Non-linear dynamical systems approach to out of equilibrium hydrodynamical attractors: the Gubser flow case

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Abstract

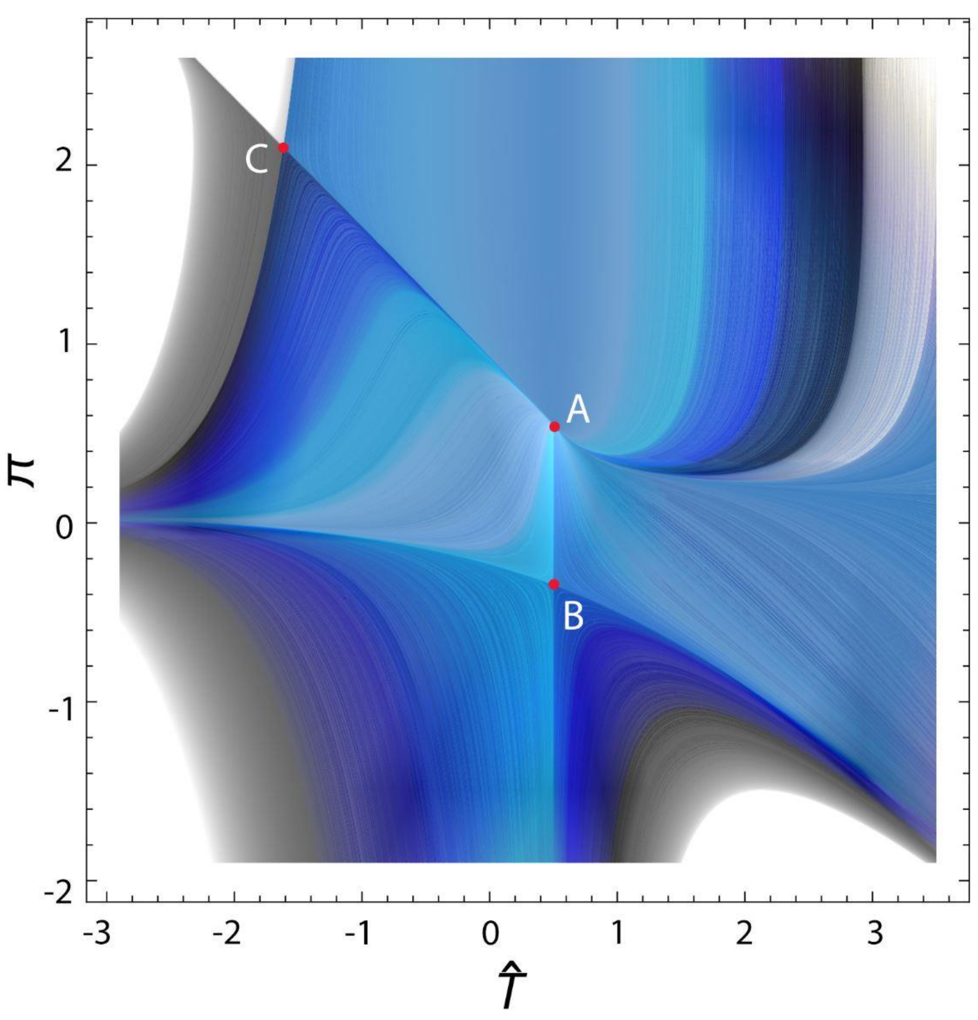
The non-equilibrium attractors of systems undergoing Gubser flow within relativistic kinetic theory are studied. In doing so we employ well-established methods of nonlinear dynamical systems which rely on finding the fixed points, investigating the structure of the flow diagrams of the evolution equations near the stable fixed points. We obtain the attractors of anisotropic hydrodynamics, Israel-Stewart (IS) and transient fluid (DNMR) theories and show that they are indeed non-planar and the basin of attraction is essentially three dimensional. The asymptotic attractors of each hydrodynamical model are compared with the one obtained from the exact Gubser solution of the Boltzmann equation within the relaxation time approximation. We observe that the anisotropic hydrodynamics is able to match up to high numerical accuracy the attractor of the exact solution while the second order hydrodynamical theories fail to describe it. We show that the IS and DNMR asymptotic series expansion diverge. Our findings indicate that the reorganization of the expansion series carried out by anisotropic hydrodynamics resums the Knudsen and inverse Reynolds numbers to all orders and thus, it can be understood as an effective theory for the far-from-equilibrium fluid dynamics.

Dynamical system analysis for IS theory

$\tau = \tanh\rho$

2d system

$$\frac{d\hat{T}}{d\tau} = \frac{\tau\hat{T}}{3(1-\tau^2)}\left(\bar{\pi}(\tau)-2\right), \quad \frac{d\bar{\pi}}{d\tau} = -\frac{1}{1-\tau^2}\left(\frac{4}{3}\bar{\pi}^2(\tau)\,\tau + \frac{1}{c}\,\bar{\pi}(\tau)\,\hat{T}(\tau) - \frac{4}{15}\,\tau\right).$$



Null-line condition

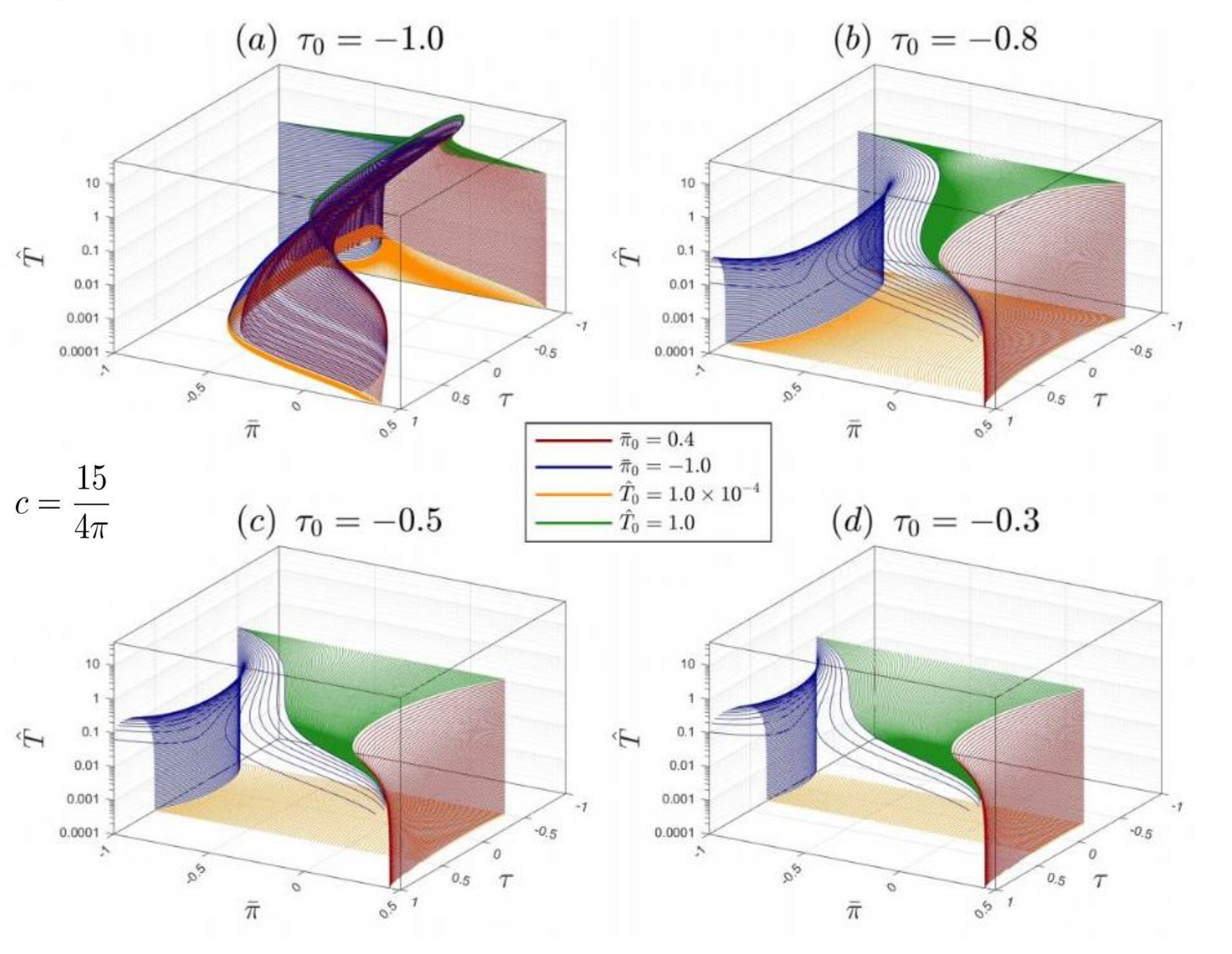
$$\frac{d\hat{T}}{d\tau} = 0 \qquad \frac{d\bar{\pi}}{d\tau} = 0$$

Fixed points

$$\bar{\pi}_c^{\pm} = \pm\frac{1}{\sqrt{5}}, \quad \hat{T}_c = 0, \qquad \bar{\pi}_c = 2, \quad \hat{T}_c = -\frac{38c}{15}.$$

3d system

$$\frac{d\hat{T}}{d\rho} = \frac{1}{3}\hat{T}(\bar{\pi}-2)\tau, \quad \frac{d\bar{\pi}}{d\rho} = \frac{4}{3}\left(\frac{1}{5}-\bar{\pi}^2\right)\tau - \frac{1}{c}\bar{\pi}\hat{T}, \quad \frac{d\tau}{d\rho} = 1-\tau^2.$$



Attractor: $\mathcal{A} \sim \hat{T}_0\, e^{\lambda_{\hat{T}}\rho}\mathbf{u_1} + (\frac{1}{\sqrt{5}} - \bar{\pi}_0\, e^{\lambda_{\bar{\pi}}\rho})\mathbf{u_2} + \mathbf{u_3}$

Lyapunov exponents: $\lambda_{\hat{T}} = -\frac{2}{3} + \frac{1}{3\sqrt{5}}, \quad \lambda_{\bar{\pi}} = -\frac{8}{3\sqrt{5}}, \quad \lambda_{\tau} = -2.$

The Gubser flow

- Gubser flow is invariant under the symmetry group

$$SO(3)_q \otimes SO(1,1) \otimes Z_2$$

$SO(3)_q$: Special conformal transformation + rotations; $SO(1,1)$: Longitudinal boost invariance; Z_2: Reflections along the beam line

- Symmetries become manifest by considering the conformal map between Minkowski and the 3 dimensional De Sitter space times a line.
- In De Sitter space, the fluid velocity is a static flow, $\hat{u}^{\mu} = (1,0,0,0)$

$$ds^2 = -d\tau^2 + \tau^2 d\varsigma^2 + dr^2 + r^2 d\phi^2$$

Weyl rescaling: $\tau = \sqrt{t^2-z^2}$, $\varsigma = \tanh^{-1}(z/t)$

$$d\hat{s}^2 = \frac{ds^2}{\tau^2}$$

Coordinate transformation: $\rho = -\sinh^{-1}\left(\frac{1-q^2\tau^2+q^2r^2}{2q\tau}\right)$, $\theta = \tanh^{-1}\left(\frac{2qr}{1+q^2\tau^2-q^2r^2}\right)$

$$d\hat{s}^2 = \underbrace{-d\rho^2 + \cosh^2\rho\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)}_{dS_3} + \underbrace{d\varsigma^2}_{R}$$

Boltzmann equation and hydrodynamical models

The Boltzmann equation within Relaxation Time Approximation is

$$p_\mu \partial^\mu f(x^\mu, p_i) = -\frac{1}{\tau_{rel}}\left(f(x^\mu,p_i) - f_{eq.}(x^\mu,p_i)\right)$$

$$\tau_{rel} = \frac{c}{T(x^\mu)} \qquad c = 5\frac{\eta}{\mathcal{S}}$$

Remember that:

$$Kn,\, Re^{-1} \sim \frac{\tanh\rho}{\hat{T}}$$

The fluid dynamical equations for the following truncation schemes (Israel-Steward (IS), transient fluid (DNMR) and anisotropic hydrodynamics with P_L prescription (aHydro)) are:

IS / DNMR:

$$\hat{\tau}_\pi\left(\partial_\rho\bar{\pi} + \frac{4}{3}(\bar{\pi})^2\tanh\rho\right) + \bar{\pi} = \frac{4}{3}\frac{\eta}{s\hat{T}}\tanh\rho + \frac{10}{7}\hat{\tau}_\pi\bar{\pi}\tanh\rho$$

aHydro:

$$\partial_\rho\bar{\pi} + \frac{\bar{\pi}}{\hat{\tau}_r} = \frac{4}{3}\tanh\rho\left(\frac{5}{16} + \bar{\pi} - \bar{\pi}^2 - \frac{9}{16}\mathcal{F}(\bar{\pi})\right)$$

Due to energy-momentum conservation for all theories

$$\frac{\partial_\rho\hat{T}}{\hat{T}} + \frac{2}{3}\tanh\rho = \frac{\bar{\pi}}{3}\tanh\rho$$

Universal asymptotic attractor

Introducing $w = \tanh\rho/\hat{T}$ and

$$\mathcal{A}(w) = \frac{1}{\tanh\rho}\frac{\partial_\rho\hat{T}}{\hat{T}} = \frac{d\log(\hat{T})}{d\log(\cosh\rho)}$$

we can combine the hydrodynamical equations as

$$3w\left(\coth^2\rho - 1 - \mathcal{A}(w)\right)\frac{d\mathcal{A}(w)}{dw} + H(\mathcal{A}(w),w) = 0$$

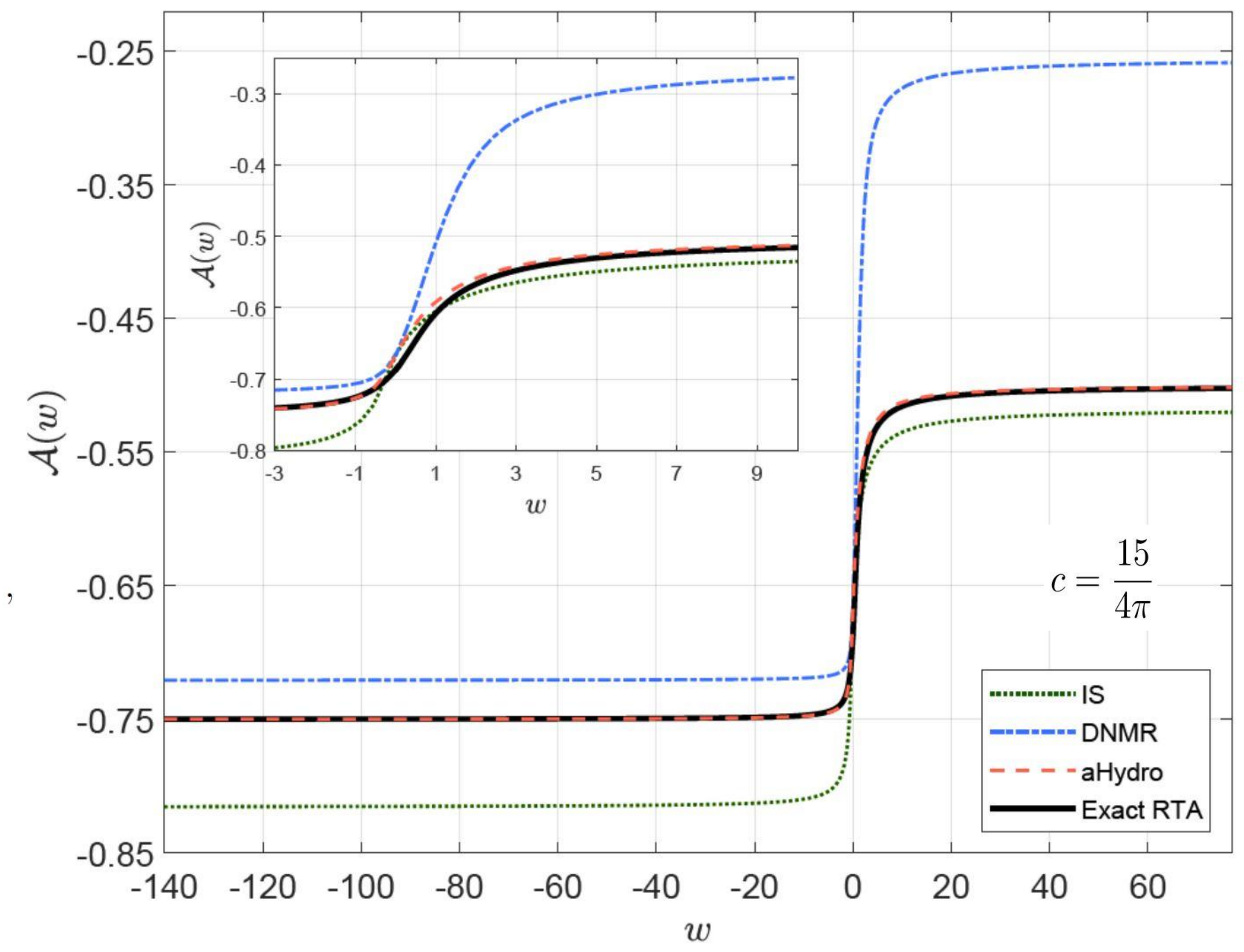
where for each hydro theory

$$H_{\rm IS} = \tfrac{4}{3}\left(3\mathcal{A}(w)+2\right)^2 + \tfrac{3\mathcal{A}(w)+2}{cw} - \tfrac{4}{15},$$

$$H_{\rm DNMR} = \tfrac{4}{3}\left(3\mathcal{A}(w)+2\right)^2 + \left(3\mathcal{A}(w)+2\right)\left[\tfrac{1}{cw} - \tfrac{10}{7}\right] - \tfrac{4}{15},$$

$$H_{\rm aHydro} = \tfrac{4}{3}\left(3\mathcal{A}(w)+2\right)^2 + \left(3\mathcal{A}(w)+2\right)\left[\tfrac{1}{cw} - \tfrac{4}{3}\right] - \tfrac{5}{12} + \tfrac{3}{4}\mathcal{F}\left(3\mathcal{A}(w)+2\right).$$

The asymptotic attractor is found based on slow-roll approximation via two steps:
1) Find roots of H. 2) Solve the differential eq.



Divergence of IS and DNMR theories

Using the following ansatz

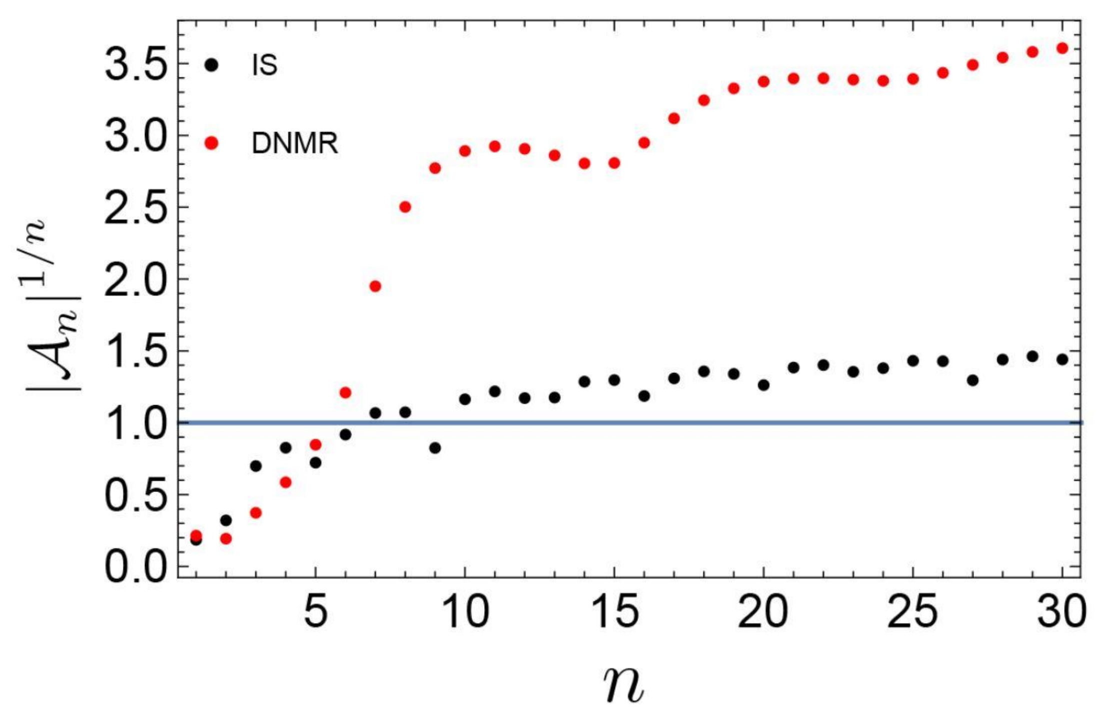
$$\mathcal{A}(w) = \sum_{n=0}^{\infty}\mathcal{A}_n w^{-n}$$

We obtain the recursive relations ($k > 2$)

IS: $$\sum_{n=0}^{k}(n+4)\mathcal{A}_n\mathcal{A}_{k-n} + \frac{\mathcal{A}_{k-1}}{c} + \frac{16\mathcal{A}_k}{3} = 0$$

DNMR: $$\sum_{n=0}^{k}(n+4)\mathcal{A}_n\mathcal{A}_{k-n} + \frac{\mathcal{A}_{k-1}}{c} + \frac{82\mathcal{A}_k}{21} = 0$$

From them we observe a divergent behavior which can be resumed via resurgence approach. [4]



Conclusions

- The stability properties of the IS theory were studied by considering well known methods of non-linear dynamical systems: fixed points, flow-lines, Lyapunov exponents, and dimensionality of the basin of attraction (3 dimensional for the Gubser flow).
- The asymptotic attractor does not depend on the Knudsen number or inverse Reynolds number.
- Anisotropic hydrodynamics is able to describe the exact asymptotic attractor to high numerical accuracy and resums effectively the Knudsen and inverse Reynolds number to all orders.

Selected references

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