The QCD equation of state at high temperatures

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Introduction

Earlier results on the equation of state

Lattice QCD setup

Trace anomaly

Cutoff dependence of the pressure

Continuum limit/estimate

Conclusion
Study response of the system to change of external parameters, i.e. temperature and baryon density, asymptotic freedom suggests a weakly interacting phase.

Experimental program: RHIC, LHC, FAIR, NICA

High-temperature phase: deconfinement, restoration of chiral symmetry

$^1$2+1 flavor QCD equation of state at zero baryon density has been recently calculated up to $T = 400 - 500$ MeV

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QCD phase diagram

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- Experimental program: RHIC, LHC, FAIR, NICA
- High-temperature phase: deconfinement, restoration of chiral symmetry
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Earlier results on the equation of state

- First perturbative EoS calculation\(^2\) (left)
- First lattice pure gauge \(SU(2)\) EoS calculation\(^3\) (right)

\(^2\)Kapusta (1979)
\(^3\)Engels et al. (1981)
Recent results up to $T = 400$ MeV

- Comparison of the continuum results with HISQ$^4$ and stout$^5$ for the trace anomaly, pressure and entropy density
- About $2\sigma$ deviations in the integrated quantities at the highest temperature

$^4$Bazavov et al. [HotQCD] (2014)
$^5$Borsanyi et al. [WB] (2014)
Approach to the perturbative limit

- The trace anomaly (left) and pressure (right) compared with (HTL)\(^6\) and Electrostatic QCD (EQCD)\(^7\) calculations
- The black line is the HTL calculation with the renormalization scale \(\mu = 2\pi T\)

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- THIS TALK: Extension of the 2+1 flavor equation of state to higher temperatures

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Lattice QCD

- Switch from Minkowski to Euclidean space – imaginary time formalism
- Define the theory on discrete space-time lattice $N_s^3 N_\tau$
- This is a gauge-invariant regularization scheme with the momentum cut-off $\pi/a$, $a$ – lattice spacing
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- Temperature is set as $T = 1/(aN_\tau)$
- Fix $N_\tau$, dial the lattice spacing to cover a temperature range
- The continuum limit is reached as $1/N_\tau \to 0$
Trace anomaly

- The partition function

\[ Z = \int DUD\bar{\psi}D\psi \exp\{-S\}, \quad S = S_g + S_f \]
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- The trace anomaly

\[ \Theta^{\mu\mu} \equiv \epsilon - 3p = -\frac{T}{V} \frac{d \ln Z}{d \ln a} \quad \Rightarrow \quad \frac{p}{T^4} - \frac{p_0}{T_0^4} = \int_{T_0}^T dT' \frac{\epsilon - 3p}{T'^5} \]
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- Requires subtraction of UV divergences (subtract divergent vacuum contribution evaluated at the same values of the gauge coupling):

\[ \frac{\varepsilon - 3p}{T^4} = R_\beta [\langle S_G \rangle_0 - \langle S_G \rangle_T] - R_\beta R_m [2m_l (\langle \bar{\ell}l \rangle_0 - \langle \bar{\ell}l \rangle_T) + m_s (\langle \bar{s}s \rangle_0 - \langle \bar{s}s \rangle_T)] \]

\[ R_\beta (\beta) = -a \frac{d \beta}{da}, \quad R_m (\beta) = \frac{1}{m} \frac{dm}{d\beta}, \quad \beta = \frac{10}{g^2} \]
HISQ data sets

- We use the Highly Improved Staggered Quarks\(^8\) action for two degenerate light quarks and physical-mass strange quark and the tree-level Symanzik-improved gauge action.

- Previous data set:

\[
\begin{align*}
m_l &= \frac{m_s}{20} \\
N_T &= 6, 8, 10, 12 \\
\beta &= 5.9, \ldots, 7.825
\end{align*}
\]

\(^8\)Follana et al. [HPQCD] (2007)
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- New data set:

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\begin{align*}
m_l &= m_s/5 \\
N_T &= 4, 6, 8, 10, 12 \\
\beta &= 8, 8.2, 8.4
\end{align*}
\]

\textsuperscript{8} Follana et al. [HPQCD] (2007)
The trace anomaly with HISQ $m_l = m_s/20$
The trace anomaly with HISQ $m_l = m_s/20$ and $m_l = m_s/5$ at $T > 400\,\text{MeV}$
We have improved the low-temperature region by adding $T = 123$ MeV at $N_\tau = 10$ and $T = 133, 140$ MeV at $N_\tau = 12$
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Bands are interpolations of the lattice data and the lines are Hadron Resonance Gas model results with the cutoff dependent spectrum.
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$N_T = 12$ has large statistical uncertainty.
The cutoff dependence of the pressure with HISQ and p4 is very similar to the cutoff dependence of quark number susceptibilities:

\[ \chi_{2n}^q = \frac{\partial^{2n} p(T, \mu_q)}{\partial \mu_q^{2n}}, \quad n = 1, 2, \quad q = u, d \]
Cutoff dependence of the pressure

- The cutoff dependence of the pressure with HISQ and p4 is very similar to the cutoff dependence of quark number susceptibilities:

\[ \chi_{2n}^q = \frac{\partial^2 p(T, \mu_q)}{\partial \mu_q^{2n}}, \quad n = 1, 2, \quad q = l, s \]

- At high temperatures where the weak-coupling picture is expected to hold we can write pressure as the sum of the quark and gluon pressures: 
\[ p(T) = p^q(T) + p^g(T) \]
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- The gluonic pressure is known to have negligible cutoff dependence for improved actions, thus we assume

\[ p(T) = p(T, N_\tau) + \text{corr}(T, N_\tau) \]

\[ \text{corr}(T, N_\tau) = p^q(T) \left( 1 - \frac{p^q(T, N_\tau)}{p^q(T)} \right) \]
We approximate the cutoff dependence of the quark pressure by the one of the second order susceptibilities $\chi^l_{2\,9}$

\[
\frac{p^q(T, N_\tau)}{p^q(T)} \approx \frac{\chi^l_{2}(T, N_\tau)}{\chi^l_{2}(T)}
\]

\(^9\)calculated in Bazavov et al. (2013)
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- We use the ideal quark pressure as an estimate for $p^q(T)$.

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- The overall estimate

\[
corr(T, N_\tau) \approx p^{q, id-15%}(T) \left(1 - \frac{\chi_2^l(T, N_\tau)}{\chi_2^l(T)}\right)
\]

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Corrected pressure at fixed cutoff and the continuum limit/estimate (black boxes)
Pressure: continuum limit

- At high temperatures the dominant cutoff dependence is the one of the ideal quark gas, thus the approach to continuum is
  \( \sim \frac{1}{N_T^4} \)

- At low temperatures the cutoff effects are due to taste-symmetry breaking of staggered fermions and scale like
  \( \sim \frac{1}{N_T^2} \)

- We use \( \frac{1}{N_T^4} \) fit for \( T > 200 \text{ MeV} \)

- In \( 200 \text{ MeV} < T < 660 \text{ MeV} \) we have four lattice spacings to perform continuum extrapolations, in \( 660 \text{ MeV} < T < 800 \text{ MeV} \) – three, and for \( T > 800 \text{ MeV} \) we can only provide a continuum estimate

- The continuum result for the pressure agrees with the corrected results: an important cross-check for continuum extrapolations for \( T < 1330 \text{ MeV} \)
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Trace anomaly above 800 MeV

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The $N_T = 4$ and $6$ results for the trace anomaly lie below this continuum estimate

We rescale the $N_T = 4$ and $6$ results on the trace anomaly by factors $1.2$ and $1.4$, respectively, to bring them in agreement with the continuum estimate for $800$ MeV $< T < 1000$ MeV
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Trace anomaly above 800 MeV

Perform a spline interpolation of the combined $N_\tau = 12, 10, 8, 6$ and 4 data in the temperature interval $400 \text{ MeV} < T < 2000 \text{ MeV}$

Integrate the trace anomaly from $T = 660 \text{ MeV}$ to 2000 MeV to get the pressure and the entropy density
Perform a spline interpolation of the combined $N_T = 12, 10, 8, 6$ and 4 data in the temperature interval $400 \text{ MeV} < T < 2000 \text{ MeV}$
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Weak-coupling expansions

- Left: Comparison of the pressure obtained on the lattice with the HTL\textsuperscript{10} and EQCD\textsuperscript{11} results
- Right: Comparison of the entropy density obtained on the lattice with the HTL and NLA\textsuperscript{12} results

\textsuperscript{10}Haque et al. (2014)  
\textsuperscript{11}Laine and Schroder (2006)  
\textsuperscript{12}Rebhan (2003)
Conclusion

- Previous results by the HotQCD collaboration for the 2+1 QCD equation of state at zero baryon chemical potential have been extended to higher temperatures.
- At temperatures above 400 MeV we use ensembles with $m_l = m_s/5$, the quark mass effects below the statistical uncertainties.
- Up to 660 MeV we perform the continuum limit with four lattice cutoffs.
- At high temperatures the cutoff effects in the pressure are similar to the ones in quark number susceptibilities.
- We use two different methods to estimate the continuum pressure up to 1330 MeV.
- In the interval 660 to 2000 MeV we provide a continuum estimate based on the rescaled $N_T = 4$ and 6 results.
- Reasonable agreement between the weak-coupling results and the lattice at high temperature.