Special Relativity, Quantum Mechanics & Quantum Field Theory

Benjamin Fuks

LPTHE / Sorbonne Université

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Outline

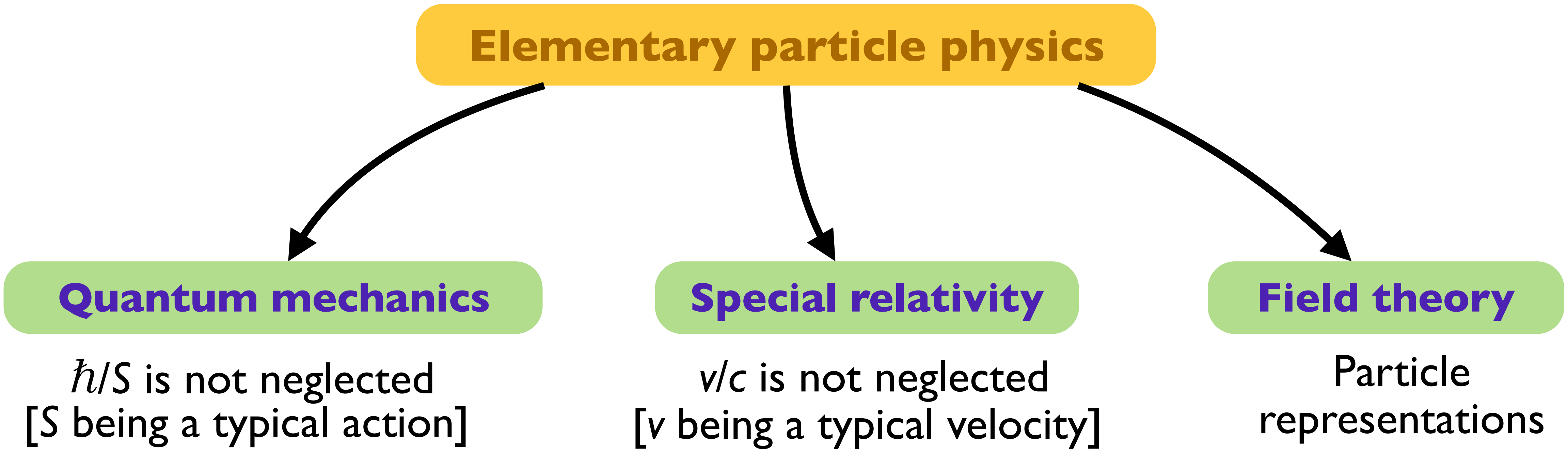
1. Introduction
2. Special relativity
3. Some quantum mechanics
4. Relativistic quantum mechanics
5. Gauge symmetries
6. Scattering theory
7. Summary

Goals

- Some of the goals of this school
  - An overview of what is fundamental physics, including some of its applications
    - Requires some prerequisites
- Aims of this lecture
  - Covering the basics
    - Special relativity
    - Quantum mechanics
    - Field theory
    - Gauge theories
    - Scattering theory
  - Ask questions anytime!

Setting up the stage

✦ In a nutshell



Elementary particle physics

Quantum mechanics

$\hbar/S$ is not neglected
[S being a typical action]

Special relativity

v/c is not neglected
[v being a typical velocity]

Field theory

Particle
representations

✦ The elementary particles and their interactions are linked to **symmetries**

- Poincaré invariance ≡ particle types (scalars, fermions, vectors, …)
- Gauge symmetries ≡ electromagnetism, weak and strong interactions, …

✦ Other tools

- Symmetry breaking ≡ mass generation
- Scattering theory ≡ observables, testing a theory against data

More on symmetries

- Definition
  - A symmetry operation leaves the laws of physics invariant
    - Newton's law is the same in any inertial frame $\boxed{\mathbf{F} = m\ \mathbf{a}}$
- Important classes of symmetries for particle physics
  - External (spacetime) symmetries: rotations, translations, boosts
  - Internal symmetries: like in quantum mechanics $\boxed{|\Psi\rangle \to e^{i\alpha}|\Psi\rangle}$
- Nœther theorem
  - A conserved charge can be associated with each symmetry
  - Exampes of conservation laws: electric charge, energy, angular momentum, *etc*.

- Wigner: a symmetry operator G is (anti-)unitary
  - For unitary operators, we introduce a set of Hermitian matrices g_i (generators)
    - ★ $G = \exp[a^i\, g_i]$ (e.g., a rotation $R(\boldsymbol{\alpha}) = \exp[i\, \boldsymbol{\alpha} \cdot \mathbf{J}]$)
    - ★ The $\{g_i\}$ forms an algebra $[g_i, g_j] = f_{ij}{}^k\, g_k$ (e.g., $[J_i, J_j] = i\, J_k$ for the rotations)
    - ★ The $\{G_i\}$ forms a group

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Kinematics

Fundamentals

✦ Einstein postulates

- The laws of physics are identical in all inertial frames
  - ➢ The outcome of any experiment is independent of the frame
- The speed of light in the vacuum *c* is constant

✦ The consequences of these postulates have been verified experimentally

✦ An infinitesimal space time interval is invariant in all inertial frames

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}r^2 \equiv c^2\mathrm{d}\tau^2$$

- This measure can be used to classify events $E_i \equiv (ct_i, \boldsymbol{r}_i)$
- The proper time τ allows for relativistic generalization of velocities and momenta

✦ Time is not absolute

- Unlike with Newtonian dynamics
- Introduction of four-vectors

Handy notations: four vectors (with c=1)

- Space and time are unified into a single object: the position four-vector

$$x^\mu \equiv (t, \mathbf{r}) \quad \text{with} \quad c = 1$$

  - Generalization of the tridimensional vector
  - The infinitesimal spacetime interval hence becomes

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \quad \text{with} \quad \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Einstein summation conventions

- The metric $\eta_{\mu\nu}$ allows to lower indices (and raise indices with $\eta^{\mu\nu}$)

$$x_\mu = \eta_{\mu\nu}x^\nu \ , \qquad x^\mu = \eta^{\mu\nu}x_\nu \quad \text{with} \quad x_\mu \equiv (t, -\mathbf{r}) \ \text{ and } \ \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- An invariant scalar product can be defined

$$x \cdot y = x^\mu y_\mu = \eta_{\mu\nu}x^\mu y^\nu$$

Derivatives, velocity and momenta

- Derivatives with respect to *x* can be defined

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \equiv \left(\frac{\partial}{\partial t}, \nabla\right) \quad \text{and} \quad \partial^\mu = \frac{\partial}{\partial x_\mu} \equiv \left(\frac{\partial}{\partial t}, -\nabla\right)$$

- The velocity and momentum can be defined from generalizing the 3D case

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \ , \qquad p^\mu = m\ u^\mu = (E, \mathbf{p})$$

- The norm of the four-momentum relates energy, momentum and mass

$$E^2 = m^2 + p^2 \qquad \text{aka} \qquad E^2 = m^2c^4 + p^2c^2$$

'*c*=*1*' is a practical choice

Poincaré and Lorentz transformations

Lorentz transformations

✦ Lorentz transformations are

- ✤ The ensemble of transformations preserving the space time interval ds = ds'
- ✤ The ensemble of transformations preserving the scalar product $x \cdot y = x' \cdot y'$

✦ A four-vector *x* is transformed as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\mu$$

Λ is a 4x4 matrix

- ✤ det **Λ** = ± 1
- ✤ The **Λ** matrix can be inverted (there exists an inverse transform)
- ✤ The **Λ** matrix must obey $\Lambda^t \eta \Lambda = \eta$
  - ★ 3 rotations (mixing of two spatial coordinates)
  - ★ 3 boosts (mixing of the time and one spatial coordinates)

6 basis elements

★ Rotation of angle α around the z axis

Wigner

$$R(\alpha) = \boxed{\exp\left[i\alpha J^{12}\right]} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

★ Boost of velocity -tanh φ in the x direction

$$B(\varphi) = \boxed{\exp\left[i\varphi J^{01}\right]} = \begin{pmatrix} \cosh\varphi & \sinh\varphi & 0 & 0 \\ \sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Lorentz and Poincaré algebras

- By perturbatively developing Λ around the identity, one gets

$$\Big[J^{\mu\nu}, J^{\rho\sigma}\Big] = -i\Big(\eta^{\nu\sigma}J^{\rho\mu} - \eta^{\mu\sigma}J^{\rho\nu} + \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma}\Big)$$

  - The 6 (independent) *J* matrices are the generators of the Lorentz algebra
    - Rotations: $J^{ij} = J^k$ with (i,j,k) a cyclic permutation of $(1,2,3)$
    - Boosts: $J^{0j} = K^j$ with $j=1,2,3$
  - They form a basis for the antisymmetric matrices
  - $J^{\mu\nu} = - J^{\nu\mu}$

- The Poincaré algebra is obtained by adding translations

$$\Big[J^{\mu\nu}, J^{\rho\sigma}\Big] = -i\Big(\eta^{\nu\sigma}J^{\rho\mu} - \eta^{\mu\sigma}J^{\rho\nu} + \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma}\Big)$$

$$\Big[J^{\mu\nu}, P^{\rho}\Big] = -i\Big(\eta^{\nu\rho}P^{\mu} - \eta^{\mu\rho}P^{\nu}\Big)$$

$$\Big[P^{\mu}, P^{\nu}\Big] = 0$$

Particle masses and spins

✦ The quadratic Casimir operators of the Poincaré algebra

- ✤ A Casimir operator is an operator commuting with all generators of the algebra
- ✤ The P^2 operator (norm of the four-momentum operator)
  - ★ $P^2 = m^2$ commutes with all J,K,P generators
  - ★ The masses are the eigenvalues of this operator
  - ★ A particle will be characterized by its mass
- ✤ The W^2 operator (where $W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}P^\nu J^{\rho\sigma}$ is the Pauli-Lubanski operator)
  - ★ Determines the particle spin in the massive case
  - ★ Determines the particle helicity in the massless case

Reminders: the rotation algebra

- The rotation algebra is given by

$$\left[J^i,J^j\right]=i\epsilon^{ij}{}_kJ^k=\begin{cases} iJ_k & \text{with } (i,j,k) \text{ a cyclic permutation of } (1,2,3). \\ -iJ_k & \text{with } (i,j,k) \text{ an anticyclic permutation of } (1,2,3). \end{cases}$$

- The representations are characterized by two quantum numbers
  - A half-integer or integer number *j* (**J**^{2} = (J^1)^2 + (J^2)^2+ (J^3)^2 is a Casimir operator)
  - *m* ∈ *[-j, j]* (the eigenvalue of J^3)
  - The *J* matrices are *(2j+1) x (2j+1)* matrices
    - *j=1/2*: the Pauli matrices (over 2)
    - j=1: the usual rotation matrices in Euclidean space

- A state is given by $|j,m\rangle$. With *ħ=1*:

$$J^3\big|j,m\rangle=m\big|j,m\rangle \qquad \text{and} \qquad \mathbf{J}^2\big|j,m\rangle=j(j+1)\big|j,m\rangle$$

Important for the spin

Spin definition

✦ Back to the Lorentz algebra

$$\left[J^{\mu\nu}, J^{\rho\sigma}\right] = -i\Big(\eta^{\nu\sigma}J^{\rho\mu} - \eta^{\mu\sigma}J^{\rho\nu} + \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma}\Big)$$

★ Rotations: $J^{ij} = J^k$
★ Boosts: $J^{0j} = K^j$

✦ Let us define two new operators

$$N^i = \frac{1}{2}(J^i + iK^i) \quad \text{and} \quad \bar{N}^i = \frac{1}{2}(J^i - iK^i)$$

✤The Lorentz algebra can be rewritten as twice the Lorentz algebra

$$\left[N^i, N^j\right] = -iN^k\ , \qquad \left[\bar{N}^i, \bar{N}^j\right] = -i\bar{N}^k \qquad \text{and} \qquad \left[N^i, \bar{N}^j\right] = 0$$

✦ Towards a definition of the spin

$$\{N_i\} \oplus \{\bar{N}_i\} = \mathfrak{sl}(2) \oplus \overline{\mathfrak{sl}(2)} \sim \mathfrak{so}(3) \oplus \mathfrak{so}(3)$$

- ➢ so(3) consist in rotations
- ➢ the representations are known

$$\begin{cases} \{N^i\} \to & S \\ \{\bar{N}^i\} \to & \bar{S} \end{cases} \Rightarrow J^i = N^i + \bar{N}^i \to \text{spin} \ = S + \bar{S}$$

(0,0) ≡ scalar
(1/2,0) ≡ left-handed spinor
(0,1/2) ≡ right-handed spinor
(1/2,1/2) ≡ vector

Fields and Lagrangians

Fields and Lagrangians

- For each spin state, we consider a specific field $\varphi(x^\mu)$
  - Depend on the spacetime coordinates
  - Has a dynamics driven by a Lagrangian density $\mathscr{L}$
    - The Lagrangian density depends on the field φ and its derivative $\partial_\mu\varphi$
- Generalization of the classical physics concepts

- In classical physics, the fundamental object is the action *S*

$$S = \int \mathrm{d}t \;\; L\big(x(t), \dot{x}(t), t\big)$$

- ★ The Lagrangian *L* depends on the position, on the velocity and on time

considered independent

- System evolution: the trajectory *x(t)* for which the action is extremal

- Continuous systems (introduction of fields) and Poincaré invariance
  - We make use of spacetime coordinates
  - We introduce a Lagrange density (4D integral)
  - Positions and velocities and are replaced by fields and their derivatives

$$S = \int \mathrm{d}^4x \; \mathcal{L}\big(\varphi(x), \partial_\mu\varphi(x)\big)$$

considered independent

Symmetric Lagrangians

- Lagrangians are required to satisfy some invariance (or be symmetric)
  - We consider an operator *G* associated with a symmetry acting on a field φ

$$\varphi(x) \to G\varphi(x)$$

  - The Lagrangian is symmetric if it satisfies

$$\mathcal{L} \to \mathcal{L} + \partial_\mu(\ldots)$$

  - This leaves the action invariant (the total derivative vanishes after integration)

$$S \to S = \int \mathrm{d}^4x \; \mathcal{L}\big(\varphi(x), \partial_\mu\varphi(x)\big)$$

Euler-Lagrange equations

✦ Properties of the action

$$S = \int \mathrm{d}^4x \; \mathcal{L}\big(\varphi(x), \partial_\mu\varphi(x)\big)$$

- ✤ The action is invariant under Poincaré transformations
  - ➢ The Lagrangian is invariant under Poincaré transformations
- ✤ The least action principle
  - ➢ Derivation of the Euler-Lagranges equations

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi\right)} = 0$$

- ★ Obtained from the minimization of the action
- ★ The Euler-Lagrange equations describe the dynamics of the system
- ★ They include the laws of nature

Euler-Lagrange equations for electromagnetism

- The (four-vector) electromagnetic potential and current are

$$A^\mu(x) = \Big(V(t,\mathbf{x}),\ \mathbf{A}(t,\mathbf{x})\Big)$$
$$j^\mu(x) = \Big(\rho(t,\mathbf{x}),\ \mathbf{j}(t,\mathbf{x})\Big)$$

- The corresponding Lagrangian density reads

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - A_\mu j^\mu \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

- The Euler-Lagrange equations induce two of the Maxwell's equations

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)} = 0 \rightsquigarrow \partial_\mu F^{\mu\nu} = j^\nu \rightsquigarrow \left\{ \begin{array}{lcl} \nabla\cdot\mathbf{E} & = & \rho \\ \nabla\times\mathbf{B} & = & \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t} \end{array} \right.$$

- Remark: the Jacobi identities imply the two other Maxwell's equations

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0 \rightsquigarrow \left\{ \begin{array}{lcl} \nabla\cdot\mathbf{B} & = & 0 \\ \nabla\times\mathbf{E} & = & -\frac{\partial \mathbf{B}}{\partial t} \end{array} \right.$$

Applications

- ✦ We will construct Lagrangians for the different considered spin states
  - ✤ Scalars: Higgs bosons, supersymmetric sfermions, *etc*.
  - ✤ Vectors: force carriers
  - ✤ Fermions: matter particles
- ✦ We will determine the what are the fields and their properties

- ✦ Some reminders from quantum mechanics are necessary

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Goals of quantum mechanics

- Explain or predict the evolution of a system over time
- Three necessary ingredients
  - A definition of the state of a physical system
  - How undertaken measurements should be interpreted
  - How the system evolves with time

Theoretical physics in a nutshell

- In classical mechanics:
  - Provide the position and velocity of all constituent of the system
  - The precision of any measurement is bounded by the experimental apparatus
  - The evolution of the system stems from Newton's law

Classical physics is deterministic.
This is different in the quantum world

The postulates of quantum mechanics

✦ Definition of the state of a system containing a single particle

- ✤ A system is characterized by a wave function $\psi(\mathbf{r},t)$
  - ★ $\psi(\mathbf{r},t)$ is a complex function of norm 1
  - ★ $\rho(\mathbf{r},t) = |\psi(\mathbf{r},t)|^2$ ➢ probability density that the particle is at $\mathbf{r}$ at a time t

✦ Measurements

- ✤ A measurable physical quantity is described by an observable $\mathcal{A}$
- ✤ The results of a measurement has to be one of the eigenvalues a of $\mathcal{A}$
- ✤ The probability to get a as a result is given by

$$\mathcal{P}(a,t) = |\langle u_a|\Psi(t)\rangle|^2 = \left|\int u_a^*(\mathbf{r})\Psi(\mathbf{r},t)\,\mathrm{d}\mathbf{r}\right|^2$$

where u_a is the eigenvector associated with a

- ✤ After the measurement, the wave function is reduced to u_a

✦ Evolution

- ✤ The evolution of the system is driven by the Schrödinger equation

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = H\Psi(\mathbf{r},t)$$

where H is the observable associated with the system energy

Building Hamiltonians

- The complicated task consists in constructing realistic Hamiltonians *H*
  - Partial derivation from classical mechanics
    - Classical mechanics is a limit of quantum mechanics
    - Quantum effects without any classical counterparts cannot be obtained (e.g. spin)

- The correspondence principle
  - Physical quantities are replaced by operators

$$\mathbf{p} \to -i\hbar\nabla \ , \qquad E \to i\hbar\frac{\partial}{\partial t} \qquad \text{and} \qquad \mathbf{r} \to \mathbf{r}$$

  - Operators do not necessarily commute
    - Classical expressions must be symmetrized

$$[x_j, p_k] = i\hbar\delta_{jk} \qquad \Rightarrow \qquad \mathbf{r}\cdot\mathbf{p} \to \frac{1}{2}\big[\mathbf{r}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{r}\big] \to -\frac{1}{2}i\hbar\left(\mathbf{r}\cdot\nabla + \nabla\cdot\mathbf{r}\right)$$

- Example: particle of mass *m* in a potential *V(**r**,t)*

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\Delta + V(\mathbf{r},t)\right)\Psi(\mathbf{r},t) \quad \text{as} \quad H = \frac{p^2}{2m} + V(\mathbf{r},t)$$

Time-independent potentials

✦ Schrödinger equation can (often) be solved with the Fourier method

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\Delta + V(\mathbf{r})\right)\Psi(\mathbf{r},t) \qquad \text{with} \qquad \Psi(\mathbf{r},t) = \chi(t)\psi(\mathbf{r})$$

✤ One gets two equations for χ and ψ

$$i\hbar\frac{1}{\chi(t)}\frac{\mathrm{d}\chi(t)}{\mathrm{d}t} = \frac{1}{\psi(\mathbf{r})}\left(-\frac{\hbar^2}{2m}\Delta + V(\mathbf{r})\right)\psi(\mathbf{r}) = E = \mathrm{cst}$$

✤ The temporal equation is easy to solve

$$\Psi(\mathbf{r},t) = \psi(\mathbf{r})\, e^{-iEt/\hbar}$$

✦ The superposition principle

✤Any linear combination of solutions is a solution

$$\Psi(\mathbf{r},t) = \sum_n E_n\ \psi_n(\mathbf{r})\, e^{-iE_nt/\hbar}$$

Probability density and current

- The Schrödinger equation leads to the quantization of the energy levels
  - The energy *E* can only take well-defined values E_n
- The superposition principle leads to interesting phenomena
  - The probability density includes interferences

$$\rho(\mathbf{r},t) = \Big|\Psi(\mathbf{r},t)\Big|^2 \neq \sum_n \rho_n(\mathbf{r},t) = \sum_n \Big|\Psi_n(\mathbf{r},t)\Big|^2$$

  - Verified in many experiments
    - ★ The wave function is not material and corresponds to a probability density

- As in fluid mechanics: if a density varies with time, then currents appear
  - Example in the free particle case:

$$\begin{aligned}\frac{\partial}{\partial t}\rho(\mathbf{r},t) =& -\frac{1}{i\hbar}\,[H\Psi(\mathbf{r},t)]^*\Psi(\mathbf{r},t) + \frac{1}{i\hbar}\,\Psi^*(\mathbf{r},t)[H\Psi(\mathbf{r},t)]\\ =& -\frac{\hbar}{2im}\nabla\Big(\Psi^*(\mathbf{r},t)\,\nabla\Psi(\mathbf{r},t) - \nabla\Psi^*(\mathbf{r},t)\,\Psi(\mathbf{r},t)\Big)\end{aligned}$$

$$\boxed{\frac{\partial}{\partial t}\rho(\mathbf{r},t) + \nabla\cdot\mathbf{J}(\mathbf{r},t) = 0}$$

Free states

✦ Stationary solutions of the Schrödinger equation

- Some solutions cannot be normalized (e.g. free states) (the wave function does not vanish at infinity)
- They can however be used to construct physical wave packets

✦ 1D example: a non-confining potential V(x) (independent of the time)

$$V(x) \underset{x\to\pm\infty}{\to} V_\pm < \infty$$

✦ There is a continuous part to the energy spectrum

- The corresponding stationary solutions of the Schrödinger equation satisfy

$$H\psi_k(x) = \frac{\hbar^2 k^2}{2m}\,\psi_k(x) = \hbar\omega(k)\,\psi_k(x)$$

- Their asymptotic behavior reads

$$\psi_k(x) \underset{x\to-\infty}{\to} \frac{1}{\sqrt{2\pi}} e^{ikx}$$

A superposition of such states coud yield physical states

One-dimensional wave packets

✦ Superposition *Ψ* of free states

the energy $E_k=\hbar\omega$

$$\Psi(x,t) = \int_{-\infty}^{+\infty} g(k)\psi_k(x)e^{-i\omega(k)t}\mathrm{d}k \qquad \text{where} \qquad H\psi_k(x) = \boxed{\frac{\hbar^2k^2}{2m}}\psi_k(x)$$

- ✤ The physical context in which the system is prepared (at *t*=0) determines *g(k)*

$$g(k) = \int_{-\infty}^{+\infty} \psi_k^*(x)\Psi(x,0)\mathrm{d}x$$

- ✤ One can show that *Ψ* satisfies the Schrödinger equation
- ✤ If *g(k)* is square-integrable, *Ψ* is square-integrable and thus physical

✦ Free wave packets

- ✤ The solutions are plane waves

the momentum *p*

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(k)\ e^{-i\left[\omega(k)t-kx\right]}\mathrm{d}k \qquad \text{as} \qquad -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\psi_k(x) = \boxed{\hbar k}\psi_k(x)$$

Three-dimensional wave packets

✦ The 3D generalization is straightforward

$$\Psi(\mathbf{r},t) = \frac{1}{\sqrt{8\pi^3}} \int \mathrm{d}\mathbf{k}\ g(\mathbf{k})\ e^{-i\left[\omega(\mathbf{k})t - \mathbf{k}\cdot\mathbf{r}\right]} \quad \text{where} \quad E(\mathbf{k}) = \overset{=\hbar\omega}{\boxed{\frac{\hbar^2 k^2}{2m}}} \quad \text{and} \quad \mathbf{p} = \hbar\mathbf{k}$$

- ✤ The physical context in which the system is prepared (at *t*=*0*) determines *g*(***k***)
- ✤ The function *g*(***k***) ≡ coordinates in a continuous basis of exponentials

✦ Second quantization: the fields will get quantized

- ✤ Inclusion of harmonic and fermionic oscillators

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Scalar fields (spin 0)

Free scalar fields

✦ Definition

- (0,0) representation of the Lorentz algebra (spin S=0)
- Invariant under a Lorentz transformation

$$\varphi(x) \to \varphi'(x') = \varphi(x)$$

Examples: Higgses, squarks, *etc.*

✦ Wave equation: the Klein-Gordon equation

- The relativistic version of the correspondance principle

$$\mathbf{p} \to -i\hbar\nabla \quad \text{and} \quad E \to i\hbar\frac{\partial}{\partial t} \quad \Rightarrow \quad p_\mu \to i\partial_\mu$$

- Application of the correspondance principle to the definition of the energy

$$E^2 = \boxed{\mathbf{p}^2 + m^2} \quad \Leftrightarrow \quad p_\mu p^\mu - m^2 = 0 \quad \Leftrightarrow \quad \Big(\Box + m^2\Big)\varphi(x) = 0$$

The Hamiltonian *H*

$$H^2\varphi(x) = E^2\varphi(x)$$

- Euler-Lagrange: the Klein-Gordon Lagrangian

$$\mathcal{L}_{\rm KG} = (\partial^\mu\varphi)^\dagger(\partial_\mu\varphi) - m^2\varphi^\dagger\varphi$$

Solution of the Klein-Gordon equation

✦ Solutions are trivial: plane waves

$$\Big(\Box + m^2\Big)\varphi(x) = 0 \quad \Leftrightarrow \quad \Big(\frac{\partial^2}{\partial t^2} - \Delta + m^2\Big)\varphi(x) = 0$$

$$\Rightarrow \varphi(x) = Ne^{-ip\cdot x} = Ne^{-i(Et-\mathbf{p}\cdot\mathbf{r})}$$

- ✤ One degree of freedom: scalar field of mass *m*, momentum **p** and energy *E*

✦ Problems

- ✤ Inserting the solution in the KG equation: $E = \pm\sqrt{\mathbf{p}^2 + m^2}$
- ✤ The energy can be negative
- ✤ We can calculate the probability density ρ: not positively-defined

$$\frac{\partial}{\partial t}\Bigg[\underbrace{i\Big(\varphi^\dagger\frac{\partial\varphi}{\partial t} - \varphi\frac{\partial\varphi^\dagger}{\partial t}\Big)}_{\rho}\Bigg] + \nabla\Bigg[\underbrace{-i\Big(\varphi^\dagger\nabla\varphi - \varphi\nabla\varphi^\dagger\Big)}_{\mathbf{j}}\Bigg] = 0$$

- ✤ We lose the notion of a probability density

Particles and antiparticles

✦ Solutions: antiparticles (particles going backward in time)

✤ Second quantization

$$\varphi(x) = \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega}}\Big[\underbrace{\boxed{a_{\mathbf{p}}\ e^{-ip\cdot x}}}_{\text{Particle annihilation}} + \underbrace{\boxed{b^\dagger_{\mathbf{p}}\ e^{ip\cdot x}}}_{\text{Antiparticle creation}}\Big] \quad \text{with} \quad \omega = \sqrt{m^2+\mathbf{p}^2}$$

✦ The Stückelberg and Feynman interpretation

✤ *E t = (-E) (-t)*

★ A particle with negative energy going backwards in time

★ An antiparticle with positive energy going forward in time

✦ The probability density can be seen as a charge density

$$\rho(x) = iq\left(\varphi^\dagger\frac{\partial\varphi}{\partial t} - \varphi\frac{\partial\varphi^\dagger}{\partial t}\right)$$

$$\mathbf{j}(x) = \ -iq\left(\varphi^\dagger\nabla\varphi - \varphi\nabla\varphi^\dagger\right)$$

No need for being positive definite

Propagators

✦ Definition

- ✤ A propagator consists in an amplitude
  - ★ Associated with the propagation of a particle (or antiparticle) from *x* to *y*
  - ★ Corresponds to the destruction of the (anti)particle in *x* and creation in *y*

✦ Feynman propagator

- ✤ A particle cannot be recreated before it has been destroyed

$$G(x-y) = \theta(x^0-y^0)\underbrace{\boxed{\langle 0|\varphi(x)\varphi^\dagger(y)|0\rangle}}_{\text{Particle destroyed in } x \text{ and created in } y} + \theta(y^0-x^0)\underbrace{\boxed{\langle 0|\varphi^\dagger(y)\varphi(x)|0\rangle}}_{\text{Antiparticle destroyed in } y \text{ and created in } x}$$

- ✤ From the field definition and the properties of the annihilation/creation operators

$$\begin{aligned} G(x-y) &= \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega}}\,\Big[\theta(x^0-y^0)\;e^{-ip\cdot(x-y)} + \theta(y^0-x^0)\;e^{ip\cdot(x-y)}\Big] \\ &= \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{1}{p^2-m^2+i\varepsilon}\;e^{-ip\cdot(x-y)} \end{aligned}$$

Feynman rules for scalar fields

✦ Calculations involve complicated integrals

- ✤ We can work directly in momentum space
- ✤ Using Feynman rules to simplify the derivation of any scattering amplitude
- ✤ Integration performed as the very last step

✦ External scalars

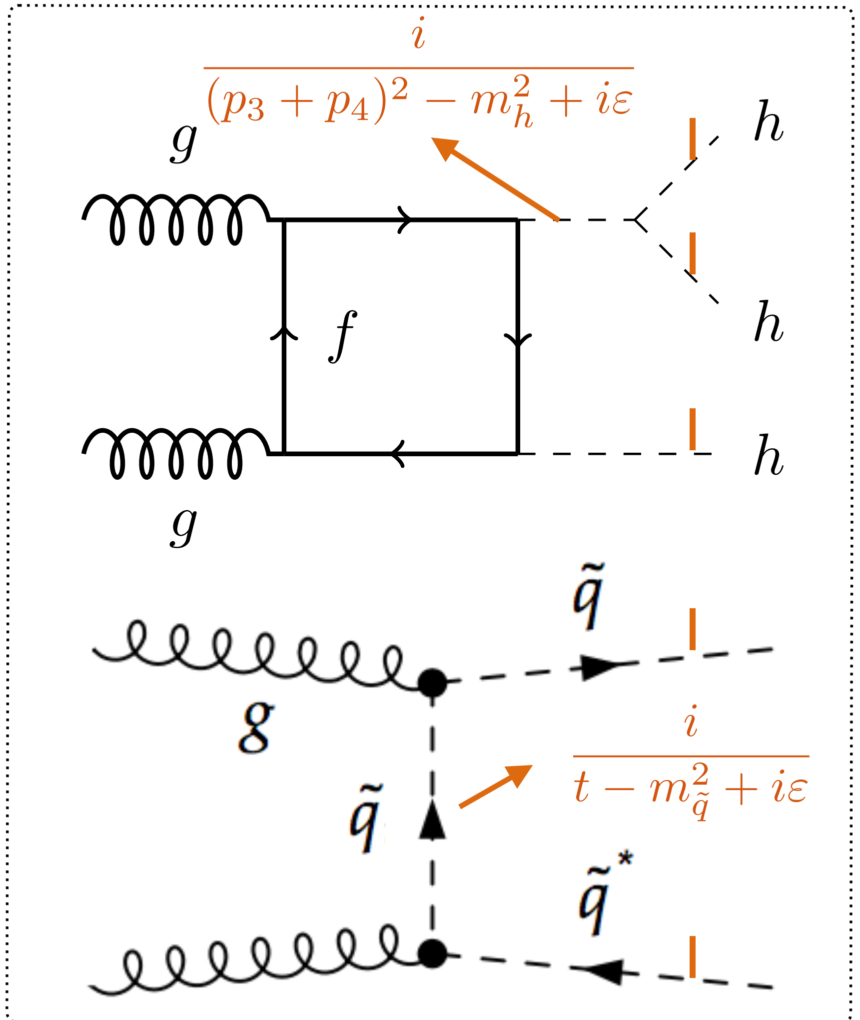
- ✤ Feynman rules derived from the field expression

$$\varphi(x) = \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega}}\Big[\underset{\text{Coefficient}\atop =1}{\boxed{a_{\mathbf{p}}\; e^{-ip\cdot x}}} + \underset{\text{Coefficient}\atop = 1}{\boxed{b^\dagger_{\mathbf{p}}\; e^{ip\cdot x}}}\Big]$$

✦ Propagators

- ✤ The Fourier transform gives the Feynman rule

$$G(x-y) \;=\; \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\underset{\frac{i}{p^2-m^2+i\varepsilon}}{\boxed{\frac{1}{p^2-m^2+i\varepsilon}}}\,e^{-ip\cdot(x-y)}$$



Fermions (spin 1/2)

Towards the Dirac equation

- The Klein-Gordon introduced several '*problems*'
  - Solutions with a negative energy
  - Not positively-defined probability density
- Dirac tried to solve these issues
  - Removal of the second-order time derivative

$$\Big(i\frac{\partial}{\partial t} - H_D\Big)\Psi(x) = 0$$

  - Imposing Lorentz invariance and that the energy satisfies the definition of Einstein

$$\Big(i\frac{\partial}{\partial t} + i\alpha\cdot\nabla - \beta m\Big)\Psi(x) = 0 \quad \text{with} \quad \{\alpha_i, \alpha_j\} = 2\delta_{ij}\ , \quad \{\alpha_i, \beta\} = 0\ , \quad \beta^2 = 1$$

  - H_D is Hermitian ➢ the α and β matrices are traceless and with ±1 eigenvalues

- Lowest dimensional solution: the α and β matrices are 4x4 matrices
  - The Dirac matrices

The Dirac equation

- The Dirac equation reads

$$\Big(i\gamma^\mu\partial_\mu - m\Big)\Psi(x) = 0$$

- The Dirac matrices are defined (in the chiral representation) by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \quad \text{with} \quad \begin{matrix} \sigma^0 = \bar\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \sigma^1 = -\bar\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma^2 = -\bar\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \sigma^3 = -\bar\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{matrix}$$

  - They satisfy the algebra

$$\{\gamma^\mu,\gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\ \eta^{\mu\nu}$$

The function Ψ is a four-dimensional object. But what does it represent?

The vector representation of the Lorentz algebra

✦ Back to the Lorentz algebra

$$\left[J^{\mu\nu}, J^{\rho\sigma}\right] = -i\Big(\eta^{\nu\sigma}J^{\rho\mu} - \eta^{\mu\sigma}J^{\rho\nu} + \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma}\Big)$$

★ Rotations: $J^{ij} = J^k$

★ Boosts: $J^{0j} = K^j$

- Act on four vectors
- The generators form a set of 6 independent 4x4 matrices given by

$$(J^{\mu\nu})^\rho{}_\sigma = -i\Big(\eta^{\rho\mu}\delta^\nu{}_\sigma - \eta^{\rho\nu}\delta^\mu{}_\sigma\Big)$$

Basis for the antisymmetric matrices

✦ Finite Lorentz transformations G (reminder: $G = \exp[a^i g_i]$)

$$\Lambda_{(\frac{1}{2},\frac{1}{2})} = \exp\left[\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right]$$

ω are real parameters

- Rotation of angle α around the z axis
  $\alpha = \omega_{12} = -\omega_{21}$

$$R(\alpha) = \exp\left[i\alpha J^{12}\right] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Boost of velocity $v = -\tanh\varphi$
  $\varphi = \omega_{01} = -\omega_{10}$

$$B(\varphi) = \exp\left[i\varphi J^{01}\right] = \begin{pmatrix} \cosh\varphi & \sinh\varphi & 0 & 0 \\ \sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The spinor representations of the Lorentz algebra

✦ We can construct two sets of 6 independent 2x2 matrices

$$(\sigma^{\mu\nu})_\alpha{}^\beta = -\frac{i}{4}\Big(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu\Big)_\alpha{}^\beta \qquad (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = -\frac{i}{4}\Big(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu\Big)^{\dot\alpha}{}_{\dot\beta}$$

- Act on two-component objects: left-handed and right-handed spinors
- Those sets of matrix satisfy the Lorentz algebra

$$\Big[\sigma^{\mu\nu},\sigma^{\rho\sigma}\Big] = -i\Big(\eta^{\nu\sigma}\sigma^{\rho\mu} - \eta^{\mu\sigma}\sigma^{\rho\nu} + \eta^{\nu\rho}\sigma^{\mu\sigma} - \eta^{\mu\rho}\sigma^{\nu\sigma}\Big)$$

$$\Big[\bar\sigma^{\mu\nu},\bar\sigma^{\rho\sigma}\Big] = -i\Big(\eta^{\nu\sigma}\bar\sigma^{\rho\mu} - \eta^{\mu\sigma}\bar\sigma^{\rho\nu} + \eta^{\nu\rho}\bar\sigma^{\mu\sigma} - \eta^{\mu\rho}\bar\sigma^{\nu\sigma}\Big)$$

✦ Finite Lorentz transformations

$\Lambda_{(\frac{1}{2},0)} = \exp\Big[\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\Big]$ and $\Lambda_{(0,\frac{1}{2})} = \exp\Big[\frac{i}{2}\omega_{\mu\nu}\bar\sigma^{\mu\nu}\Big]$ ω are real parameters

✦ Complex conjugation maps left-handed and right-handed spinors

Dirac and Majorana spinors

- Four-component Dirac spinors are constructed from two-component ones

$$\psi_D = \begin{pmatrix} \boxed{\psi_\alpha} \\ \boxed{\bar\chi^{\dot\alpha}} \end{pmatrix}$$

Left-handed component

Right-handed component

Majorana spinor: $\psi = \chi$
(the left and right-handed
components are conjugate)

- We can construct 6 independent 4x4 matrices acting on spin space

$$\gamma^{\mu\nu} = -\frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar\sigma^{\mu\nu} \end{pmatrix}$$

  - The non-zero upper element act on the left-handed spinor ψ_α
  - The non-zero lower element act on the right-handed spinor $\bar\chi^{\dot\alpha}$

- Finite Lorentz transformations

$$\Lambda_{(\frac{1}{2},0)\oplus(0,\frac{1}{2})} = \exp\left[\frac{i}{2}\omega_{\mu\nu}\gamma^{\mu\nu}\right] = \begin{pmatrix} \Lambda_{(\frac{1}{2},0)} & 0 \\ 0 & \Lambda_{(0,\frac{1}{2})} \end{pmatrix}$$

ω are real parameters

Back to the Dirac equation

- The Dirac equation reads

$$\Big(i\gamma^\mu\partial_\mu - m\Big)\Psi(x) = 0$$

- The Dirac equation has to be invariant under Lorentz transformations
  - The equations of motion are Lorentz-invariant
  - This implies that Ψ is a four-component spinor:

$$\Psi(x) \to \Lambda_{(\frac{1}{2},0)\oplus(0,\frac{1}{2})}\Psi(x) \quad \Rightarrow \quad \Big(i\gamma^\mu\partial'_\mu - m\Big)\Psi'(x') = 0$$

- Euler-Lagrange equations: the Dirac Lagrangian

$$\mathcal{L}_D = \bar\Psi(i\gamma^\mu\partial_\mu - m)\Psi \quad \text{where} \quad \bar\Psi = \Psi^\dagger\gamma^0$$

- Conserved current

$$j^\mu = \bar\Psi\gamma^\mu\Psi \quad \Rightarrow \quad \partial_\mu j^\mu = 0$$

The notion of probability density is recovered $\rho = j^0 = \Big|\Psi\Big|^2 > 0$

Solution of the Dirac equation

- Each component of a Dirac spinor obeys to the Klein-Gordon equation

$$\Big(i\gamma^\mu\partial_\mu - m\Big)\Psi(x) = 0 \quad \Leftrightarrow \quad \Psi(x) = u(p)e^{-ip\cdot x}$$

  - We must have a linear combination of plane waves

- Simplification: $p^\mu = (m, 0, 0, 0)$ with a positive energy
  - The *u(p)* spinor can be easily derived: two linearly-independent solutions

$$u(p) = \sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix} \qquad \text{with} \qquad \xi = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}0\\1\end{pmatrix}$$

  - Two degrees of freedom

- General solution obtained with a Lorentz transformation

$$u(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\ \xi\\ \sqrt{p\cdot\bar\sigma}\ \xi\end{pmatrix} \qquad \text{with} \qquad \xi = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}0\\1\end{pmatrix}$$

- Problems: there are also 2 other solutions, for a negative energy

$$\Psi(x) = v(p)e^{ip\cdot x} \quad \text{where} \quad v(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\ \eta\\ -\sqrt{p\cdot\bar\sigma}\ \eta\end{pmatrix} \qquad \text{with} \qquad \eta = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}0\\1\end{pmatrix}$$

Dirac equation: interpretation

- The Dirac equation solves one of the issues of the Klein-Gordon equation
  - The probability density interpretation is again viable
  - However, antiparticles are still there (the goal of Dirac was to get rid of them)

- There exists one observable commuting with H_D
  - The helicity operator:

$$h = \frac{1}{2}\, 1_{\mathbf{p}} \cdot \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

★ The 2 independent particle/antiparticle solutions have *h*=±1/2

The function *Ψ* represents a fermion
(two independent spin states)

- Lorentz transformations can change the helicity of massive particles
  - Chirality is preferred

$$\Psi_L = P_L \Psi = \frac{1-\gamma_5}{2}\begin{pmatrix}\psi \\ \bar{\chi}\end{pmatrix} = \begin{pmatrix}\psi \\ 0\end{pmatrix}$$

$$\Psi_R = P_R \Psi = \frac{1+\gamma_5}{2}\begin{pmatrix}\psi \\ \bar{\chi}\end{pmatrix} = \begin{pmatrix}0 \\ \bar{\chi}\end{pmatrix}$$

$$\text{with} \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \qquad \Big(\{\gamma^5, \gamma^\mu\} = 0\Big)$$

Feynman rules for fermionic fields

- External scalars (rules derived from the coefficients in the field expression)

$$\Psi(x) = \sum_{s=1,2}\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega}}\left[\boxed{u(p,s)}a^s_{\mathbf{p}}\ e^{-ip\cdot x} + \boxed{v(p,s)}b^{s\dagger}_{\mathbf{p}}\ e^{ip\cdot x}\right]$$

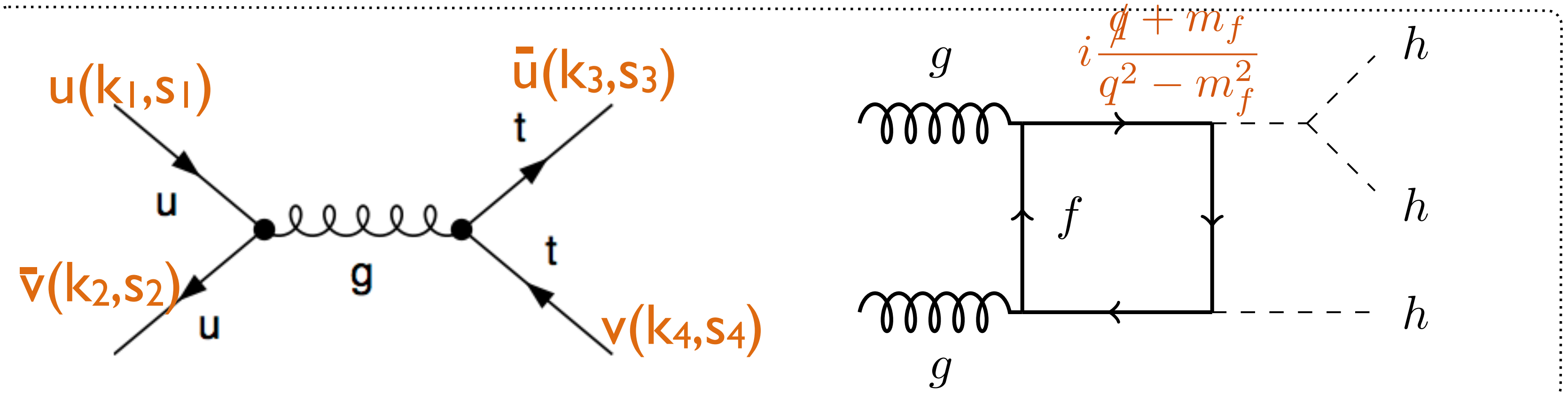
$$\bar\Psi(x) = \sum_{s=1,2}\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega}}\left[\boxed{\bar u(p,s)}a^{s\dagger}_{\mathbf{p}}\ e^{ip\cdot x} + \boxed{\bar v(p,s)}b^{s}_{\mathbf{p}}\ e^{-ip\cdot x}\right]$$

- Propagators (Fourier transform of the Green function for Dirac equation)

$$\Big(i\gamma^\mu\partial_\mu - m\Big)G(x-y) = i\delta^{(4)}(x-y)$$

$$G(x-y) = \int\frac{\mathrm{d}^4p}{(2\pi)^4}\boxed{i\frac{\not p + m}{p^2 - m^2 + i\varepsilon}}e^{-ip\cdot(x-y)}$$

- Examples



Vector fields (spin 1)

(Abelian) vector fields

✦ Definition

- (1/2,1/2) representation of the Lorentz algebra (spin S=1)
- Lorentz transformation of a vector field *A*

$$A^\mu(x) \to A'^\mu(x') = \left(\Lambda_{(\frac{1}{2},\frac{1}{2})}\right)^\mu{}_\nu \; A^\nu(x)$$

  - ★ Where $\mathbf{\Lambda}$ denotes a transformation matrix in the vector representation

Examples: gauge bosons

✦ Maxwell's equations

- The relativistic version of Maxwell's equations is given by

$$\partial_\mu F^{\mu\nu} = j^\nu \qquad \text{where} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- From Euler-Lagrange equations, one can derive the Maxwell Lagrangian

$$\mathcal{L}_{\rm M} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A^\mu j_\mu$$

Could be generalized to other Abelian interactions

Solutions of (free) Maxwell's equations

✦ We can rewrite Maxwell's equations as

$$\partial_\nu \partial^\nu A^\mu - \partial^\mu \partial_\nu A^\nu = 0$$

✤ The solution can be written as a linear combination of plane waves

$$A^\mu(x) = \varepsilon^\mu e^{-ip\cdot x} \quad \text{with} \quad p^2 = 0$$

✤ Gauge choice: Lorentz gauge

$$\partial_\mu A^\mu(x) = 0 \quad \Leftrightarrow \quad p \cdot \varepsilon = 0$$

★ The ***ε*** quantity is left with three degrees of freedom

✤ Residual gauge freedom:

$$A^\mu \to A^\mu + \partial^\mu \chi \quad \text{with} \quad \Box \chi = 0$$

★ Preserves the Lorentz gauge

★ Can be used to get rid of a degree of freedom $\chi = iae^{-ip\cdot x}$

★ We chose *a* is such that $\varepsilon^0 + ap^0 = 0$

✤ The ***ε*** quantity has **two** degrees of freedom and the photon is **transverse**

Photon polarization

- The photon has two degrees of freedom and is transversely polarized
  - Different possible bases for the polarization vectors

$$\varepsilon^\mu(x) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \ , \quad \varepsilon^\mu(y) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

or

$$\varepsilon^\mu(\pm1) = \frac{1}{\sqrt{2}}\Big(\varepsilon(x) \pm i\varepsilon(y)\Big)$$

  - More general and complicated sets exist

Feynman rules for Abelian vector fields

- Second quantification: Feynman rules derived from the field expression)

$$\Psi(x) = \sum_\lambda \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega}}\,\Big[\boxed{\varepsilon(p,\lambda)}\,a^\lambda_{\mathbf{p}}\ e^{-ip\cdot x} + \boxed{\varepsilon^*(p,\lambda)}\,b^{s\dagger}_{\mathbf{p}}\ e^{ip\cdot x}\Big]$$

- Propagators: Fourier transform of the Green function for Maxwell's equations

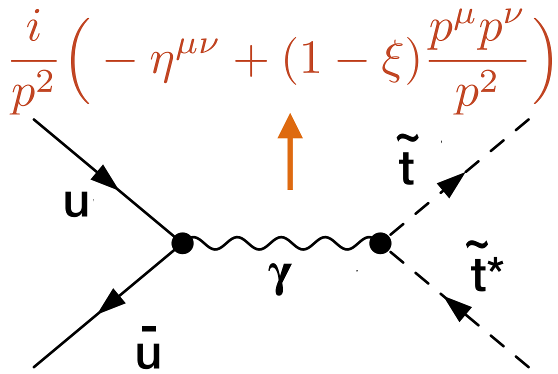
$$\Big(\Box\eta_{\mu\nu} - \partial_\mu\partial_\nu\Big)G^{\nu\rho}(x-y) = i\delta^{(4)}(x-y)\delta_\mu{}^\rho$$

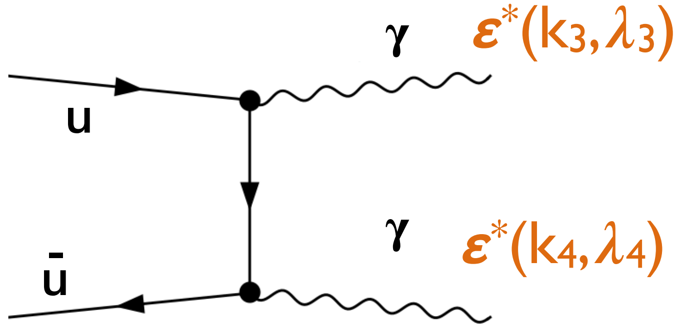
  - Not calculable: need to add a gauge fixing term to the Lagrangian

$$\mathcal{L} \to \mathcal{L} - \frac{1}{2\xi}\Big(\partial_\mu A^\mu\Big)^2 \qquad \Big(= \mathcal{L}\Big)$$

$$G^{\mu\nu}(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\,\boxed{\frac{i}{p^2}\Big(-\eta^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}\Big)}\ e^{-ip\cdot(x-y)}$$

- Examples





Non-Abelian interactions: the *SU(N)* example

- Maxwell's equations describe interactions based on an Abelian gauge group

- Many common gauge theories are based on the *SU(N)* group
  - ✤ The algebra is generated by N^2-1 matrices T_a with $a=1, \ldots, N^2-1$.

$$\big[T_a, T_b\big] = i f_{ab}{}^c\, T_c$$ where the $f_{ab}{}^c$ are the structure constants of the group

- Commonly used representations
  - ✤ Fundamental (and anti fundamental): *NxN* matrices such that

$$\mathrm{Tr}(T_a) = 0 \qquad \text{and} \qquad T_a^\dagger = T_a$$

  - ✤ Adjoint representation: $(N^2-1)x(N^2-1)$ matrices such that

$$(T_a)_b{}^c = -i f_{ab}{}^c$$

  - ✤ For a given representation

$$\mathrm{Tr}\big(T_a T_b\big) = \tau_{\mathcal{R}} \delta_{ab}$$ where τ_R is the Dynkin index of the representation

Building non-Abelian interactions

✦ Recipe (more on gauge theories below)

1. We select a gauge group *SU(N)*
2. We define a coupling constant *g*
3. We assign representations to the matter fields
4. The interactions are mediated by N^2-*1* gauge bosons $A^\mu = A^{\mu a} T_a$

Relativistic wave equations for the gauge bosons?

✦ Generalization of the field strength (now includes a non-Abelian part)

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig\big[A_\mu, A_\nu\big] \\ &= \Big[\partial_\mu A^c_\nu - \partial_\nu A^c_\mu + g\ f_{ab}{}^c A^a_\mu A^b_\nu\Big]\ T_c \end{aligned}$$

✤ The Lagrangian is the Yang-Mills Lagrangian (➢ Yang-Mills equations)

$$\mathcal{L}_{\rm YM} = -\frac{1}{4\tau_{\mathcal{R}}} {\rm Tr}(F_{\mu\nu} F^{\mu\nu})$$

✦ Similar Feynman rules as for the Abelian case (but with extra gauge indices)

Massive (Abelian) vector fields

- ✦ The generalization to the massive case is immediate

$$\boxed{\Big(\Box + m^2\Big) A^\mu - \partial^\mu \partial_\nu A^\nu = 0}$$

- ✤ We cannot use the Lorentz gauge choice to reduce the number of freedoms

$$\boxed{\partial_\mu A^\mu(x) = 0}$$

★ Automatically realized

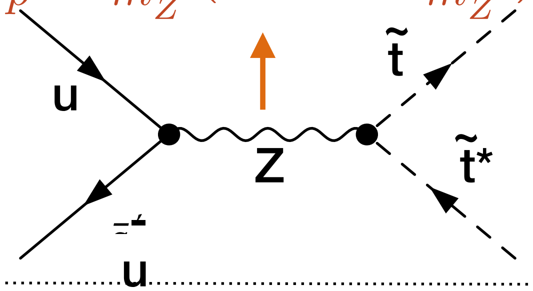
- ✤ The $\boldsymbol{\varepsilon}$ quantity has **three** degrees of freedom (**transverse** and **longitudinal)**

- ✦ Polarization vectors

$$\varepsilon^\mu(\pm 1) = \frac{1}{\sqrt{2}}\Big(\varepsilon(x) \pm i\varepsilon(y)\Big) \qquad \varepsilon^\mu(z) = \frac{1}{m}\big(|\mathbf{p}|, 0, 0, E\big)$$

- ✦ Propagator Feynman rule

$$\frac{i}{p^2 - m_Z^2}\Big(-\eta^{\mu\nu} + \frac{p^\mu p^\nu}{m_Z^2}\Big)$$



Outline

1. Introduction
2. Special relativity
3. Some quantum mechanics
4. Relativistic quantum mechanics
5. **Gauge symmetries**
6. Scattering theory
7. Summary

Gauge symmetries

Global symmetries of the Dirac Lagrangian

✦ Toy model

- Gauge group: *SU(N)*
- We assign the fundamental representation to a fermionic field ***Ψ***

$$\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix} \ , \qquad \bar\Psi = \begin{pmatrix} \bar\psi_1 & \cdots & \bar\psi_N \end{pmatrix}$$

Fundamental representation of *SU(N)*:
N^2-*1* (*NxN*) matrices T_a

- Dirac Lagrangian

$$\mathcal{L}_D = \bar\Psi (i\gamma^\mu \partial_\mu - m)\Psi$$

✦ Global *SU(N)* invariance

- We define a global *SU(N)* transformation of parameters ω^a ($a=1,\ldots,N^2-1$)

$$\Psi(x) \to \Psi'(x) = \exp\Big[+ ig\omega^a T_a \Big] \Psi(x) \equiv U \ \Psi(x)$$

$$\bar\Psi(x) \to \bar\Psi'(x) = \bar\Psi(x) \exp\Big[- ig\omega^a T_a \Big] \equiv \bar\Psi(x) \ U^\dagger$$

- The Lagrangian is invariant ($UU^\dagger = U^\dagger U = 1$)

Gauge symmetries of the Dirac Lagrangian

✦ Promotion of the global symmetry to a local one

- ✤ The transformation parameters are not constant anymore: $\omega^a \to \omega^a(x)$

$$\Psi(x) \to \Psi'(x) = U(x)\ \Psi\ , \qquad \bar{\Psi}(x) \to \bar{\Psi}'(x) = \bar{\Psi}\ U^\dagger(x)$$

✦ The Dirac Lagrangian is not invariant anymore

$$\partial_\mu \Psi(x) \quad \not\to \quad U(x)\ \partial_\mu \Psi(x)$$

$$\mathcal{L} = \bar{\Psi}\Big(i\gamma^\mu \partial_\mu - m\Big)\Psi \quad \not\to \quad \mathcal{L}$$

- ✤ The invariance is lost because of
  - ★ The presence of a derivative
  - ★ The dependence on *x* of the transformation matrix *U*

✦ Idea: modification of the derivative

- ✤ Introduction of a new field with *ad hoc* transformation rules
- ✤ Recovery of the invariance

Nœther procedure

✦ Transformation of the Lagrangian under a local transformation

$$\Psi(x) \to \Psi'(x) = U(x)\ \Psi\ , \qquad \bar\Psi(x) \to \bar\Psi'(x) = \bar\Psi\ U^\dagger(x)$$

$$\mathcal{L} = \bar\Psi\Big(i\gamma^\mu\partial_\mu - m\Big)\Psi \quad \to \quad \mathcal{L} + i\bar\Psi\gamma^\mu\boxed{\Big(ig\partial_\mu\omega^a T_a\Big)}\Psi$$

Vector

✦ Nœther procedure

- We add a vectorial field whose transformation laws compensate the extra term
- Covariantization of the derivative

$$\partial_\mu\Psi(x) \quad \to \quad D_\mu\Psi(x) = \Big(\partial_\mu - igT_aA^a_\mu\Big)\Psi(x)$$

- The gauge transformation laws are

$$\begin{aligned}\Psi(x) &\to U(x)\ \Psi(x)\\ \bar\Psi(x) &\to \bar\Psi(x)\ U^\dagger(x)\\ A_\mu(x) &\to U(x)\Big(A_\mu(x) + \frac{i}{g}\partial_\mu\Big)U^\dagger(x)\end{aligned} \quad \Rightarrow \quad \begin{aligned}F^{\mu\nu}(x) &\to U(x)F^{\mu\nu}(x)U^\dagger(x)\\ D_\mu\Psi(x) &\to U(x)D_\mu\Psi(x)\end{aligned}$$

Gauge invariance of the Dirac equation

✦ Gauge transformation of the fields

$$\Psi(x) \to \Psi'(x) = U(x)\ \Psi\ , \qquad \bar{\Psi}(x) \to \bar{\Psi}'(x) = \bar{\Psi}\ U^\dagger(x)$$

$$F^{\mu\nu}(x) \to U(x) F^{\mu\nu}(x) U^\dagger(x) \qquad D_\mu\Psi(x) \to U(x) D_\mu \Psi(x)$$

✦ Gauge transformation of the Lagrangian

$$\mathcal{L} = \underbrace{\boxed{-\frac{1}{4\tau_R}\mathrm{Tr}\big[F_{\mu\nu}F^{\mu\nu}\big]}}_{\text{Gauge-invariant vector boson kinetic term}} + \underbrace{\boxed{\bar{\Psi}\Big(i\gamma^\mu D_\mu - m\Big)\Psi}}_{\text{Gauge-invariant fermion kinetic term}} \quad \to \quad \mathcal{L}$$

✦ Remarks

- ✤ Imposing gauge invariance yields the fundamental interactions
- ✤ The vector field A_μ is the potential associated with the considered interaction

Feynman rules fore the interactions

- The Lagrangian includes the gauge interactions of the model

$$\mathcal{L} = -\frac{1}{4\tau_R}\mathrm{Tr}\Big[F_{\mu\nu}F^{\mu\nu}\Big] + \bar{\Psi}\Big(i\gamma^\mu D_\mu - m\Big)\Psi$$

$$\frac{1}{2}\, g\,\partial^\nu A^\mu_a\ f_{cb}{}^a\ \Big[A^b_\mu A^c_\nu - A^c_\mu A^b_\nu\Big]$$

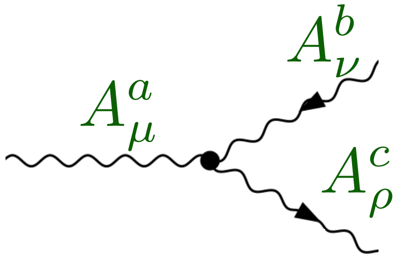
$$\bar{\Psi}\Big[ig\gamma^\mu T_a\Big]\Psi A^a_\mu$$

  - The derivatives act on the exponentials of the field

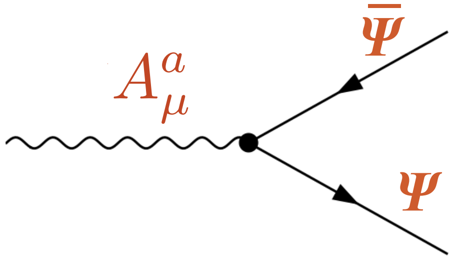
$$\Psi(x) = \int \mathrm{d}^4p\ \Big[(\ldots)e^{-ip\cdot x} + (\ldots)e^{+ip\cdot x}\Big]$$

  - Symmetrization if identical particles

- Feynman rules



$$g f_{cb}{}^a\ \Big[(k_1-k_2)^\rho\eta^{\mu\nu} + (k_2-k_3)^\mu\eta^{\nu\rho} + (k_3-k_1)^\nu\eta^{\rho\mu}\Big]$$



$$ig\ T_a\ \gamma^\mu$$

Symmetry breaking

Necessity for breaking the gauge symmetries

- Particles are in general massive
  - Lagrangian mass terms are in general forbidden by the gauge symmetries
  - The gauge symmetry concept works well with respect to data

- Spontaneous symmetry breaking
  - Allows for mass term generation
  - Relies on the gauge symmetry concept

A global U(1) toy model with a scalar field

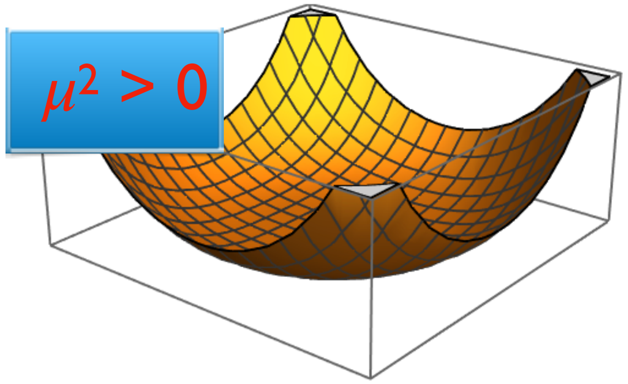
✦ Toy model

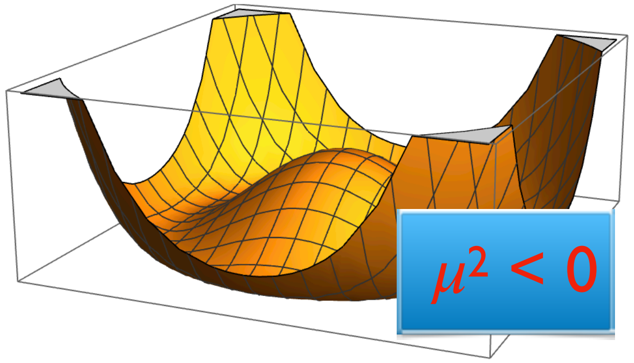
- ✤ We consider a complex scalar field φ
- ✤ The most general (renormalizable) Lagrangian is given

$$\mathcal{L} = \underbrace{\boxed{\partial_\mu \varphi^\dagger \partial^\mu \varphi}}_{\text{Kinetic terms}} - V(\varphi) \quad \text{with} \quad \underbrace{\boxed{V(\varphi) = \mu^2 \varphi^\dagger \varphi + \lambda \big(\varphi^\dagger \varphi\big)^2}}_{\text{Scalar potential}}$$

- ✤ This Lagrangian is invariant under a *U(1)* global symmetry

$$\varphi \quad \to \quad e^{i\alpha} \varphi$$





✦ Like in classical physics, the ground state lies at the minimum of the potential

- ✤ Let us minimize the potential

Minimization of the scalar potential

✦ Potential

$$V(\varphi) = \mu^2 \varphi^\dagger \varphi + \lambda\big(\varphi^\dagger\varphi\big)^2 \quad \text{with a complex scalar field} \quad \varphi = \frac{1}{\sqrt{2}}\Big(\varphi_1 + i\varphi_2\Big)$$

✤ We search for the minima of the potential $\varphi = \varphi_0$

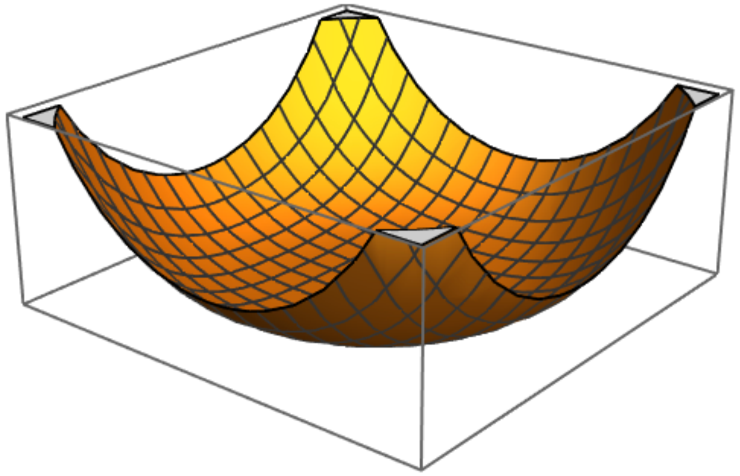
✦ The potential has to be bounded from below: $\lambda > 0$

✦ 2 cases for the bilinear term

✤ $\mu^2 > 0$

- ★ Then $V(\varphi) \geq 0$
- ★ The minimum lies in $|\varphi| = 0$
- ★ The solution $\varphi_0 = 0$ is U(1)-invariant

$$\varphi_0 \to e^{i\alpha}\varphi_0 = 0$$



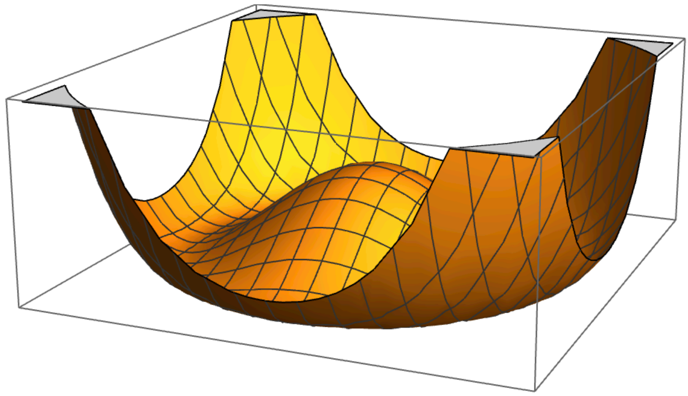
✤ $\mu^2 < 0$

- ★ A '*Mexican hat*' potential
- ★ The minimum lies in $|\varphi| = -\mu^2/(2\lambda)$
- ★ One specific solution is not U(1)-invariant

$$\varphi_0 \;\to\; e^{i\alpha}\varphi_0 \neq \varphi_0$$

$$\frac{1}{\sqrt{2}}\Big(\varphi_1, \varphi_2\Big) \;\to\; \frac{1}{\sqrt{2}}\Big(c_\alpha\varphi_1 + s_\alpha\varphi_2, -s_\alpha\varphi_1 + c_\alpha\varphi_2\Big)$$

The symmetry is spontaneously broken



Interpretation (for $\mu^2 < 0$)

✦ Nature chooses a fundamental state, and we study it perturbatively

$$\varphi_0 = \langle\varphi\rangle = \frac{1}{\sqrt{2}}(v,0) \qquad \text{and} \qquad \varphi = \varphi_0 + \frac{1}{\sqrt{2}}(\eta,\xi)$$

- v is the vacuum expectation value (or vev) of the scalar field
- η and ξ are real scalar fields

✦ Lagrangian

One real massless scalar

$$\mathcal{L} = \boxed{\frac{1}{2}\partial_\mu\eta\partial^\mu\eta} + \boxed{\frac{1}{2}\partial_\mu\xi\partial^\mu\xi} \boxed{- \lambda v^2\eta^2} - \lambda\Big(\frac{\eta^4}{4} + \frac{\xi^4}{4} + \frac{1}{2}\eta^2\xi^2 + v\eta^3 + v\eta\xi^2\Big)$$

One real massive scalar

- Two interacting scalar fields
- The massless field is a Goldstone boson
  (associated with the spontaneous breaking of a continus symmetry)

Spontaneous breaking of a gauge symmetry

✦ We consider a U(1)$_X$ gauge symmetry

- ✤ Massless gauge boson *X* and gauge coupling *g*
- ✤ We charge a set fermionic field Ψ_j under the gauge group
- ✤ We add a scalar field φ, charged under U(1)$_X$, and its associated scalar potential

✦ Kinetic terms of the Lagrangian

$$\mathcal{L} = -\frac{1}{4}X^{\mu\nu}X_{\mu\nu} + \bar{\Psi}_j\ i\gamma^\mu D_\mu\ \Psi^j + (D_\mu\varphi)^\dagger(D^\mu\varphi) \\ + \mu^2\varphi^\dagger\varphi - \lambda(\varphi^\dagger\varphi)^2 - \Big[\varphi\bar{\Psi}\mathbf{y}\Psi + \text{h.c.}\Big]$$

- ✤ Line 1: kinetic and gauge interaction terms
- ✤ Mass terms are forbidden (except for vector-like fermions)
- ✤ Line 2: scalar potential and Yukawa interactions (in flavor space)

✦ We minimize the potential and shift the scalar field by its vev

Gauge boson and scalar mass terms

✦ The scalar field is shifted by its vacuum expectation value

$$\varphi_0 = \langle\varphi\rangle = \frac{1}{\sqrt{2}}(v,0) \qquad \text{and} \qquad \varphi = \varphi_0 + \frac{1}{\sqrt{2}}(\eta,\xi)$$

✦ The scalar potential

$$\mathcal{L}_{\rm V} = -\lambda v^2\eta^2 - \lambda\Big(\frac{\eta^4}{4} + \frac{\xi^4}{4} + \frac{1}{2}\eta^2\xi^2 + v\eta^3 + v\eta\xi^2\Big)$$

- ✤ The η field gets massive: a Higgs bosons
- ✤ The ξ field is massless: a Goldstone mode
- ✤ We get multiscalar interactions

✦ The scalar kinetic term

$$\begin{aligned} D^\mu\varphi^\dagger\ D_\mu\varphi &= \Big[\big(\partial_\mu + ig_X q_\varphi X_\mu\big)\varphi^\dagger\Big]\Big[\big(\partial^\mu - ig_X q_\varphi X^\mu\big)\varphi\Big] \\ &= \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}\partial_\mu\xi\partial^\mu\xi - g_x v\partial_\mu\xi X^\mu + \frac{1}{2}g_X^2 v^2 X_\mu X^\mu + \ldots \end{aligned}$$

- ✤ Mass term for the gauge boson together with its longitudinal polarization
- ✤ The Goldstone mode is absorbed

Fermionic mass terms

- The scalar field is shifted by its vacuum expectation value

$$\varphi_0 = \langle\varphi\rangle = \frac{1}{\sqrt{2}}(v,0) \qquad \text{and} \qquad \varphi = \varphi_0 + \frac{1}{\sqrt{2}}(\eta,\xi)$$

- The Yukawa Lagrangian generates fermion mass terms

$$\mathcal{L}_{\rm Yuk} = -\varphi\bar\Psi\mathbf{y}\Psi \to \frac{1}{\sqrt{2}}\ v\ \bar\Psi\mathbf{y}\Psi + \frac{1}{\sqrt{2}}\ (\eta + i\ \xi)\ \bar\Psi\mathbf{y}\Psi$$

  - The fermions are now massive

Outline

1. Introduction
2. Special relativity
3. Some quantum mechanics
4. Relativistic quantum mechanics
5. Gauge symmetries
6. **Scattering theory**
7. Summary

From Lagrangian to practical computations

✦ Some reminder about scattering theory

- ✤ We consider an initial state *i(t)* at a give time *t*
- ✤ We are interested by the evolution a a time *t'* into a final state *f(t')*
- ✤ The transition is given by the corresponding matrix element of the *S* matrix

$$S_{fi} = \langle f(t') \mid i(t') \rangle = \langle f(t') \mid S \mid i(t) \rangle$$

✦ S_{fi} can be calculated perturbatively

- ✤ Connection to the path integral

$$\int \mathrm{d}\big(\text{fields}\big)\ e^{i \int \mathrm{d}^4x \mathcal{L}(x)}$$

- ✤ Perturbative expansion

$$S_{fi} = \underbrace{\boxed{\delta_{fi}}}_{\substack{\text{No interaction}\\ i=f}} + \overbrace{\boxed{i\Big[\underbrace{\boxed{\int \mathrm{d}^4x \mathcal{L}(x)}}_{\text{One interaction}}\Big]_{fi} - \underbrace{\boxed{\frac{1}{2}\Big[\int \mathrm{d}^4x \mathrm{d}^4x' T\Big\{\mathcal{L}(x)\mathcal{L}(x')\Big\}}\Big]_{fi}}_{\text{Two interactions}} + \dots}}^{\equiv iT_{fi}}$$

Example: weak boson production

- Let us focus on the process $e^+ e^- \to Z$
  - The relevant part of the Lagrangian is schematically given by

$$\mathcal{L}_{eeZ} = \bar{\Psi}_e i\gamma^\mu (g_L P_L + g_R P_R) Z_\mu \Psi_e$$

  - The initial state reads $i = e^+ e^-$ and the final state $f = Z$

- The final state can be obtained through a single interaction (and also more)

$$S_{fi} = \delta_{fi} + i\Big[\int \mathrm{d}^4x \mathcal{L}(x)\Big]_{fi} - \frac{1}{2}\Big[\int \mathrm{d}^4x \mathrm{d}^4x' T\Big\{\mathcal{L}(x)\mathcal{L}(x')\Big\}\Big]_{fi} + \ldots$$

  - The leading contribution to S_{fi} is thus given by

$$i\Big[\int \mathrm{d}^4x \mathcal{L}(x)\Big]_{fi} = i\int \mathrm{d}^4x \, \Big[\bar{\Psi}_e \, \gamma^\mu (g_L P_L + g_R P_R) Z_\mu \Psi^e\Big]_{fi}$$

  - Easy generalization to cases with more interactions
    - For instance: $e^+ e^- \to Z^{(*)} \to \mu^+ \mu^-$
    - Chronology matters (which interaction comes first)
    - Intermediate virtual particles are allowed

How to calculate the integrals?

Momentum conservation

✦ We consider a generic collision process (2 → n)

$$\underbrace{i_1(p_a) + i_2(p_b)}_{\text{Initial state}} \to \underbrace{f_1(p_1) + \ldots + f_n(p_n)}_{n\text{-body final state}}$$

✤ The p_k indicate the four-momenta of the different particles

✦ Reminder: relativistic wave equations yield plane waves

$$\Psi(x) = \int \mathrm{d}^4p \, \Big[(\ldots)e^{-ip\cdot x} + (\ldots)e^{+ip\cdot x}\Big]$$

✤ The dots stand for annihilation and creation operators related to (anti)particles

✤ The solution are inserted back into the *S* matrix $S_{fi} = \delta_{fi} + i\Big[\int \mathrm{d}^4x \mathcal{L}(x)\Big]_{fi} + \ldots$

✤ The integral over spacetime factorizes
(the only *x*-dependence lies at the level of the exponentials)

$$\int \mathrm{d}^4x \Big[e^{-ip_a\cdot x}e^{-ip_b\cdot x}\prod_j e^{-ip_j\cdot x}\Big] = (2\pi)^4 \ \delta^{(4)}\Big(p_a + p_b - \sum_j p_j\Big)$$

We get energy-momentum conservation
We now need to work out momentum integrals

Matrix element, total & differential cross sections

- The matrix element is defined from the interacting part of the *S* matrix

$$iT_{fi} = (2\pi)^4\ \delta^{(4)}\Big(p_a + p_b - \sum_j p_j\Big) iM_{fi}$$

  - Momentum conservation is factorized

- The total rate is given after integrating over the four-momenta

$$\sigma = \frac{1}{F}\int \mathrm{dPS}^{(n)}\overline{|M_{fi}|^2}$$

  - We integrate over all final-state configurations (the phase space integral)
  - We average over all initial-state configurations (the flux factor)

- Differential cross sections are trivially derived

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\omega} = \frac{1}{F}\int \mathrm{dPS}^{(n)}\overline{|M_{fi}|^2}\delta\Big(\omega - \omega(p_a, p_b, p_1, \ldots, p_n)\Big)$$

  - Numerical methods are usually used for the integration (multidimensional)

The phase space integral and the flux factor

✦ The phase space integration reads

$$\int \mathrm{dPS}^{(n)} = \int (2\pi)^4\ \delta^{(4)}\Big(p_a + p_b - \sum_j p_j\Big) \prod_j \left[\frac{\mathrm{d}^4 p_j}{(2\pi)^4} (2\pi)\delta(p_j^2 - m_j^2)\theta(p_j^0)\right]$$

- Include momentum conservation, mass-shell conditions, positivity of the energy
- Integrals over all final-state momentum configurations

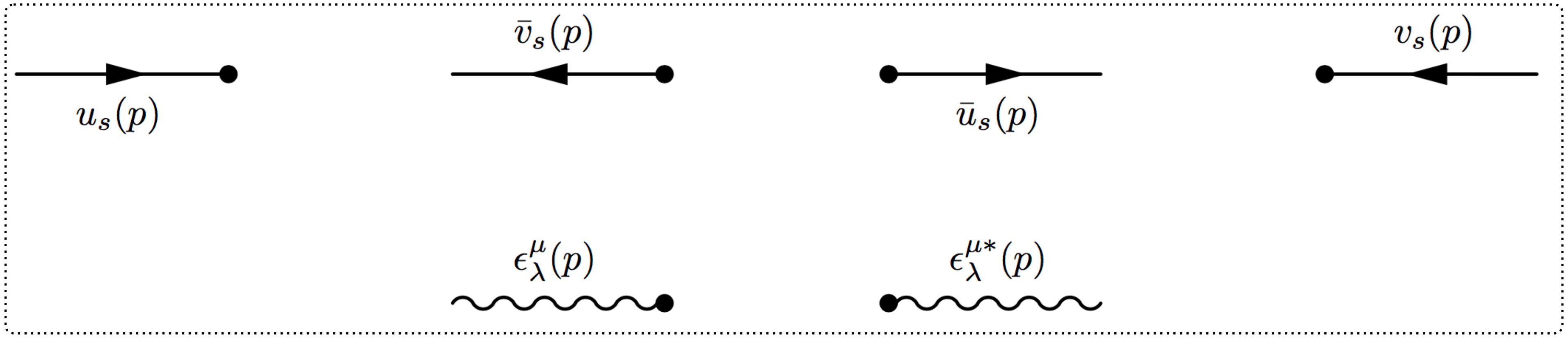
✦ The flux factor reads

$$\frac{1}{F} = \frac{1}{4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}}$$

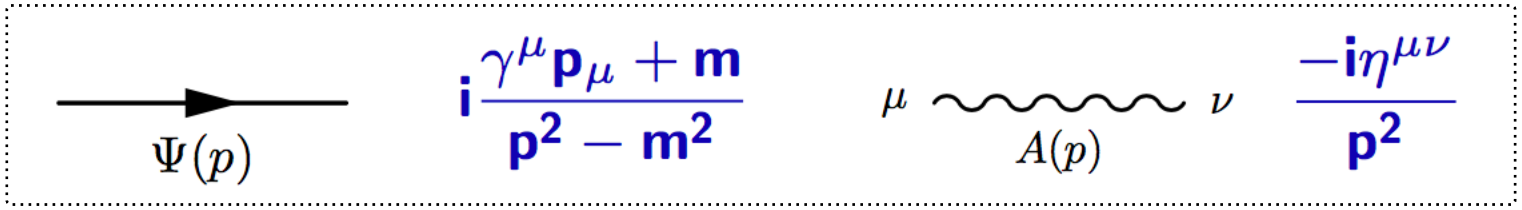
- Normalizes the cross section to the initial state density per surface unit

The matrix element

- The matrix element can be calculated from Feynman rules
  - External particles: spinors, polarization vectors, *etc.*



  - Internal particles: propagators (gauge dependent for the vectors)



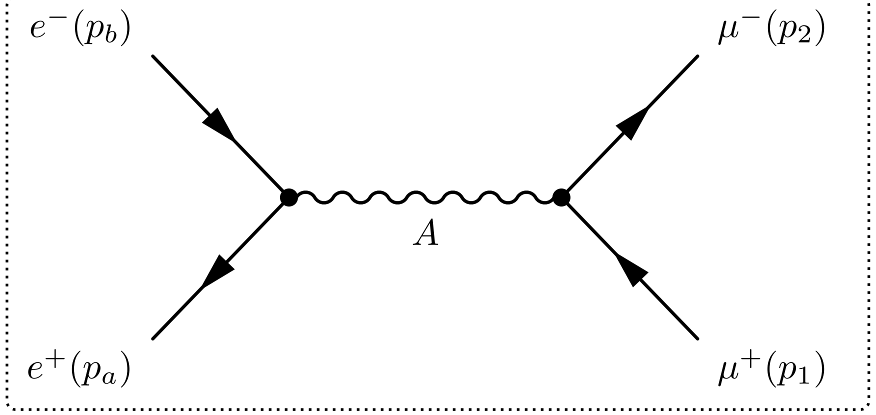
$$\mathbf{i}\frac{\gamma^\mu \mathbf{p}_\mu + \mathbf{m}}{\mathbf{p}^2 - \mathbf{m}^2} \qquad \frac{-\mathbf{i}\eta^{\mu\nu}}{\mathbf{p}^2}$$

  - Vertices: Feynman rules derived from the Lagrangian

- Once the process is fixed, one draw all possible diagrams

Example: e⁺e⁻ → $\mu^+\mu^-$ (I)

- Muon-antimuon production in electron-positron collisions



- One single diagram in QED (no Z-boson)

- The amplitude reads (after following reversely all fermion lines)

$$iM = \Big[\bar{v}_{s_a}(p_a)\ (-ie\gamma^\mu)\ u_{s_b}(p_b)\Big]\Big[\bar{u}_{s_2}(p_2)\ (-ie\gamma^\nu)\ v_{s_1}(p_1)\Big]\frac{-i\eta_{\mu\nu}}{(p_a+p_b)^2}$$

- The conjugate amplitude is given by

$$-iM^\dagger = \Big[\bar{u}_{s_b}(p_b)\ (ie\gamma^\mu)\ v_{s_a}(p_a)\Big]\Big[\bar{v}_{s_1}(p_1)\ (ie\gamma^\nu)\ u_{s_2}(p_2)\Big]\frac{i\eta_{\mu\nu}}{(p_a+p_b)^2}$$

- The (averaged) squared matrix element is then

$$\overline{|M|^2} = \frac{1}{2}\ \frac{1}{2}\ (iM)\ \big(-iM^\dagger\big)$$

★ Two 1/2 factors for the electron and positron spins

Example: e⁺e⁻ → $\mu^+\mu^-$ (2)

✦ The (averaged) squared matrix element is then

$$\begin{aligned}\overline{|M|^2} &= \frac{1}{2}\,\frac{1}{2}\,\left(iM\right)\,\left(-iM^\dagger\right)\\ &= \frac{e^4}{4(p_a+p_b)^4}\mathrm{Tr}\Big[\gamma^\mu(\not{p}_b+m_e)\gamma^\rho(\not{p}_a-m_e)\Big]\ \mathrm{Tr}\Big[\gamma_\mu(\not{p}_1-m_\mu)\gamma_\rho(\not{p}_2+m_\mu)\Big]\end{aligned}$$

- ✤ We have performed a sum over all particle spins
- ✤ Dirac and Maxwell equations allow to derive (simplifications in the calculations)

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p}+m \quad \text{and} \quad \sum_s v_s(p)\bar{v}_s(p) = \not{p}-m$$

$$\sum_\lambda \varepsilon^\mu_\lambda(p)\varepsilon^{\nu\,*}_\lambda(p) = -\eta^{\mu\nu}$$

✦ Properties of the Dirac matrices and Mandelstam variables

$$\overline{|M|^2} = \frac{8e^4}{(p_a+p_b)^4}\Big[(p_b\cdot p_1)(p_a\cdot p_2)+(p_b\cdot p_2)(p_a\cdot p_1)\Big] = \frac{2e^4}{s^2}\left[t^2+u^2\right]$$

with $s=(p_a+p_b)^2=(p_1+p_2)^2\ ,\quad t=(p_a-p_1)^2=(p_b-p_2)^2$
$u=(p_a-p_2)^2=(p_b-p_1)^2$

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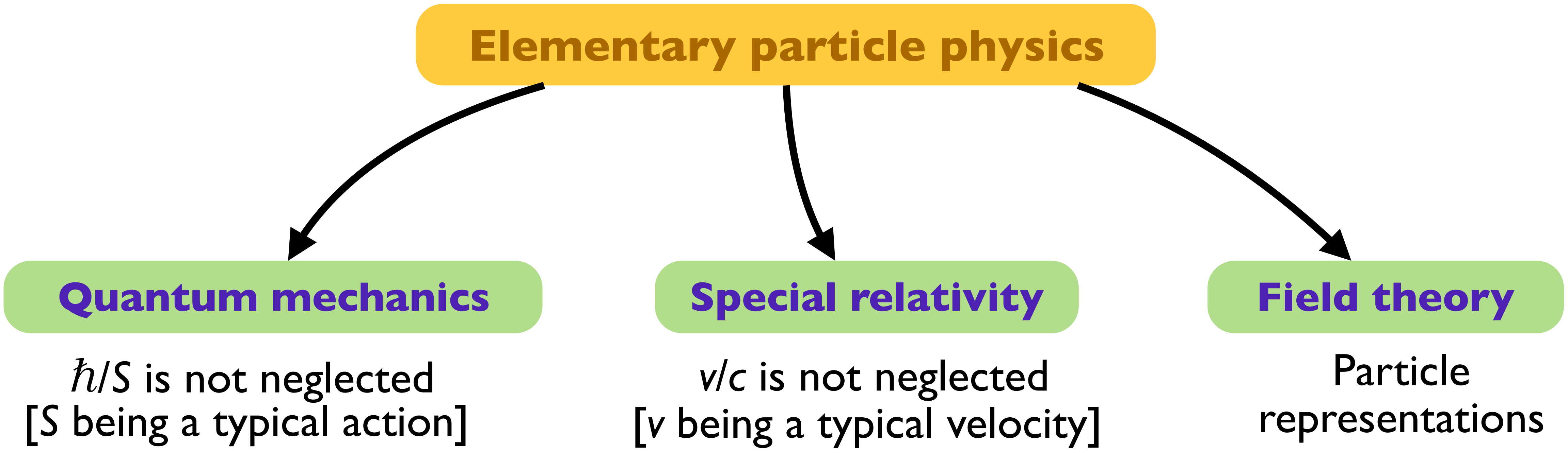
Summary

- The stage itself



Elementary particle physics

Quantum mechanics

$\hbar/S$ is not neglected
[S being a typical action]

Special relativity

v/c is not neglected
[v being a typical velocity]

Field theory

Particle representations

- The elementary particles and their interactions are linked to **symmetries**
  - Poincaré invariance ≡ particle types (scalars, fermions, vectors, …)
  - Gauge symmetries ≡ electromagnetism, weak and strong interactions, …
- Other tools
  - Symmetry breaking ≡ mass generation
  - Scattering theory ≡ observables, testing a theory against data