Metric anisotropies and emergent anisotropic hydrodynamics

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Introduction

- Expansion of a locally equilibrated fluid is considered in an anisotropic space-time given by Bianchi type I metric.
- We obtain expressions for number density, energy density and pressure components in terms of anisotropy parameters of the metric.
- In the case of an axis-symmetric Bianchi type I metric, we show that they are identical to that obtained within the setup of anisotropic hydrodynamics.
- We further consider the case when Bianchi type I metric is a vacuum solution of Einstein equation: the Kasner metric.
- For axis-symmetric Kasner metric, we discuss the implications of our results in the context of anisotropic hydrodynamics.

The metric

The most general anisotropic Bianchi type I metric is [1]

$$ds^2 = dt^2 - g_{ij}dx^i dx^j \tag{1}$$

When there is no a priori preferred direction the metric simply takes a diagonal form given as

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)dy^2 - C^2(t)dz^2. \tag{2}$$

The quantities $A(t)$, $B(t)$, and $C(t)$ are scale factors for the expansion along x, y, and z axes.

Collisionless stress-energy tensor

The stress energy tensor and conserved current is defined as

$$T^{\mu\nu} = \int \sqrt{-g}\,\frac{d^3p}{p^0}\,p^\mu p^\nu f(x,p), \qquad N^\mu = \int \sqrt{-g}\,d^3p\,p^\mu f(x,p). \tag{3}$$

We consider ultrarelativistic particles and ignore the particle masses. Therefore

$$E_0 = p_0|_{t=t_0} = \left[\left(\frac{p_1}{A(t_0)}\right)^2 + \left(\frac{p_2}{B(t_0)}\right)^2 + \left(\frac{p_3}{C(t_0)}\right)^2\right]^{1/2}. \tag{4}$$

The gas decouples from its surrounding happens at time $t = t_0$, such that after time t_0 the gas experiences a collision-less adiabatic expansion or contraction as specified by the metric. Also, Liouville's theorem guarantees that the distribution function $f(x,p)$ remains constant throughout the phase space for all time during the evolution. This in turn implies that the energy E and the temperature T, at a given time t, are red-shifted by the same amount, i.e.,

$$\frac{E}{E_0} = \frac{T}{T_0} = z \tag{5}$$

where z is the usual red-shift factor.

The 3-momenta p_i are constants of motion, i.e., $dp_i/d\tau = 0$, we get:

$$E_0 = \left[\left(\frac{A^2(t)p^1}{A(t_0)}\right)^2 + \left(\frac{B^2(t)p^2}{B(t_0)}\right)^2 + \left(\frac{C^2(t)p^3}{C(t_0)}\right)^2\right]^{1/2}. \tag{6}$$

Using the above equations, we find the red-shift factor z to be

$$z = \left[\left(\frac{A(t)\sin\theta\,\sin\phi}{A(t_0)}\right)^2 + \left(\frac{B(t)\sin\theta\,\cos\phi}{B(t_0)}\right)^2 + \left(\frac{C(t)\cos\theta}{C(t_0)}\right)^2\right]^{-1/2} \tag{7}$$

From the above equation, we see that the characteristic temperature is dependent on the direction of motion of particles. For axis-symmetric case, i.e $\frac{A(t)}{A(t_0)} = \frac{B(t)}{B(t_0)} = \xi_1$ and $\frac{C(t)}{C(t_0)} = \xi_3$, we get:

$$n = \frac{n_0}{2}\int_{-1}^{1}\left(\lambda^2(\xi_3^2 - \xi_1^2) + \xi_1^2\right)^{-3/2} d\lambda, \tag{8}$$

$$\varepsilon = \frac{\varepsilon_0}{2}\int_{-1}^{1}\left(\lambda^2(\xi_3^2 - \xi_1^2) + \xi_1^2\right)^{-2} d\lambda, \tag{9}$$

$$\mathcal{P}_\perp = \frac{3(P_\perp)_0}{4}\int_{-1}^{1}(1-\lambda^2)\left(\lambda^2(\xi_3^2 - \xi_1^2) + \xi_1^2\right)^{-2} d\lambda, \tag{10}$$

$$\mathcal{P}_\parallel = \frac{3(P_\parallel)_0}{2}\int_{-1}^{1}\lambda^2\left(\lambda^2(\xi_3^2 - \xi_1^2) + \xi_1^2\right)^{-2} d\lambda, \tag{11}$$

where $\lambda = \cos\theta$, we have defined $\mathcal{P}_x = \mathcal{P}_y = \mathcal{P}_\perp$ and $\mathcal{P}_z = \mathcal{P}_\parallel$.

Integrating Eqs. (8)-(11), we get

$$n = \frac{n_0}{\xi_1^3\xi^{1/2}}, \quad \varepsilon = \frac{\varepsilon_0\mathcal{R}(\xi)}{\xi_1^4}, \quad \mathcal{P}_\perp = \frac{3(\mathcal{P}_\perp)_0}{2\xi_1^4}\left(\frac{1+\xi(\xi-2)\mathcal{R}(\xi)}{\xi(\xi-1)}\right), \tag{12}$$

$$\mathcal{P}_\parallel = \frac{3(\mathcal{P}_\parallel)_0}{\xi_1^4}\left(\frac{\xi\mathcal{R}(\xi)-1}{\xi(\xi-1)}\right), \quad \mathcal{R}(\xi) = \frac{1}{2\xi}\left(1 + \frac{\xi\arctan\sqrt{\xi-1}}{\sqrt{\xi-1}}\right) \tag{13}$$

for $\xi > 1$ where $\xi = \frac{\xi_3^2}{\xi_1^2}$, while we substitute ξ as $\frac{1}{\xi}$ for $\xi < 1$. One also arrives at the same results by considering the collisionless Boltzmann equation [2].

The Kasner metric

We consider the vacuum solutions of Einstein's equation, the Kasner metric [3]

$$ds^2 = dt^2 - t^{2a}dx^2 - t^{2b}dy^2 - t^{2c}dz^2, \tag{14}$$

where a, b and c are three parameters related to each other by the equations

$$a + b + c = 1, \qquad a^2 + b^2 + c^2 = 1. \tag{15}$$

Since the particle current must be conserved, the number density n of particles that is measured by a co-moving observer satisfies the continuity equation

$$\frac{dn}{dt} + \Gamma^i_{i0}n = 0. \tag{16}$$

The non-vanishing Christoffel symbols for Kasner metric are:

$$\Gamma^1_{10} = \frac{a}{t}, \qquad \Gamma^2_{20} = \frac{b}{t}, \qquad \Gamma^3_{30} = \frac{c}{t}. \tag{17}$$

Using Eq. (17) in Eq. (16) we have

$$\frac{dn}{dt} + (a+b+c)\frac{n}{t} = 0 \quad \Rightarrow \quad \frac{dn}{dt} + \frac{n}{t} = 0 \quad \Rightarrow \quad n = \frac{n_0 t_0}{t} \tag{18}$$

It is interesting to note that the above equation holds for all Kasner type expansion. The Milne metric turns out to be a special case of Kasner metric.

If we impose an additional constraint of azimuthal symmetry, we have only two possibilities for (a,b,c):

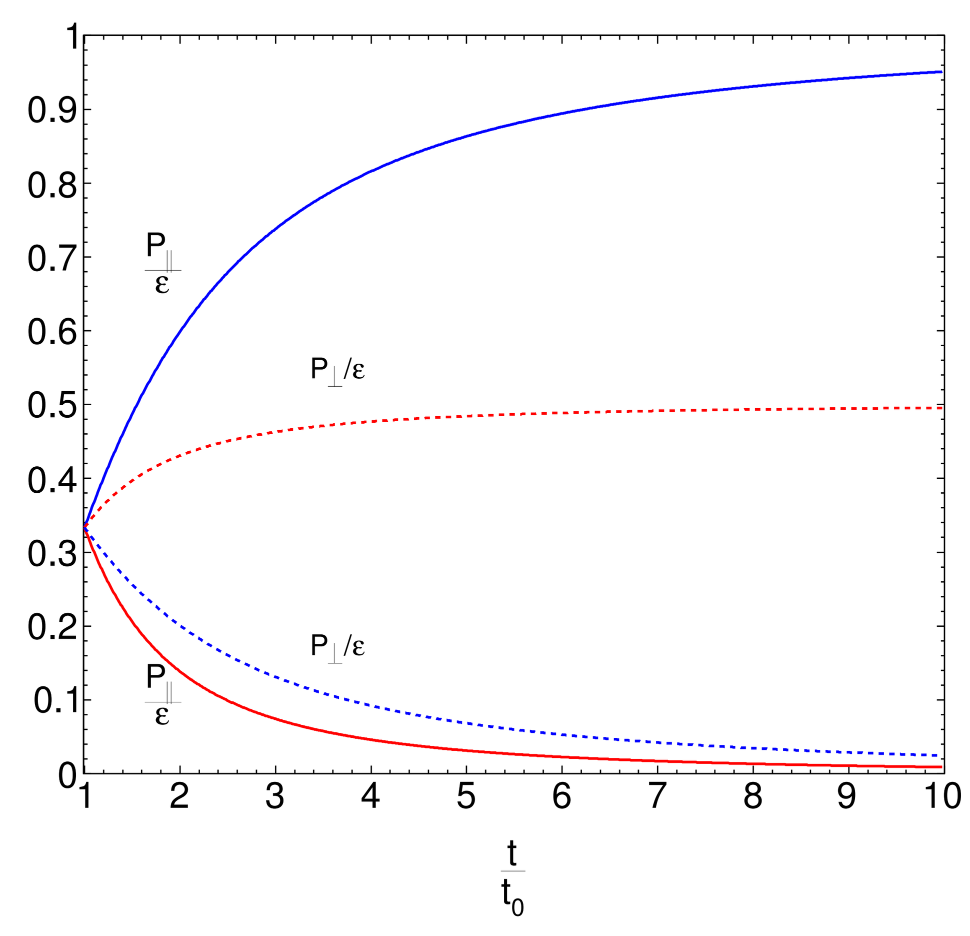
$$\text{Case I: } (0,0,1) \qquad \text{Case II: } \left(\frac{2}{3},\frac{2}{3},-\frac{1}{3}\right). \tag{19}$$

Case I: usual Milne coordinates; Case-II: a new finding in the context of azimuthally symmetric anisotropic hydrodynamics.

Imposing the condition in Eq. (19) on the variable ξ gives us

$$\text{Case I: } \xi = \frac{t^2}{t_0^2} \qquad \text{Case II: } \xi = \frac{t_0^2}{t^2} \tag{20}$$

Case I refers refers to longitudinal expansion while Case II denotes transverse expansion. We note that Case I corresponds to the usual free streaming solution in Bjorken expansion.



Evolution of longitudinal and transverse pressures, scaled by the energy density, for Case I (red) and Case II (blue).

References

[1] C. W. Misner, Astrophys. J. **151**, 431 (1968).

[2] A. Dash and A. Jaiswal, Phys. Rev. D **97**, 104005 (2018).

[3] C. W. Misner, K. S. Thorne and J. A. Wheeler, "Gravitation," W. H. Freeman and Co., (1970).