Causal Charge Diffusion and Fluctuations in Heavy-Ion Collisions:

Modeling Diffusion Without Breaking the Speed of Light

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Abstract

How does one model diffusion in heavy-ion collisions without breaking the speed of light?

Fluctuating hydrodynamics [1] - or ‘noise’ - provides a useful framework for understanding and modeling diffusion of conserved charges in heavy-ion collisions. The simplest formulations of this framework, however, are known to produce acausal signal propagation.

Here we explore one way of restoring relativistic causality in a simplified, analytically solvable model of heavy-ion evolution. We show that the causal formalism affects both the density correlations in the system, as well as experimental observables such as the charge balance functions [2, 3].

White and Colored Noise

There are two kinds of noise:

1. White noise
   (a) Produces ***ordinary*** diffusion: $\left(\partial_t + D_Q \partial_x^2\right) n_Q = 0$
   (b) Pro: simple to implement
   (c) Con: acausal signal propagation ($v_Q^2 \to \infty$)
2. Colored noise
   (a) Produces ***causal*** diffusion: $\left(\partial_t + t_Q \partial_t^2 + D_Q \partial_x^2\right) n_Q = 0$
   (b) Pro: causal ($v_Q^2 = D_Q/t_Q =$ finite)
   (c) Con: more complicated implementation

Bjørken flow

Bjørken coordinates are related to the standard Cartesian ones by

$$(t, z) = (\tau \cosh \xi, \tau \sinh \xi)\,, \tag{1}$$

so that Bjørken flow, defined by

$$u^\mu = \left(\cosh \xi, \vec{0}_\perp, \sinh \xi\right)^\mu, \tag{2}$$

is invariant under longitudinal Lorentz boosts. Derivatives orthogonal to u^μ are taken by the operator

$$\Delta^\mu = \partial^\mu - u^\mu\,(u \cdot \partial) \tag{3}$$

Electric charge diffusion

Electric charge diffuses in heavy-ion collisions according to the conserved current density [3]

$$J_Q^\mu = n_Q u^\mu + \left(\frac{D_Q T}{\chi_Q}\right) \Delta^\mu (1 + \tau_Q u \cdot \partial)^{-1} \left(\frac{n_Q}{\chi_Q T}\right) + I^\mu, \tag{4}$$

where the noise term I^μ assumes the form

$$I^\mu = s(\tau) f(\xi, \tau) \left(\sinh \xi, \vec{0}_\perp, \cosh \xi\right), \tag{5}$$

s is the entropy density and f is a dimensionless random variable which sources fluctuations of electric charge δn_Q. The rest of our notation is defined in the Figure caption.

Requiring that $\partial_\mu J_Q^\mu = 0$ and applying the Fourier transform defined by

$$X(\xi, \tau) = \int_\infty^\infty \frac{dk}{2\pi} e^{ik\xi} \tilde{X}(k, \tau) \tag{6}$$

means that

$$\frac{\partial^2}{\partial \tau^2}(\tau \delta \tilde{n}) + \left[\frac{1}{\tau_Q} - \frac{\partial}{\partial \tau} \ln\left(\frac{\chi_Q T D_Q}{\tau}\right)\right] \frac{\partial}{\partial \tau}(\tau \delta \tilde{n}) + \frac{v_Q^2 k^2}{\tau^2}(\tau \delta \tilde{n})$$
$$= -iks\left[\frac{\partial \tilde{f}}{\partial \tau} + \left(\frac{1}{\tau_Q} - \frac{1}{\tau} - \frac{\partial}{\partial \tau} \ln\left(\frac{\chi_Q T D_Q}{\tau}\right)\right) \tilde{f}\right]. \tag{7}$$

The homogeneous solutions to this equation are given in terms of Kummer’s function $M(a, b, x)$ [4]:

$$\psi_\pm(x) = x^{\lambda_\pm - 1/2} e^{-x} M\left(\lambda_\pm + \frac{3}{2}, 2\lambda_\pm + 1, x\right), \tag{8}$$

with $\psi = \tau \delta \tilde{n}$, $x = \tau/\tau_Q$, and $\lambda_\pm = \pm\sqrt{\frac{1}{4} - v_Q^2 k^2}$. Additionally, the signal propagation speed $v_Q^2 = D_Q/\tau_Q$, meaning that we can obtain (acausal) white noise from our results by taking the limit $\tau_Q \to 0^+$, corresponding to $v_Q^2 \to \infty$.

Correlation Functions and Noise

The colored noise source f fluctuates on average to zero, $\left\langle \tilde{f}(\tau, k) \right\rangle = 0$, and satisfies a modified form of the fluctuation-dissipation theorem:

$$\left\langle \left(1 + \tau_Q \frac{\partial}{\partial \tau_1}\right) \tilde{f}(\tau_1, k_1) \left(1 + \tau_Q \frac{\partial}{\partial \tau_2}\right) \tilde{f}(\tau_2, k_2) \right\rangle$$
$$= N(\tau_1) \delta(\tau_1 - \tau_2) \delta(k_1 + k_2); \quad N(\tau) = \frac{4\pi D_Q T(\tau)}{A \tau \chi_Q(\tau) s^2(\tau)}. \tag{9}$$

Equivalently, for a system created at τ_0,

$$\left\langle \tilde{f}(\tau_1, k_1) \tilde{f}(\tau_2, k_2) \right\rangle = \frac{2\pi \delta(k_1 + k_2)}{\tau_Q^2 \exp\left(|\tau_2' - \tau_1'|/\tau_Q\right)} \times \int_{\tau_0}^{\min(\tau_1', \tau_2')} d\tau N(\tau) e^{-2[\min(\tau_1', \tau_2') - \tau]/\tau_Q} \tag{10}$$

Results

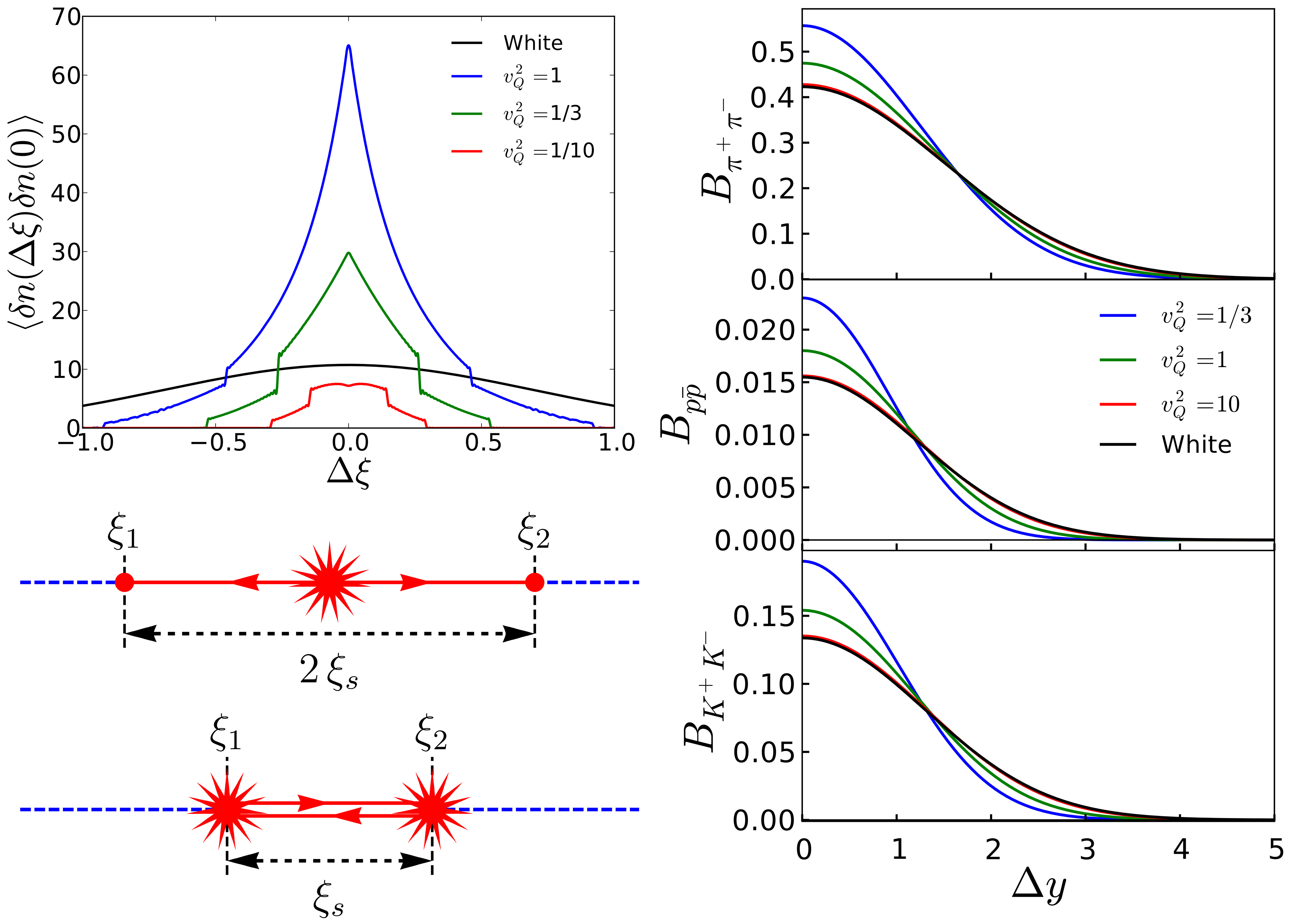

Figure: (Left column) - The evolution of density correlations, at a fixed instant in time τ, for various values of v_Q. For finite v_Q, the correlator contains wavefronts whose interpretation is illustrated schematically in the diagrams below; the outer set of wavefronts represents the “light cone" of a single fluctuation, while the inner wavefronts represent the correlations of two fluctuations occurring a distance ξ_s apart. Here $\xi_s \equiv v_Q \ln(\tau/\tau_0) \approx 0.266$. Since white noise corresponds to $v_Q \to \infty$, it does not any generate wavefronts.

(Right column) - Charge balance functions for various values of v_Q. With $v_Q \to \infty$, white noise corresponds to maximally efficient diffusion and generates the widest balance functions. Restoring relativistic causality leads to less efficient diffusion, narrow balance functions, and enhancement of charged hadron pairs near small momentum rapidity separation Δy.

Notation: Quantities with a ‘Q’ subscript describe electric charge. This includes D_Q (the diffusion coefficient), τ_Q (the noise timescale), v_Q (the speed of propagation), n_Q (the charge density), and χ_Q (the charge susceptibility). The system temperature is given by T.

Density Correlations

Density correlations are determined by

$$\langle \delta \tilde{n}(k_1, \tau_f) \delta \tilde{n}(k_2, \tau_f) \rangle$$
$$= \frac{1}{\tau_f^2} \int_{\tau_0}^{\tau_f} d\tau_1' s(\tau_1') \tilde{G}(k_1; \tau_f, \tau_1') \int_{\tau_0}^{\tau_f} d\tau_2' s(\tau_2') \tilde{G}(k_2; \tau_f, \tau_2')$$
$$\times \left\langle \tilde{f}(\tau_1', k_1) \tilde{f}(\tau_2', k_2) \right\rangle, \tag{11}$$

where the Green’s function $\tilde{G}(k; \tau_1, \tau_2)$ is a linear combination of the $\psi_\pm$ given above.

Charge Balance Functions

We determine the single-particle spectra from the Cooper-Frye procedure:

$$\frac{dN}{dy} = \frac{d_s A \tau_f}{(2\pi)^3} \int d\xi \cosh(y - \xi) \int d^2 p_\perp m_\perp \times \exp\left[-\left(m_\perp \cosh(y - \xi) - \mu\right)/T_f\right],$$

where d_s is the spin degeneracy, A the transverse area of the system, and τ_f the proper time defining the freeze-out surface. Then the charge balance function is given by [2]

$$B(\Delta y) \equiv \left\langle \delta\left(\frac{dN}{dy_1}\right) \delta\left(\frac{dN}{dy_2}\right) \right\rangle \left\langle \frac{dN}{dy} \right\rangle^{-1}, \tag{12}$$

where

$$\delta\left(\frac{dN}{dy}\right) \equiv \frac{dN}{dy} - \left\langle \frac{dN}{dy} \right\rangle \tag{13}$$

AdS/CFT and colored noise

The AdS/CFT correspondence is often used to guide the choice of transport coefficients in the strongly coupled sector of QCD (e.g., the famous KSS bound on η/s of $1/4\pi$). Matching the fluctuating current density used above onto the density motivated by holographic considerations [5, 6] leads to the estimates

$$\tau_Q \sim D_Q \sim \frac{1}{2\pi T} \Longrightarrow v_Q \sim 1 \tag{14}$$

The condition on D_Q is a rough approximation of the relevant lattice data [7], and in general, these parameters all depend on τ in a heavy-ion collision.

For simplicity, in this study both D_Q and τ_Q are chosen to be independent of τ. We take v_Q as a free parameter and use this to determine the corresponding value of τ_Q, where D_Q is taken from the lattice and averaged over the relevant temperature interval $T \sim 150 - 350$ MeV.

We also take $\chi_Q = 2T^2/3$.

Conclusions

We have focused in this study on the effects of relativistic causality, in the form of colored hydrodynamic fluctuations, on the description of diffusion and charge balance functions in a simple, analytically solvable model of heavy-ion collisions.

We find results which are intuitively very reasonable: imposing a relativistic “speed limit" reduces the efficiency of the diffusive mechanism, thereby narrowing balance functions and enhancing the number of charge pairs at small rapidity separation. We therefore expect that our results will be important both for qualitative and quantitative understanding of heavy-ion experimental data.

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This work was supported by the U. S. Department of Energy grant DE-FG02-87ER40328.