculating the first few coefficients of the Taylor series via
  - calculating Taylor coefficients directly at $\mu = 0$
  - or fitting them to restuls of simulations at imaginary μ
- But even if there is a critical point, there is no guarantee that the first few Taylor coefficients are sensitive to it

This talk

Toy model

(1) We look at a rough course action, $N_t = 4$ unimproved staggered, and consider it as a toy model. For this action there is a critical point signal in the literature: Fodor-Katz: JHEP 0404 (2004) 050. We redo the old Fodor-Katz analysis, with an exact algorithm and higher statistics, just to see if we still see the critical point signal

(2) Next we calculate the Taylor coefficients at a temperature close to T_{CEP} in the same model to see if the convergence radius estimate is close to the critical point estimate from the Lee-Yang zero analysis or not

Realistic, fine lattice

(1) We present our result for baryon fluctuations up to χ_8^B and the corresponding convergence radius estimators for the $N_t = 12$ 4stout improved lattice data

(2) We argue that the results show no sign of criticality

(3) We warn about misinterpreting apparent convergence of the ratio estimators

The pleasures of a rough lattice

Why $N_t = 4$ unimproved staggered?

(1) The sign problem is easier

$$\left\langle \theta^2 \right\rangle \propto -\# V \chi_{11}^{ud} \mu^2 + \mathcal{O}(\mu^4)$$

Continuum: $\chi_{11}^{ud} \sim -0.2$
$N_t = 4$ unimproved staggered: $\chi_{11}^{ud} \sim -0.05$

(2) The overlap problem is easier, since already at $\mu = 0$ we almost have a phase transition.

(3) Small enough lattice to use the following reduction formula:

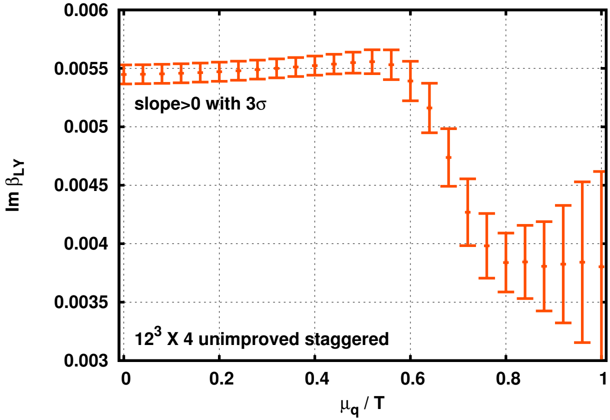
$$\det M(\mu, \beta) = e^{-3N_s^3 N_t \mu} \prod_{i=1}^{6N_s^3} \left(e^{N_t \mu} - \Lambda_i \right)$$

The Fodor-Katz analysis with higher statistics

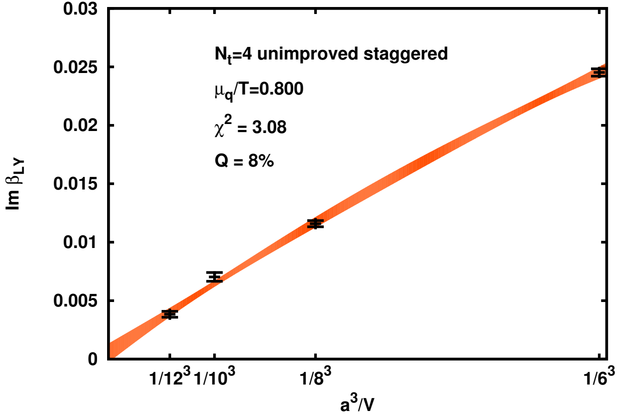
Fodor-Katz: JHEP 0404 (2004) 050

- Look explicitly for Lee-Yang zeros in complex β
- We therefore map the strength of the crossover as a function of quark chemical potential, with $\mu_u = \mu_d$ and $\mu_s = 0$
- We have redone the old analysis with two differences:
  - R algorithm $\rightarrow$ HMC (an exact algorithm)
  - higher statistics: 2,000-3,000 measurements per ensemble $\rightarrow$ 50,000-100,000 measurements per ensemble
- We got basically the same result.

The Fodor-Katz analysis with higher statistics



The Fodor-Katz analysis with higher statistics



Convergence radius estimators

(1) Ratio test for the pressure:

$$p(\mu) = p_0 + p_2\hat{\mu}^2 + p_4\hat{\mu}^4 + p_6\hat{\mu}^6 + \dots$$

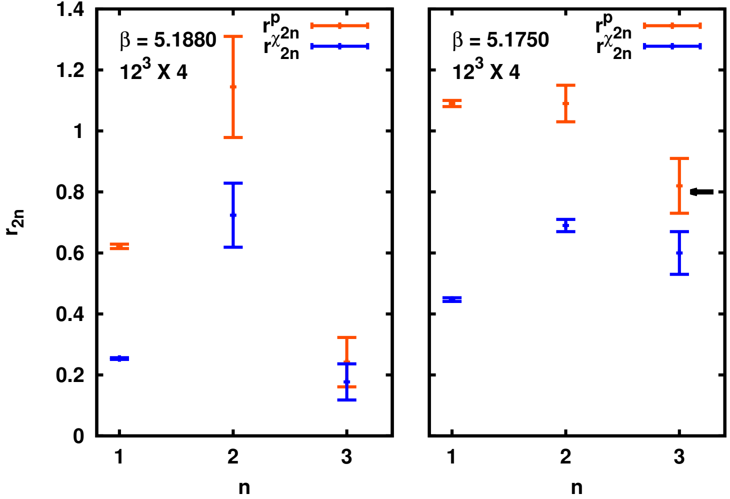
Ratio test $\rightarrow r^p_{2n} = \sqrt{\frac{p_{2n}}{p_{2n+2}}}$

(2) Ratio test for the susceptibility:

$$\chi_2(\mu) = 2p_2 + 12p_4\hat{\mu}^2 + 30p_6\hat{\mu}^4 + \dots$$

Ratio test $\rightarrow r^\chi_{2n} = \sqrt{\frac{2n(2n-1)}{(2n+1)(2n+2)}} r^p_{2n}$

Convergence of the first few coefficients? $N_t = 4$



Summary for $N_t = 4$

- The exact algorithm and the factor of 50 higher statistics did not change the old Fodor-Katz result
- The high statistics is especially important, since there might be an overlap problem
- The convergence radius estimators near β_E are actually in the same ballpark as the CEP estimate from the Lee-Yang zero analysis
- This might be coincidence, but at least it gives some hope

What about the fine lattices?

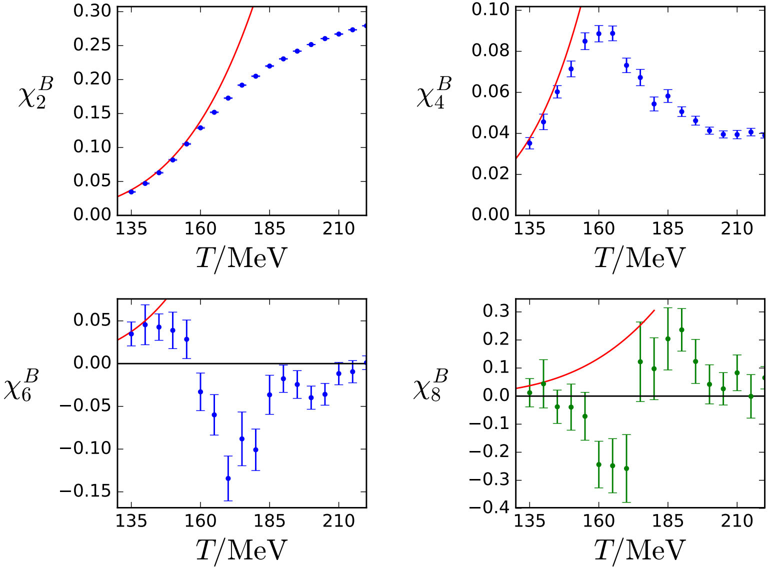
Our recent paper: hep-lat/1805.04445
On this conference:

- cross-correlators: J. Guenther
- equation of state at finite μ: Sz. Borsanyi
- here: do the results imply anything for criticality?

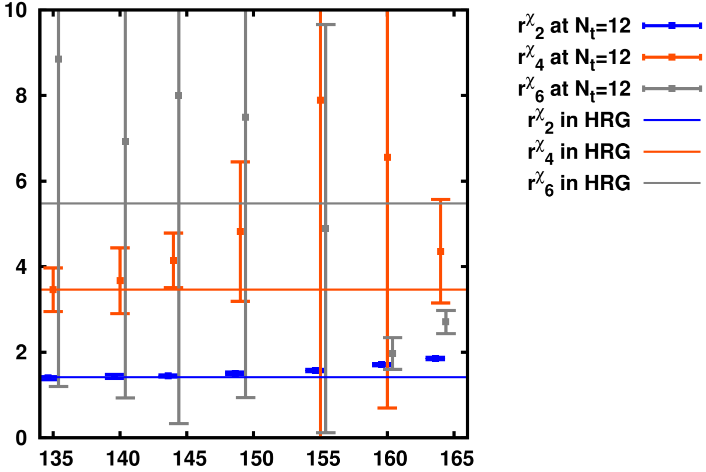
Comments on our methodology for $N_t = 12$:

- 4stout improved staggered action, $48^3 \times 12$ lattices
- The analysis uses simulations at imaginary chemical potentials combined with a Bayesian fit for the Taylor coefficients χ_2^B, χ_4^B, χ_6^B, χ_8^B and χ_{10}^B
- Only χ_8^B and χ_{10}^B have a prior. This prior:
  - allows for the HRG prediction but does not prefer it
  - allows for the coefficients to grow faster than in the HRG (a sign of criticality)

Fluctuations for the $N_t = 12$ 4stout ensembles



Ratios for the $N_t = 12$ 4stout ensembles



Why the jumping around?

Has to do with the sign structure of the fluctuations.

(1) Near and above T_c is the region where the closest Lee-Yang zero very likely has a large imaginary part

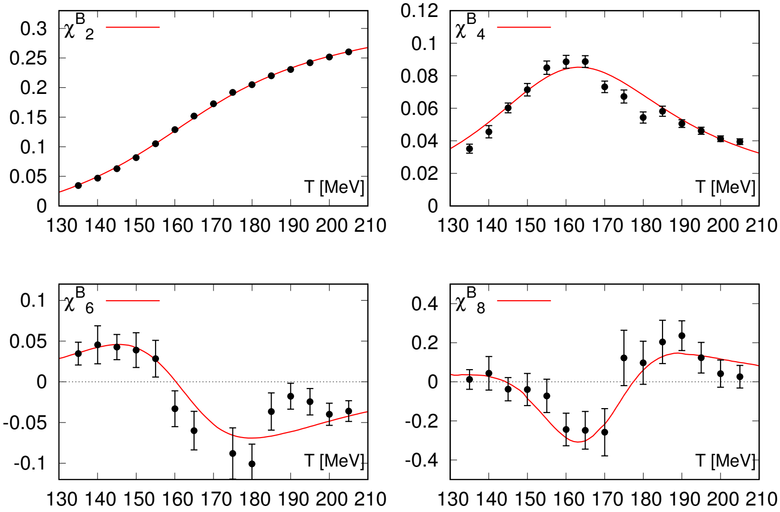
(2) Phenomenologically it has to do with the curvature of the crossover line

To understand this structure consider the following toy model:

- Start with some parametrization of the curve χ_1^B/μ_B at $\mu = 0$
- Assume that the only difference in the physics at finite μ is a shift in this curve
- The inflection point of this curve is one possible definition of T_c, so shift the curve by using the κ values found in the literature
- You now have a model prediction of χ_1^B for any finite μ, differentiate it a few times at $\mu = 0$ to get esimates of χ_4^B, χ_6^B and χ_8^B

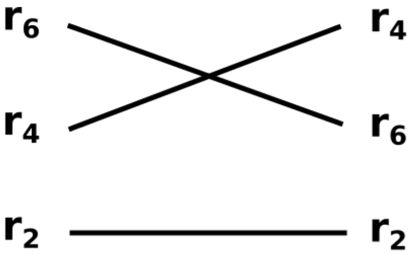
NOTE: The model assumes no criticality

Fluctuations for the $N_t = 12$ 4stout ensembles



RED CURVE: The simple model described in the previous slide This

Cartoon: Apparent convegence with no CEP



Below T_c: HRG   Near/above T_c: κ

Summary: Do the first few Taylor coefficients constrain the critical point?

- If a critical point is close, one may see it in a fast convergence. ($N_t = 4$?)
- Apparent convergence does not imply CEP. Even in the case with no CEP, the ratios $\sqrt{p_4/p_6}$ and $\sqrt{p_6/p_8}$ will show apparent convergence somewhere below T_c
- Near T_c, the curvature of the crossover implies $\sqrt{|p_2/p_4|} < \sqrt{|p_4/p_6|}$ and $\sqrt{|p_4/p_6|} > \sqrt{|p_6/p_8|}$ Here the closest Lee Yang zero most likely has a large imaginary part.
- For our fine lattice ($N_t = 12$ 4stout):
  - The sign structure of χ_6^B and χ_8^B near T_c is consistent with only a κ and no criticality.
  - At lower temperatures the data quickly become compatible with HRG
- None of these observations can be converted into a rigorous bound for the convergence radius.
- Does at least the crossover get stronger at finite μ? Maybe a little, see Sz. Borsanyi's talk.

Backup