Synchrotron Radiation
Basic physics, generation, properties

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Useful books and references

H. Wiedemann, *Synchrotron Radiation*
Springer-Verlag Berlin Heidelberg 2003

H. Wiedemann, *Particle Accelerator Physics I and II*
Springer Study Edition, 2003

A.Hofmann, *The Physics of Synchrotron Radiation*
Cambridge University Press 2004

A. W. Chao, M. Tigner, *Handbook of Accelerator Physics and Engineering*, World Scientific 1999

CERN Accelerator School Proceedings

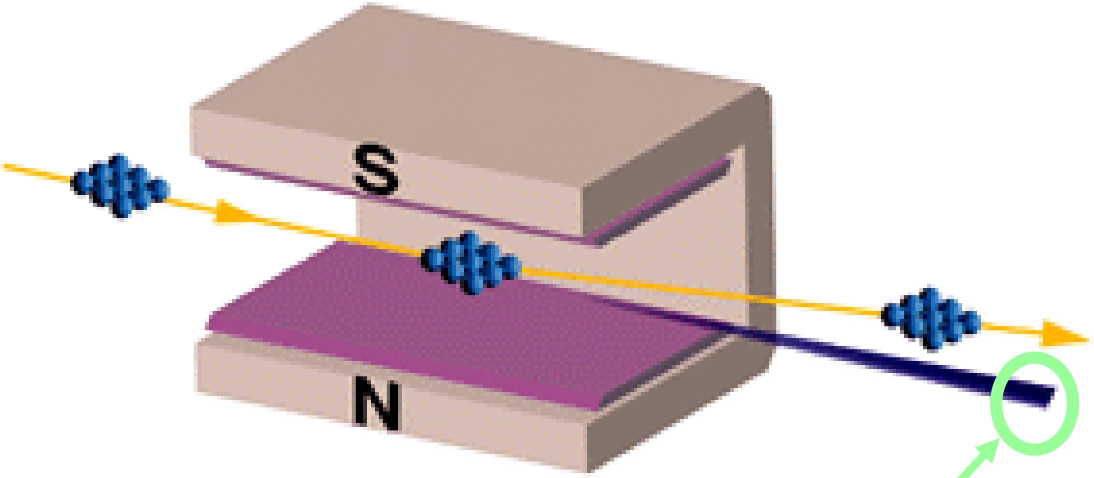
Synchrotron Radiation and Free Electron Lasers

Grenoble, France, 22 - 27 April 1996
(A. Hofmann's lectures on synchrotron radiation)
CERN Yellow Report 98-04

Brunnen, Switzerland, 2 – 9 July 2003
CERN Yellow Report 2005-012

Previous CAS Schools Proceedings

Curved orbit of electrons in magnet field



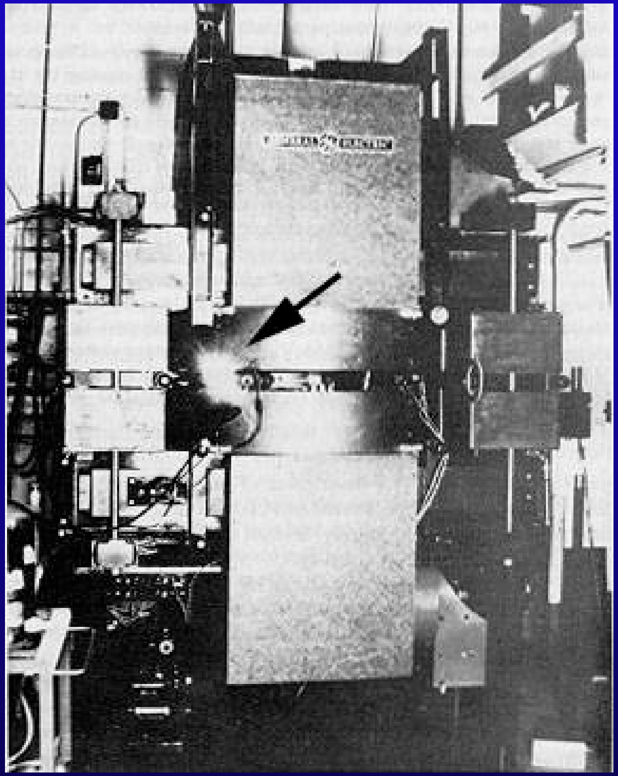
Accelerated charge → Electromagnetic radiation

Electromagnetic waves

Crab Nebula
6000 light years away

First light observed
1054 AD

GE Synchrotron
New York State



First light observed
1947

Synchrotron radiation: some dates

- 1873 Maxwell’s equations
- 1887 Hertz: electromagnetic waves
- 1898 Liénard: retarded potentials
- 1900 Wiechert: retarded potentials
- 1908 Schott: Adams Prize Essay

... waiting for accelerators …
1940: 2.3 MeV betatron,Kerst, Serber

Maxwell equations (poetry)

War es ein Gott, der diese Zeichen schrieb
Die mit geheimnisvoll verborg'nem Trieb
Die Kräfte der Natur um mich enthüllen
Und mir das Herz mit stiller Freude füllen.
Ludwig Boltzman

Was it a God whose inspiration
Led him to write these fine equations
Nature's fields to me he shows
And so my heart with pleasure glows.
translated by John P. Blewett

Synchrotron radiation: some dates

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... waiting for accelerators …
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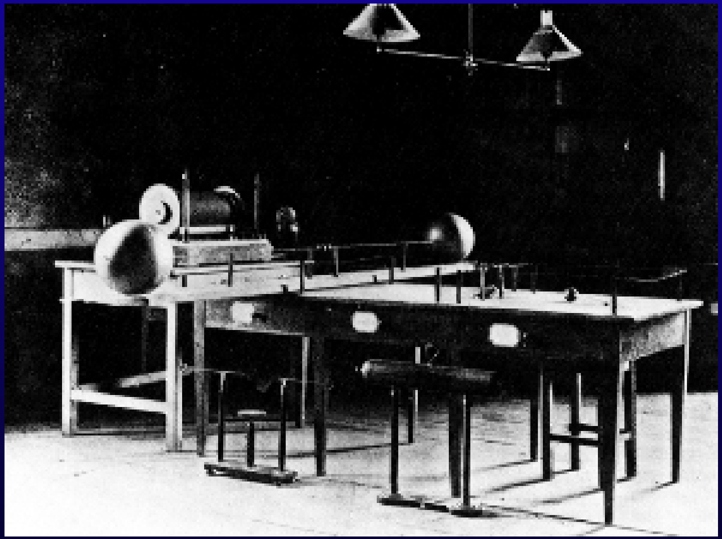
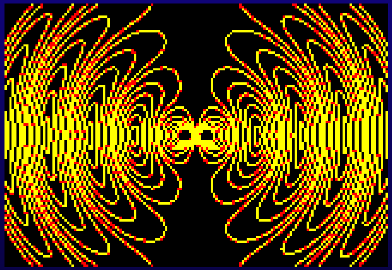
THEORETICAL UNDERSTANDING →

1873 Maxwell's equations

→ made evident that changing charge densities would result in electric fields that would radiate outward

1887 Heinrich Hertz demonstrated such waves:





It's of no use whatsoever[...] this is just an experiment that proves Maestro Maxwell was right—we just have these mysterious electromagnetic waves that we cannot see with the naked eye. But they are there.

Synchrotron radiation: some dates

1873	Maxwell’s equations
1887	Hertz: electromagnetic waves
1898	Liénard: retarded potentials
1900	Wiechert: retarded potentials
1908	Schott: Adams Prize Essay

... waiting for accelerators …

1940: 2.3 MeV betatron,Kerst, Serber

Donald Kerst: first betatron (1940)

"Ausserordentlichhochgeschwindigkeitelektronenent wickelndenschwerarbeitsbeigollitron"

Synchrotron radiation: some dates

- 1946 Blewett observes energy loss due to synchrotron radiation
  100 MeV betatron
- 1947 First visual observation of SR
  70 MeV **synchrotron**, GE Lab

NAME!

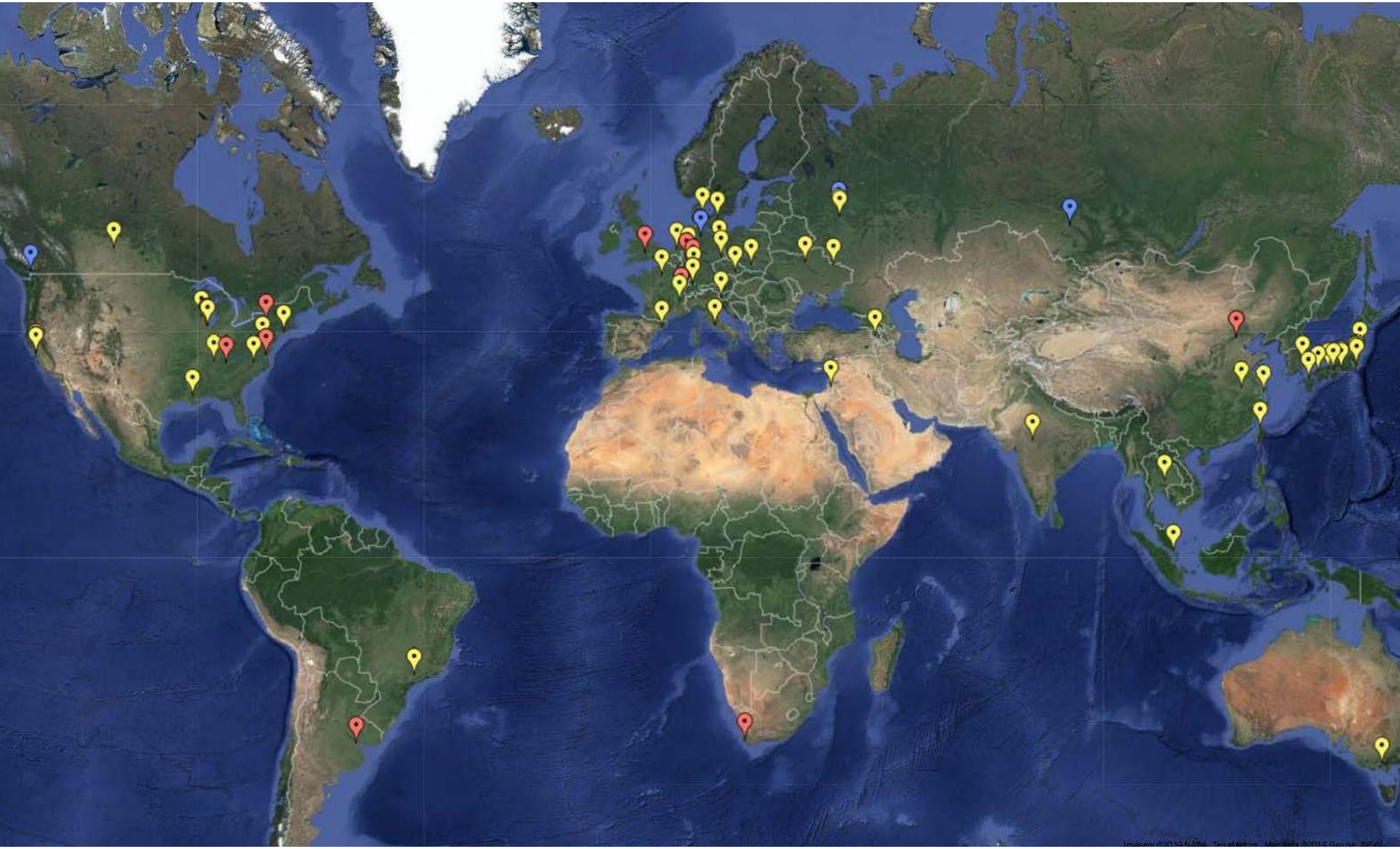
- 1949 Schwinger PhysRev paper

…

- 1976 Madey: first demonstration of Free Electron laser

Paul Scherrer Institute, Switzerland

60‘000 SR users world-wide



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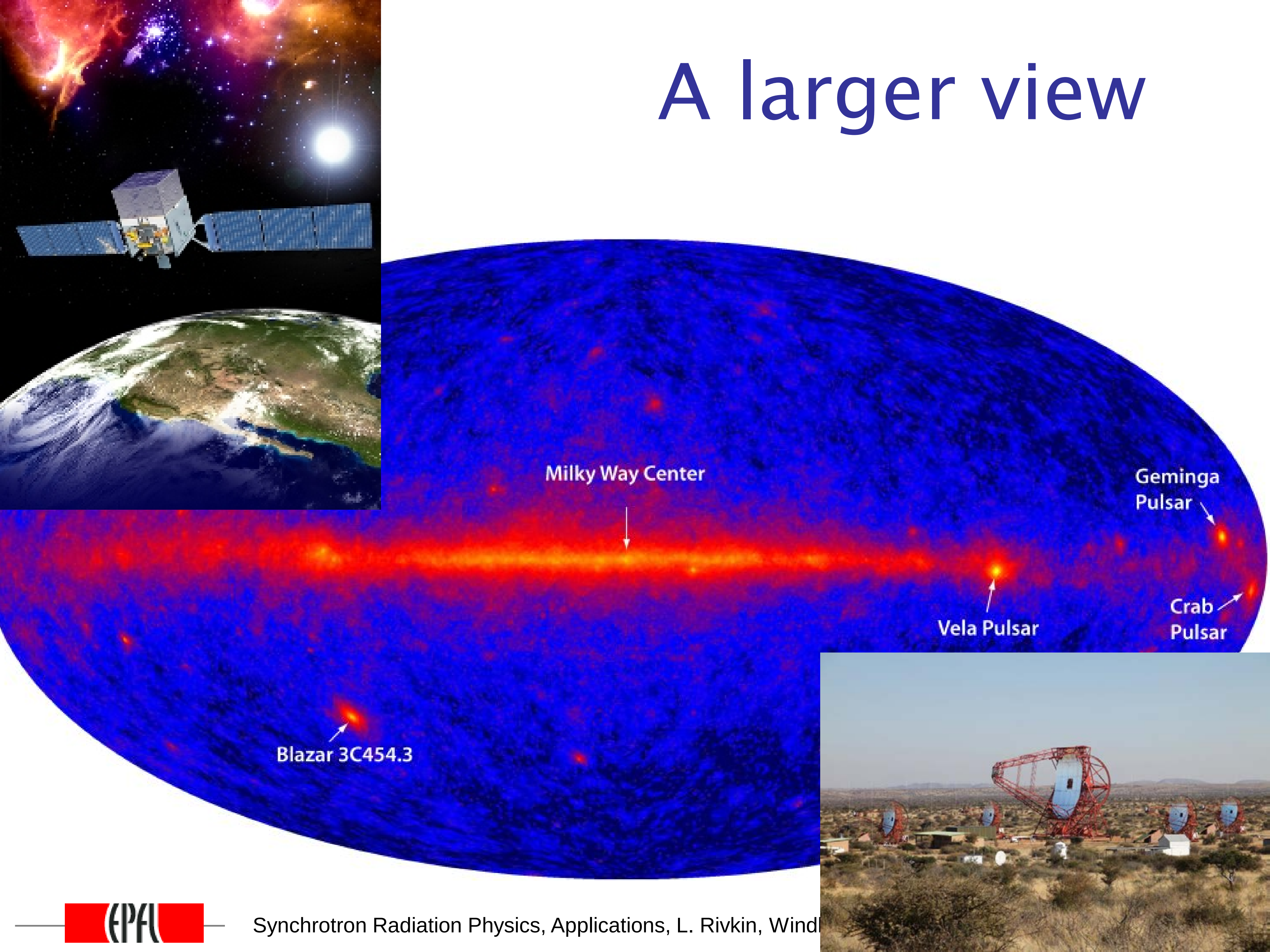
A larger view

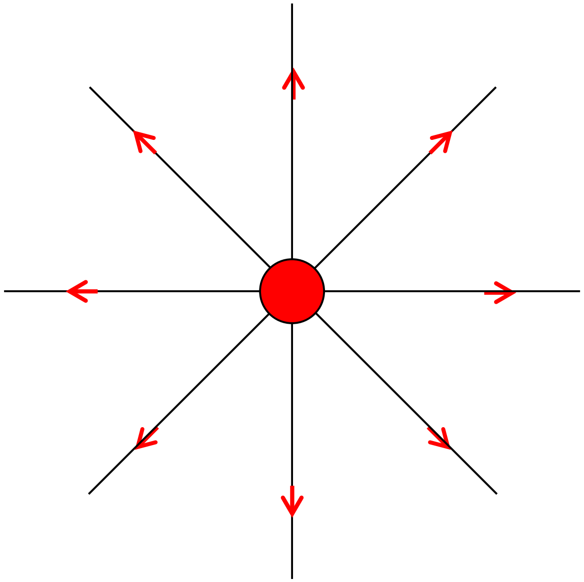


Why do they radiate?

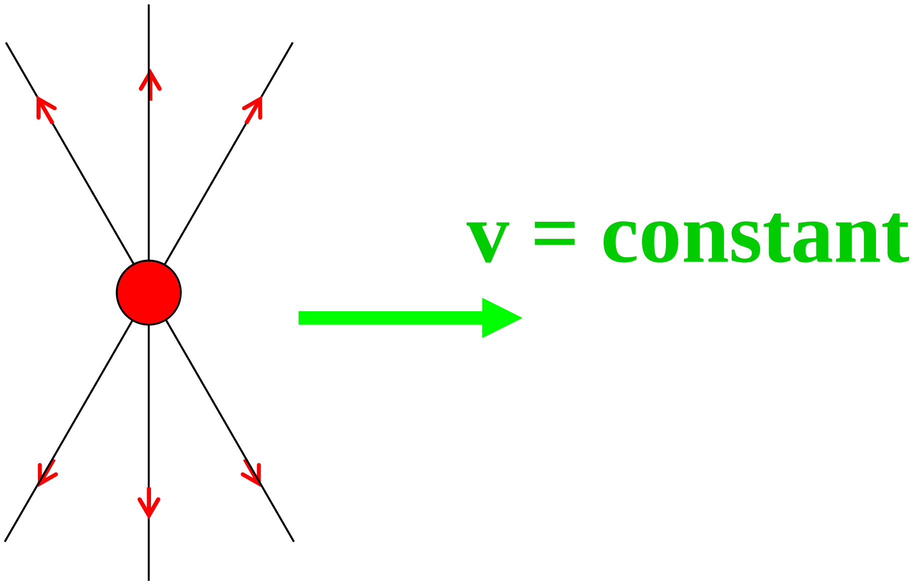
Synchrotron Radiation is not as simple as it seems

… I will try to show that it is much simpler

Charge at rest
Coulomb field, no radiation



Uniformly moving charge does not radiate



v = constant

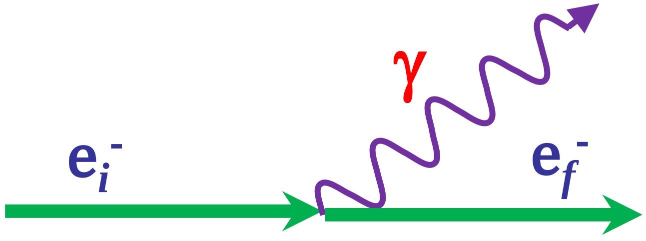
But! Cerenkov!

Free isolated electron cannot emit a photon

Easy proof using 4–vectors and relativity

- momentum conservation if a photon is emitted

$$\boldsymbol{P}_i = \boldsymbol{P}_f + \boldsymbol{P}_\gamma$$



- square both sides

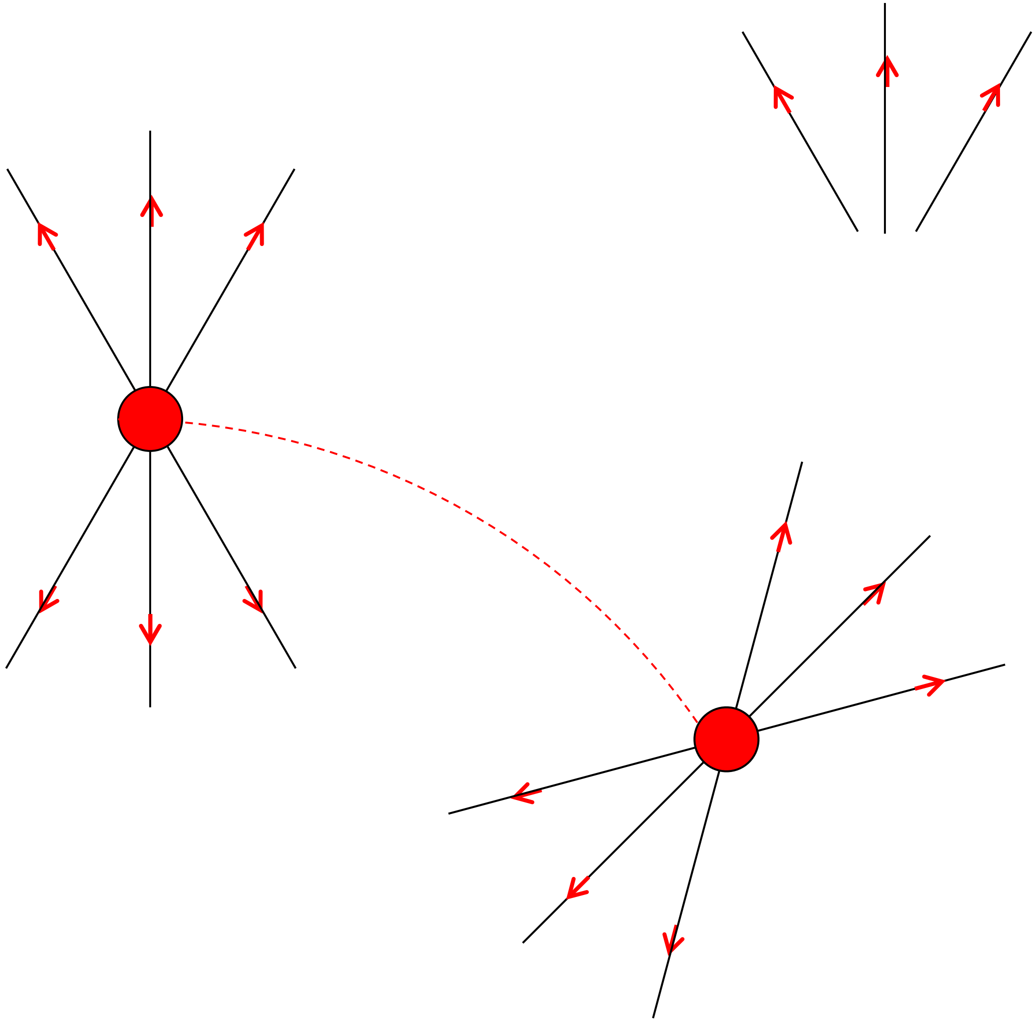
$$m^2 = m^2 + 2\boldsymbol{P}_f \cdot \boldsymbol{P}_\gamma + 0 \;\Rightarrow\; \boldsymbol{P}_f \cdot \boldsymbol{P}_\gamma = 0$$

- in the rest frame of the electron

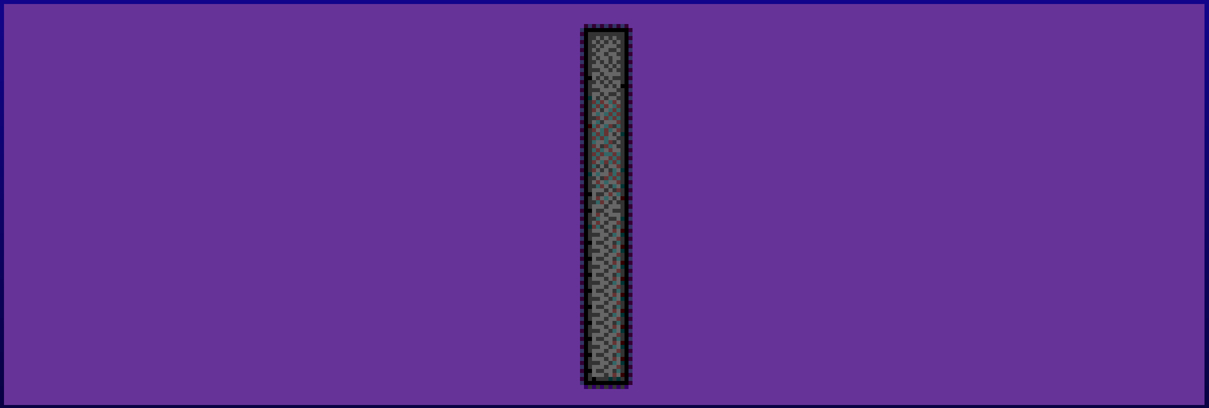
$$\boldsymbol{P}_f = (m, 0) \qquad \boldsymbol{P}_\gamma = (E_\gamma, p_\gamma)$$

this means that the photon energy must be zero.

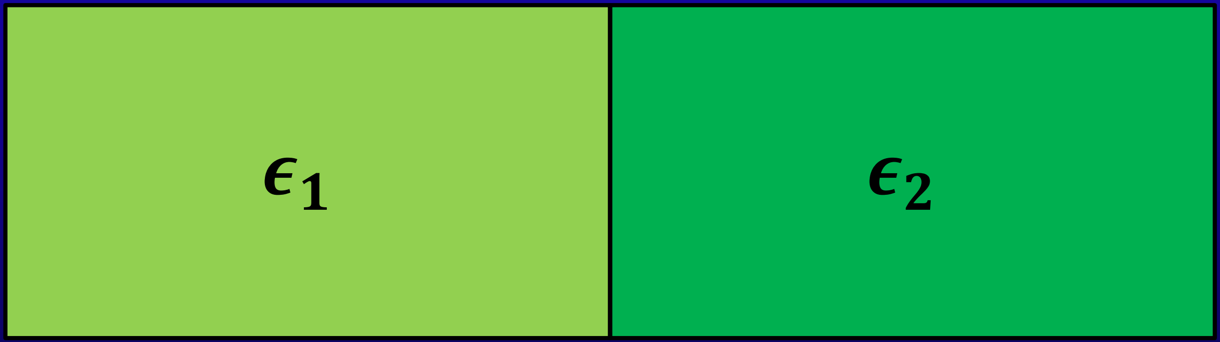
We need to separate the field from charge



Bremsstrahlung
or
“braking” radiation



Transition Radiation



$$c_1 = \frac{1}{\sqrt{\epsilon_1 \mu_1}} \qquad c_2 = \frac{1}{\sqrt{\epsilon_2 \mu_2}}$$

Liénard–Wiechert potentials

$$\varphi(t) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\left[r\left(1-\vec{\mathbf{n}}\cdot\vec{\beta}\right)\right]_{ret}} \qquad \vec{\mathbf{A}}(t) = \frac{q}{4\pi\varepsilon_0 c^2}\left[\frac{\vec{\mathbf{v}}}{r\left(1-\vec{\mathbf{n}}\cdot\vec{\beta}\right)}\right]_{ret}$$

and the electromagnetic fields:

$$\nabla\cdot\vec{\mathbf{A}} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0$$ (Lorentz gauge)

$$\vec{\mathbf{B}} = \nabla\times\vec{\mathbf{A}}$$

$$\vec{\mathbf{E}} = -\nabla\varphi - \frac{\partial\vec{\mathbf{A}}}{\partial t}$$

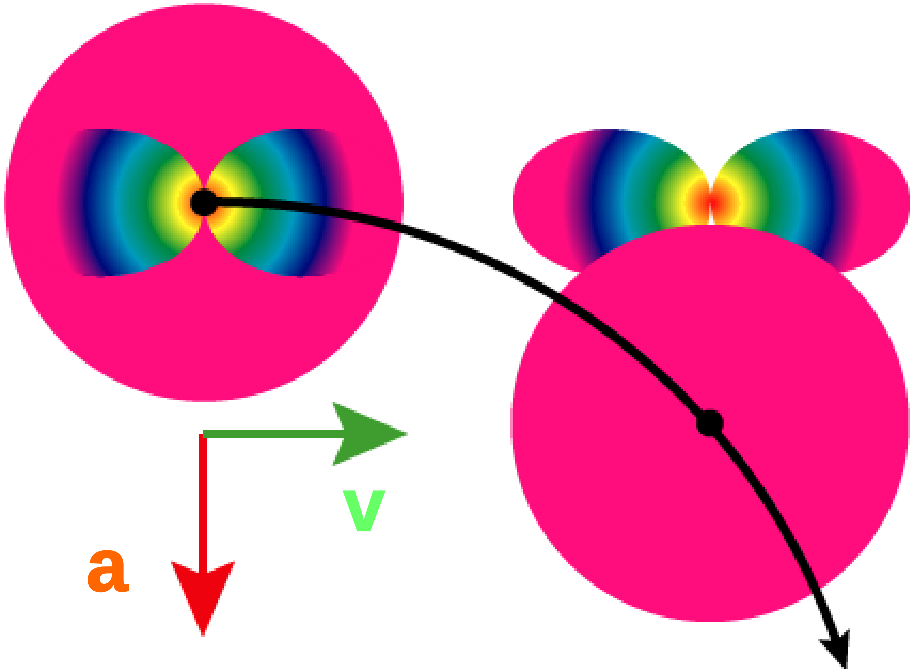
Fields of a moving charge

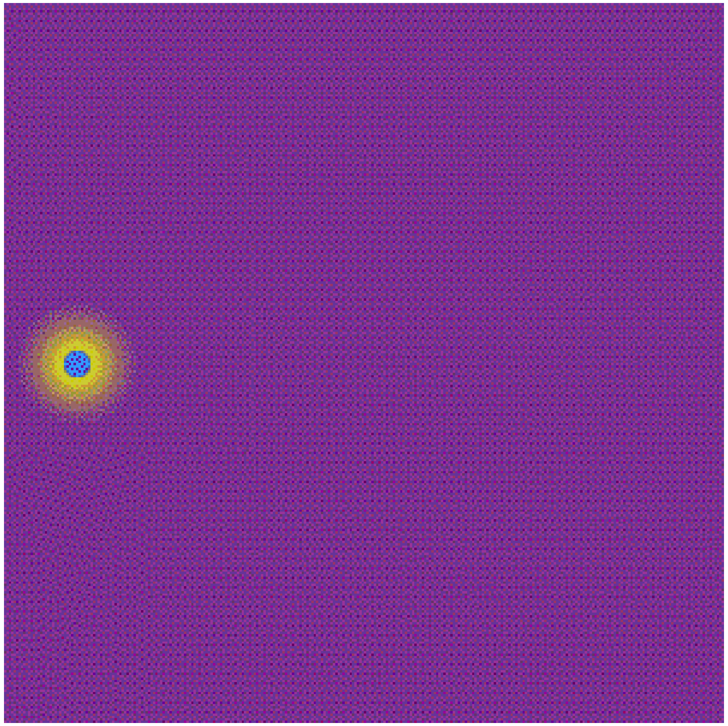
$$\vec{\mathbf{E}}(t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\vec{\mathbf{n}} - \vec{\boldsymbol{\beta}}}{\left(1 - \vec{\mathbf{n}}\cdot\vec{\boldsymbol{\beta}}\right)^3\gamma^2}\cdot\boxed{\frac{\mathbf{1}}{\mathbf{r}^2}}\right]_{ret} +$$

$$\frac{q}{4\pi\varepsilon_0 c}\left[\frac{\vec{\mathbf{n}}\times\left[\left(\vec{\mathbf{n}} - \vec{\boldsymbol{\beta}}\right)\times\boxed{\dot{\vec{\boldsymbol{\beta}}}}\right]}{\left(1 - \vec{\mathbf{n}}\cdot\vec{\boldsymbol{\beta}}\right)^3\gamma^2}\cdot\boxed{\frac{\mathbf{1}}{\mathbf{r}}}\right]_{ret}$$

$$\vec{\mathbf{B}}(\mathrm{t}) = \frac{1}{\mathrm{c}}\left[\vec{\mathbf{n}}\times\vec{\mathbf{E}}\right]$$

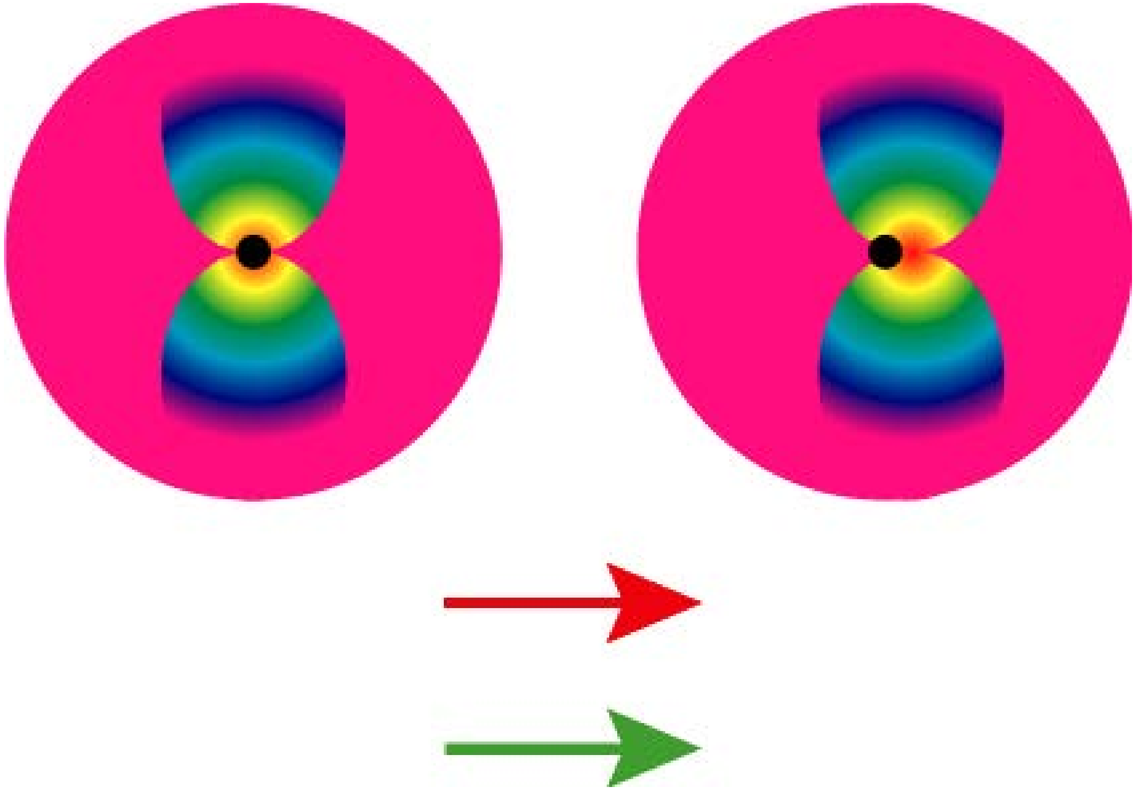
Transverse acceleration





Radiation field quickly separates itself from the Coulomb field

Longitudinal acceleration

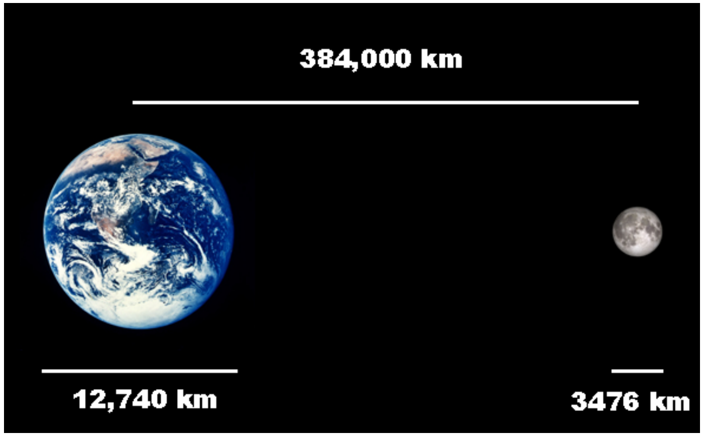


Radiation field cannot separate itself from the Coulomb field

Synchrotron Radiation Basic Properties

Beams of ultra-relativistic particles: e.g. a race to the Moon

An electron with energy of a few GeV emits a photon… a race to the Moon!



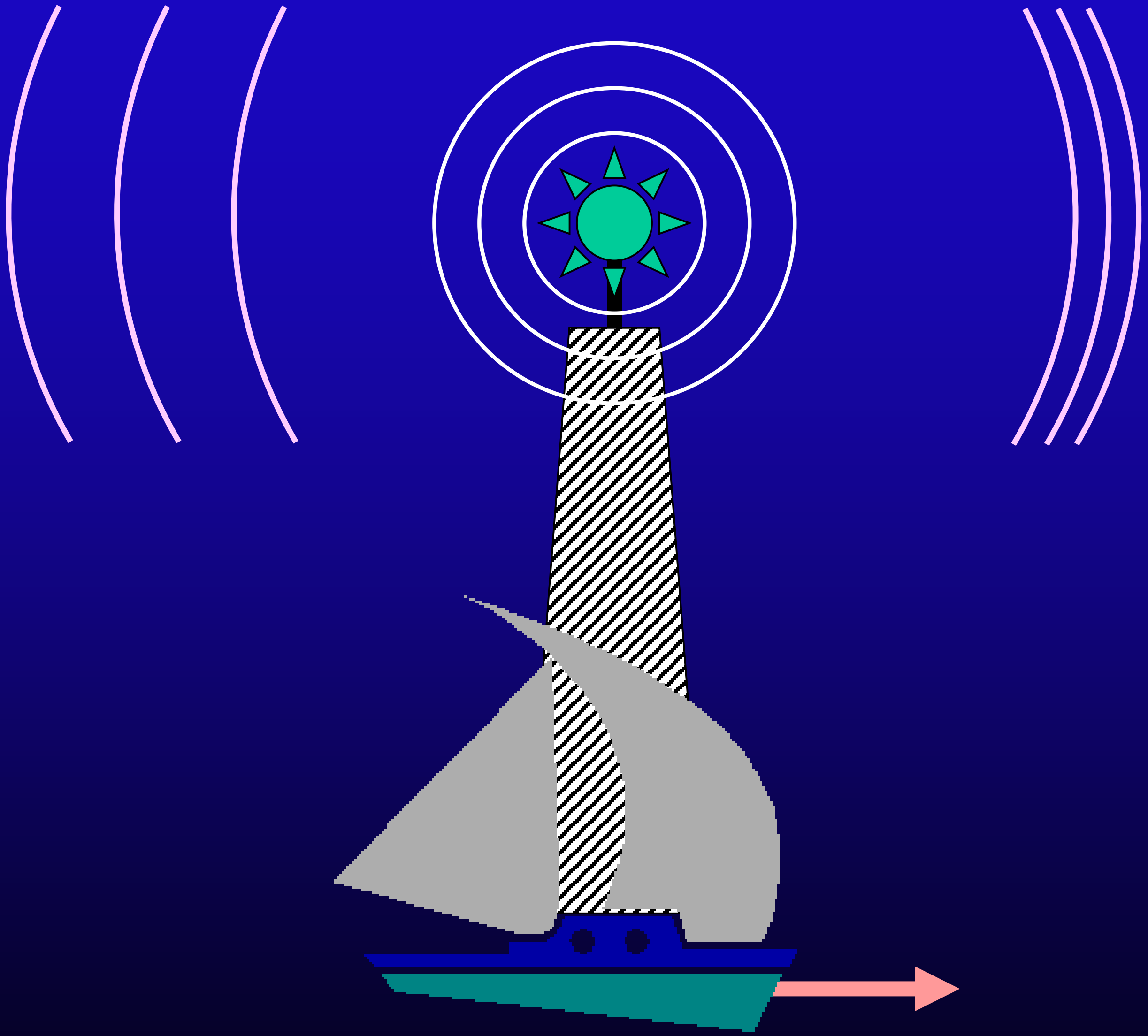
$$\Delta t = \frac{L}{\beta c} - \frac{L}{c} = \frac{L}{\beta c}(1-\beta) \sim \frac{L}{\beta c} \cdot \frac{1}{2\gamma^2}$$

Electron will lose

- by only 8 meters
- the race will last only 1.3 seconds

$$\Delta L = L(1-\beta) \cong \frac{L}{2\gamma^2}$$

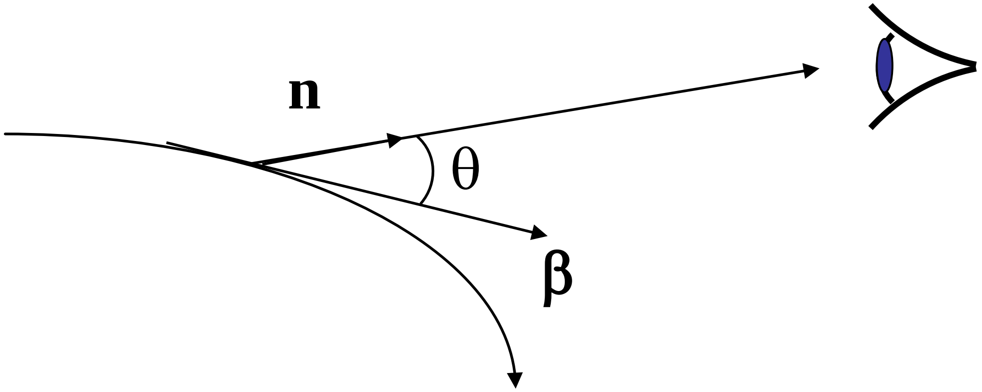
Moving Source of Waves



Cape Hatteras, 1999

Time compression

Electron with velocity β emits a wave with period T_{emit} while the observer sees a different period T_{obs} because the electron was moving towards the observer



$$T_{obs} = (1 - \mathbf{n} \cdot \boldsymbol{\beta}) T_{emit}$$

The wavelength is shortened by the same factor

$$\lambda_{obs} = (1 - \beta \cos\theta) \lambda_{emit}$$

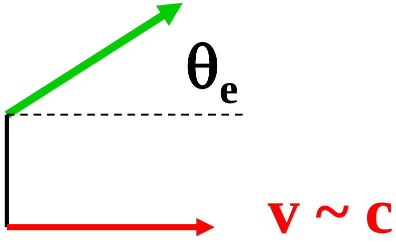
in ultra-relativistic case, looking along a tangent to the trajectory

$$\boxed{\lambda_{obs} = \frac{1}{2\gamma^2} \lambda_{emit}}$$

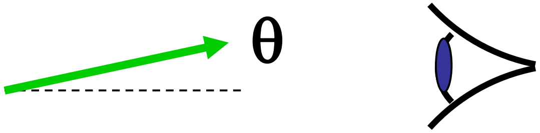
since

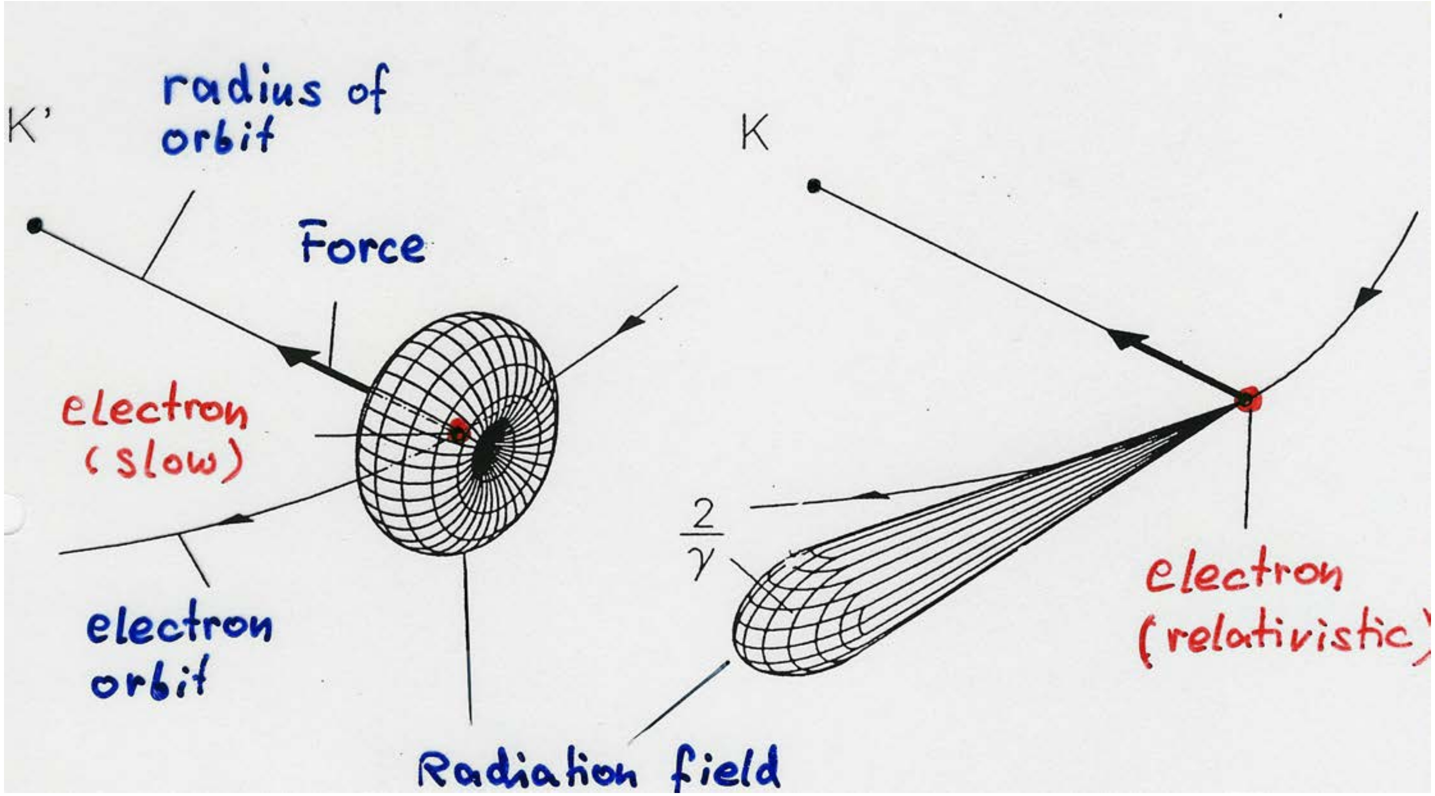
$$\boxed{1 - \beta = \frac{1 - \beta^2}{1 + \beta} \cong \frac{1}{2\gamma^2}}$$

Radiation is emitted into a narrow cone



$$\theta = \frac{1}{\gamma}\cdot\theta_e$$



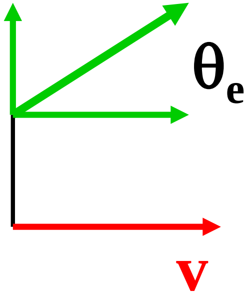


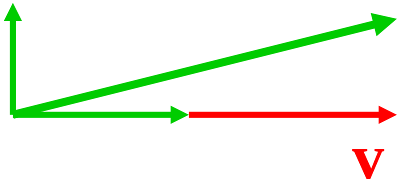
v << c

v ≈ c

Sound waves (non-relativistic)

Angular collimation





$$\theta = \frac{v_{s\perp}}{v_{s\parallel} + \mathbf{v}} = \frac{v_{s\perp}}{v_{s\parallel}} \cdot \frac{1}{1 + \frac{\mathbf{v}}{v_s}} \approx \theta_e \cdot \frac{1}{1 + \frac{\mathbf{v}}{v_s}}$$

Doppler effect (moving source of sound)

$$\lambda_{heard} = \lambda_{emitted}\left(1 - \frac{\mathbf{v}}{v_s}\right)$$

Synchrotron radiation power

Power emitted is proportional to:

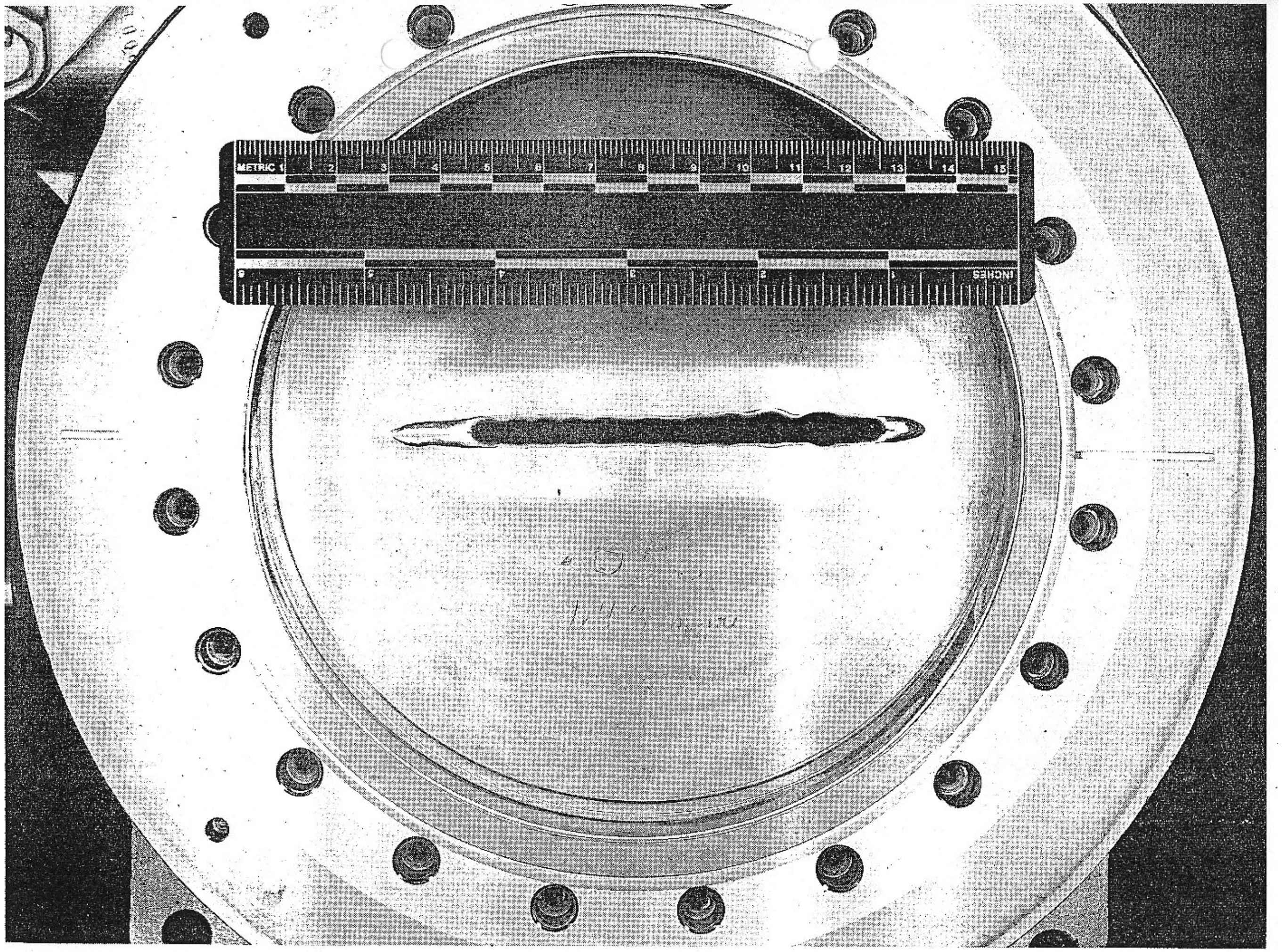
$$\boxed{P \propto E^2 B^2}$$

Energy Magnetic Field

$$\boxed{P_\gamma = \frac{cC_\gamma}{2\pi} \cdot \frac{E^4}{\rho^2}}$$

$$C_\gamma = \frac{4\pi}{3}\frac{r_e}{(m_e c^2)^3} = 8.858 \cdot 10^{-5}\left[\frac{\mathrm{m}}{\mathrm{GeV}^3}\right]$$

The power is all too real!



ig. 12. Damaged X-ray ring front end gate valve. The power incident on the valve was approximately 1 kW for a duration estimated to 2–10 min and drilled a hole through the valve plate.

Synchrotron radiation power

Power emitted is proportional to:

$$\boxed{P \propto E^2 B^2}$$

$$\boxed{P_\gamma = \frac{cC_\gamma}{2\pi} \cdot \frac{E^4}{\rho^2}}$$

$$\boxed{P_\gamma = \frac{2}{3}\alpha\hbar c^2 \cdot \frac{\gamma^4}{\rho^2}}$$

$$C_\gamma = \frac{4\pi}{3}\frac{r_e}{(m_e c^2)^3} = 8.858 \cdot 10^{-5}\left[\frac{\mathrm{m}}{\mathrm{GeV}^3}\right]$$

$$\boxed{\alpha = \frac{1}{137}}$$

$$\boxed{\hbar c = 197 \text{ Mev} \cdot \text{fm}}$$

Energy loss per turn:

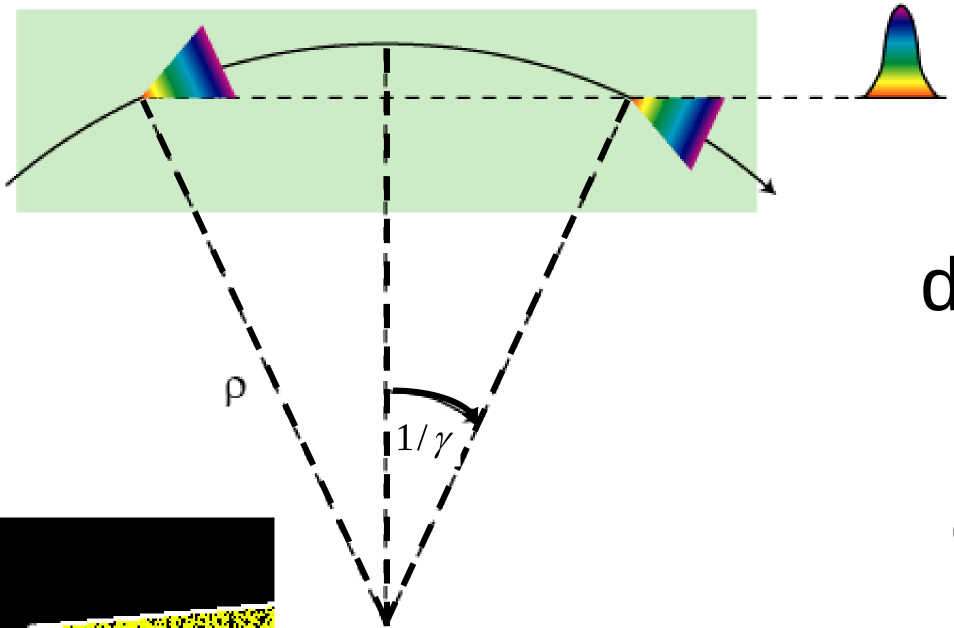
$$\boxed{U_0 = C_\gamma \cdot \frac{E^4}{\rho}}$$

$$\boxed{U_0 = \frac{4\pi}{3}\alpha\hbar c\frac{\gamma^4}{\rho}}$$

Typical frequency of synchrotron light

Due to extreme collimation of light observer sees only a small portion of electron trajectory **(a few mm)**

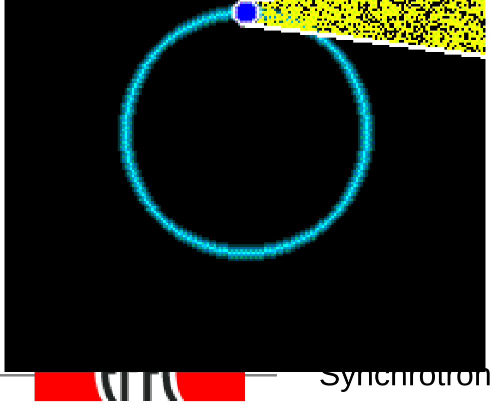
$$l \sim \frac{2\rho}{\gamma}$$



Pulse length: difference in times it takes an electron and a photon to cover this distance

$$\Delta t \sim \frac{l}{\beta c} - \frac{l}{c} = \frac{l}{\beta c}(1-\beta)$$

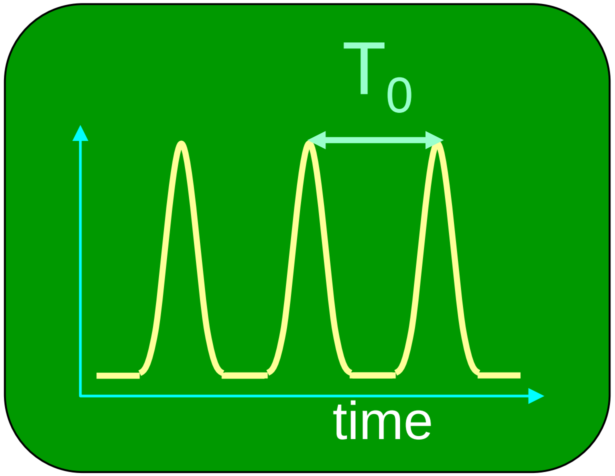
$$\Delta t \sim \frac{2\rho}{\gamma c}\cdot\frac{1}{2\gamma^2}$$



$$\omega \sim \frac{1}{\Delta t} \sim \gamma^3\omega_0$$

Spectrum of synchrotron radiation

- Synchrotron light comes in a series of flashes every T_0 (revolution period)



- the spectrum consists of harmonics of

$$\omega_0 = \frac{1}{T_0}$$

- flashes are extremely short: harmonics reach up to very high frequencies

$$\omega_{typ} \cong \gamma^3 \omega_0$$

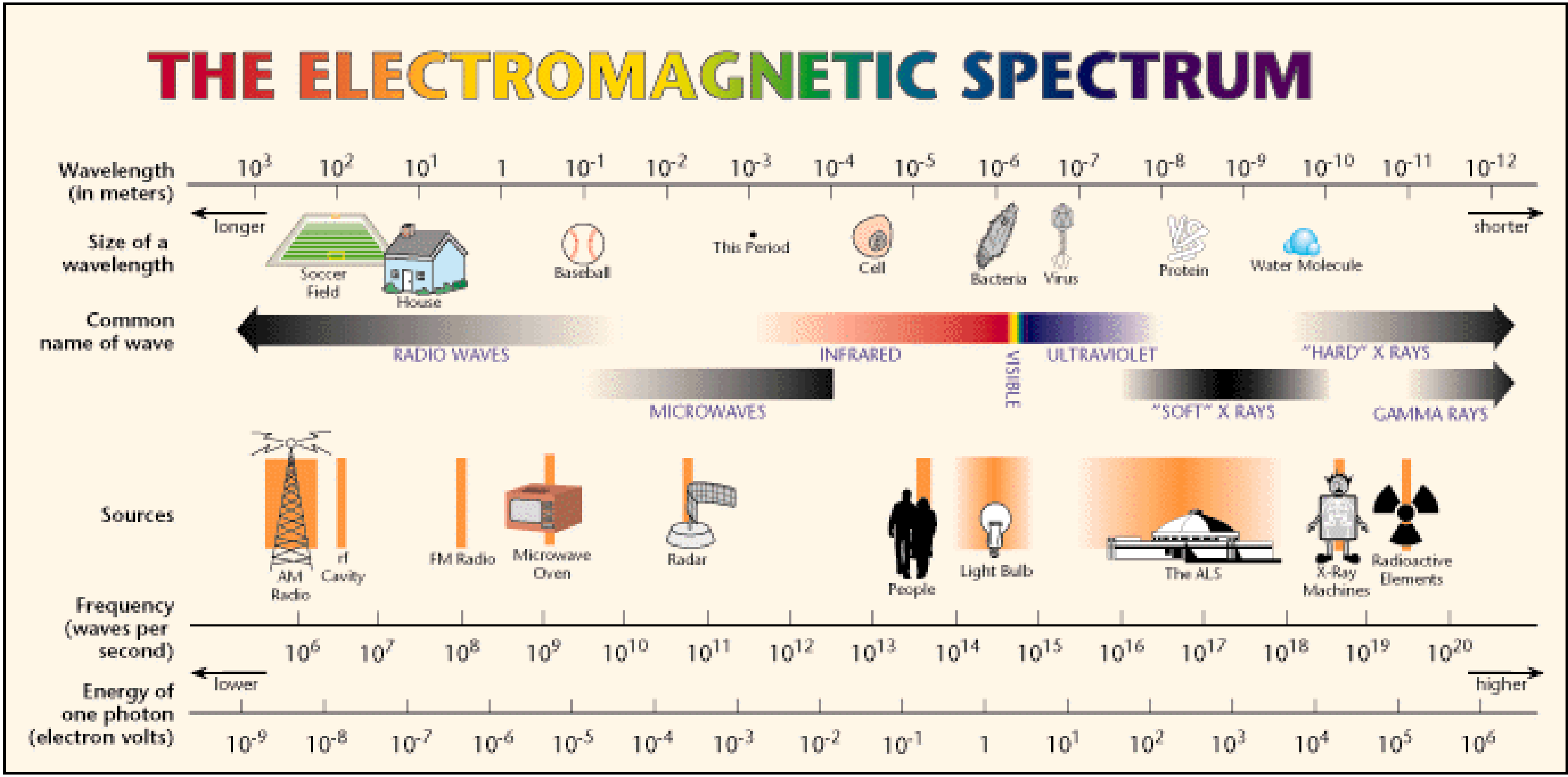
$$\omega_0 \sim 1 \text{ MHz}$$

$$\gamma \sim 4000$$

$$\omega_{\text{typ}} \sim 10^{16} \text{ Hz} !$$

- At high frequencies the individual harmonics overlap

continuous spectrum !



Wavelength continuously tunable !

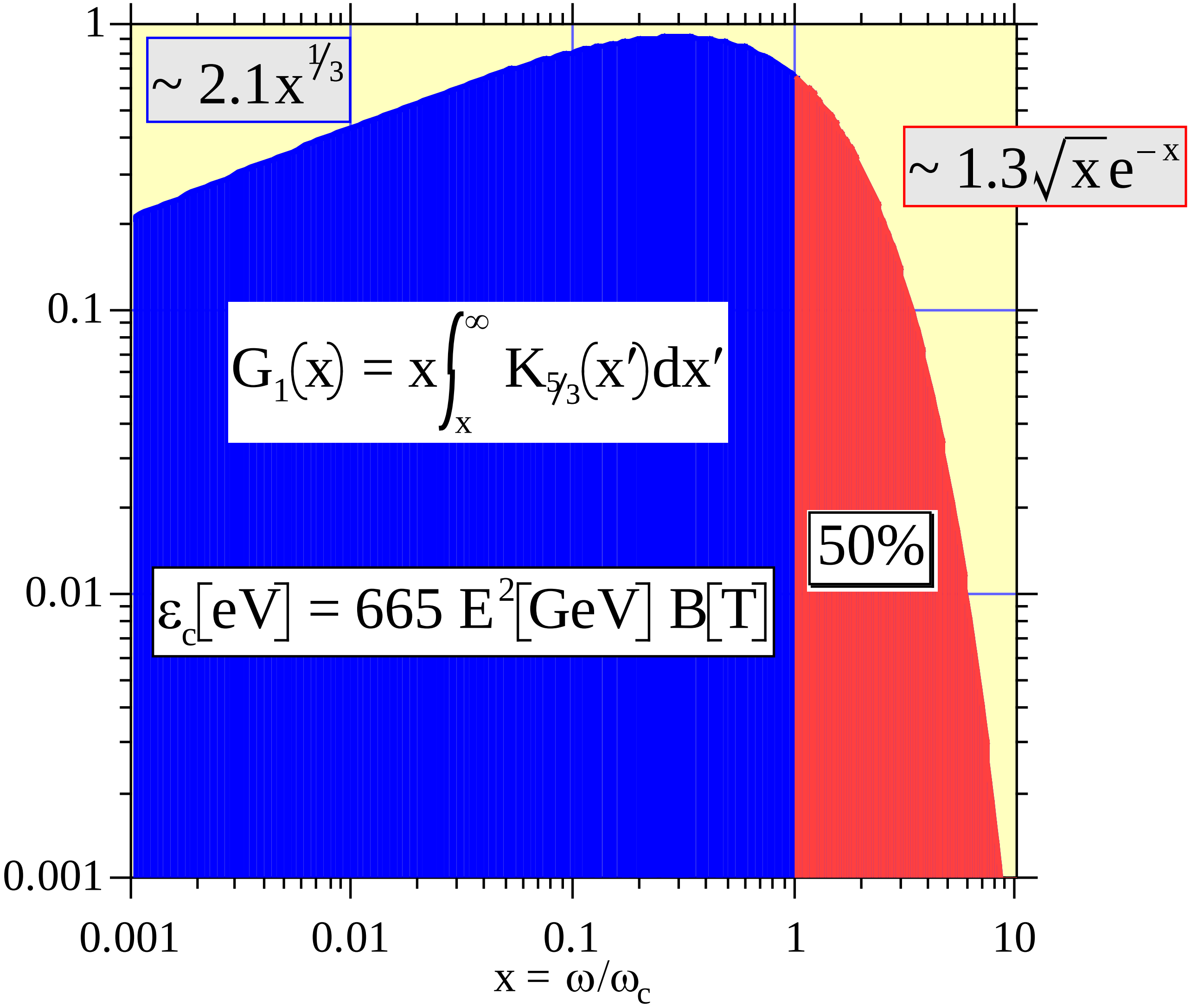
$$\boxed{\frac{dP}{d\omega} = \frac{P_{tot}}{\omega_c} S\left(\frac{\omega}{\omega_c}\right)}$$

$$S(x) = \frac{9\sqrt{3}}{8\pi}\, x \int_x^\infty K_{5/3}(x')dx'$$

$$\int_0^\infty S(x')dx' = 1$$

$$\boxed{P_{tot} = \frac{2}{3}\hbar c^2 \alpha \frac{\gamma^4}{\rho^2}}$$

$$\boxed{\omega_c = \frac{3}{2}\frac{c\gamma^3}{\rho}}$$



$$\sim 2.1 x^{1/3}$$

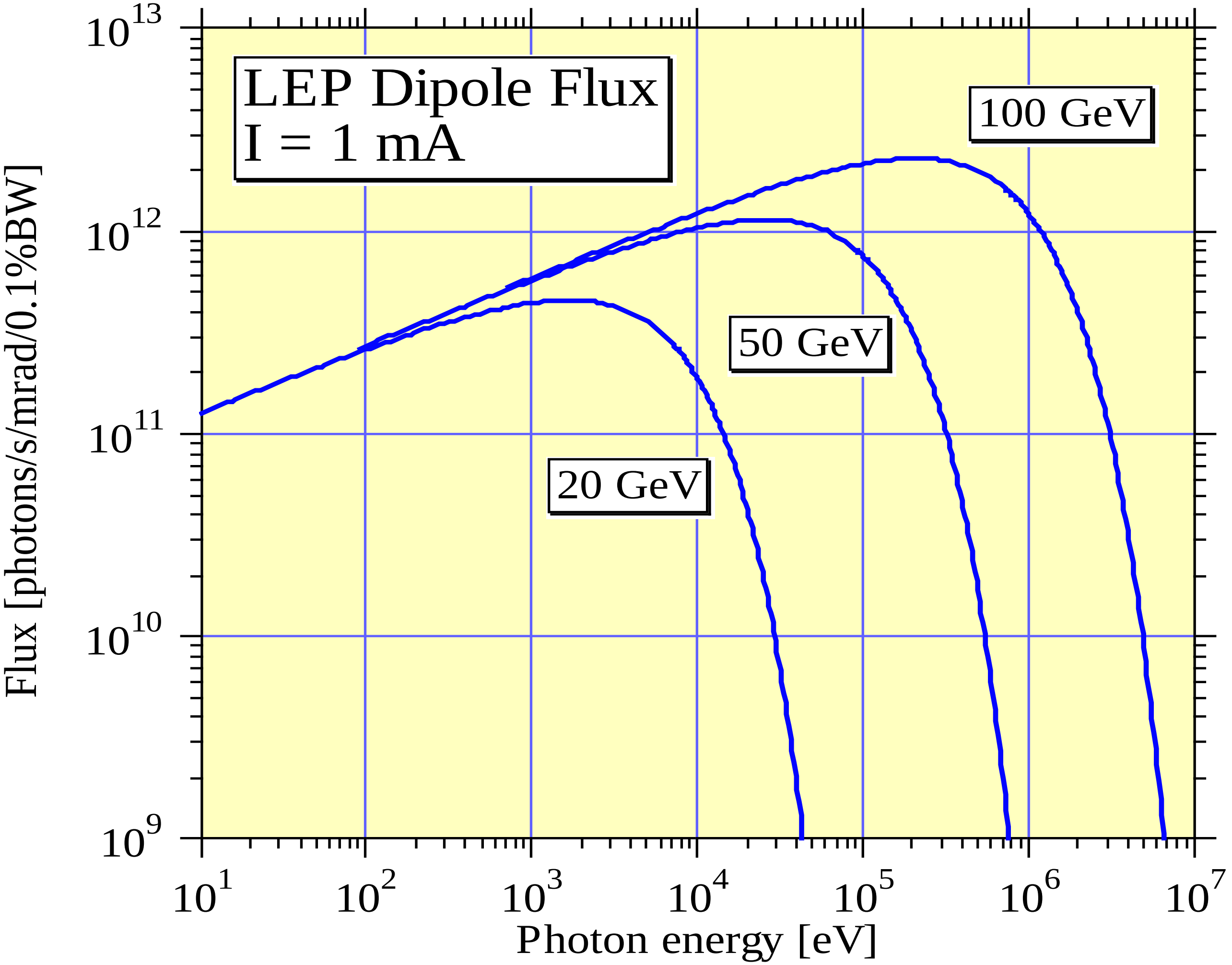
$$\sim 1.3\sqrt{x}\, e^{-x}$$

$$G_1(x) = x\int_x^\infty K_{5/3}(x')dx'$$

50%

$$\varepsilon_c[\mathrm{eV}] = 665\ E^2[\mathrm{GeV}]\ B[\mathrm{T}]$$

Synchrotron radiation flux for different electron energies



Radiation effects in electron storage rings

Average radiated power restored by RF $\boxed{U_0 \cong 10^{-3} \text{ of } E_0}$

- Electron loses energy each turn to synchrotron radiation $\boxed{V_{RF} > U_0}$
- RF cavities accelerate electrons back to the nominal energy

Radiation damping

- Average rate of energy loss produces **DAMPING** of electron oscillations in all three degrees of freedom (if properly arranged!)

Quantum fluctuations

- Statistical fluctuations in energy loss (from quantized emission of radiation) produce **RANDOM EXCITATION** of these oscillations

Equilibrium distributions

- The balance between the damping and the excitation of the electron oscillations determines the equilibrium distribution of particles in the beam

Radiation damping

Transverse oscillations

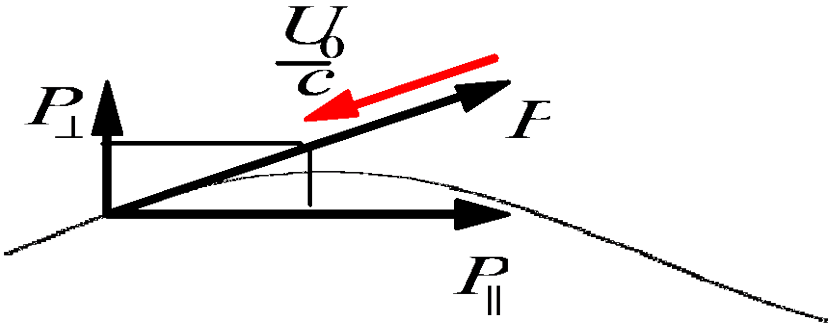
Average energy loss and gain per turn

- Every turn electron radiates small amount of energy

$$E_1 = E_0 - U_0 = E_0\left(1 - \frac{U_0}{E_0}\right)$$

- only the amplitude of the momentum changes

$$P_1 = P_0 - \frac{U_0}{c} = P_0\left(1 - \frac{U_0}{E_0}\right)$$

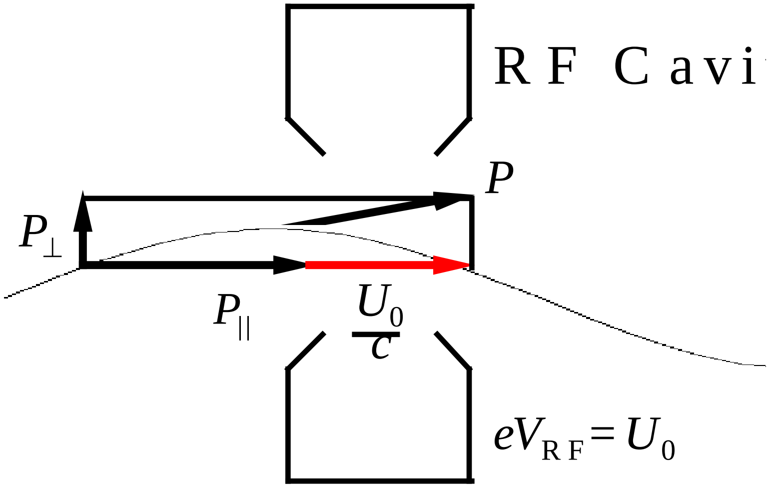


- Only the longitudinal component of the momentum is increased in the RF cavity

- Energy of betatron oscillation

$$E_\beta \propto A^2$$

$$A_1^2 = A_0^2\left(1 - \frac{U_0}{E_0}\right) \quad \text{or} \quad A_1 \cong A_0\left(1 - \frac{U_0}{2E_0}\right)$$



Damping of vertical oscillations

- But this is just the exponential decay law!

$$\frac{\Delta A}{A} = -\frac{U_0}{2E}$$

$$A = A_0 \cdot e^{-t/\tau}$$

- The oscillations are exponentially **damped** with the **damping time (milliseconds!)**

$$\tau = \frac{2E\,T_0}{U_0}$$

the time it would take particle to 'lose all of its energy'

- In terms of radiation power

$$\tau = \frac{2E}{P_\gamma}$$

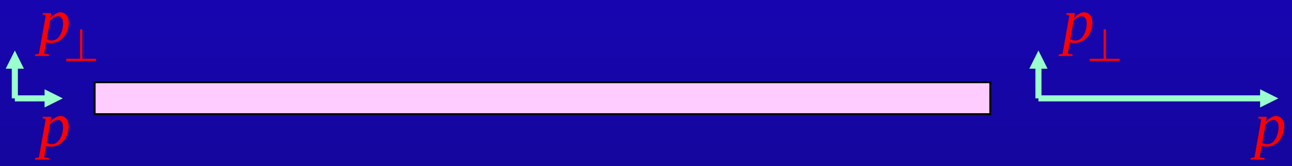
and since $P_\gamma \propto E^4$

$$\tau \propto \frac{1}{E^3}$$

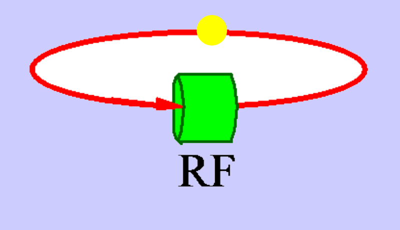
Adiabatic damping in linear accelerators

In a **linear accelerator**:

$$x' = \frac{p_\perp}{p} \quad \text{decreases} \quad \propto \frac{1}{E}$$

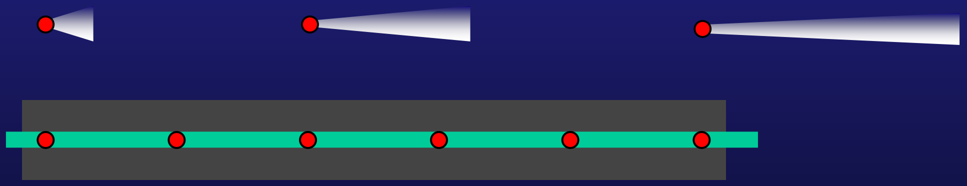


In a **storage ring** beam passes many times through same RF cavity



- **Clean loss of energy every turn (no change in x')**
- **Every turn is re-accelerated by RF (x' is reduced)**
- **Particle energy on average remains constant**

Emittance damping in linacs:



ε $\quad \frac{\varepsilon}{2}$ $\quad \frac{\varepsilon}{4}$

Ω $\quad \frac{\Omega}{2}$ $\quad \frac{\Omega}{4}$

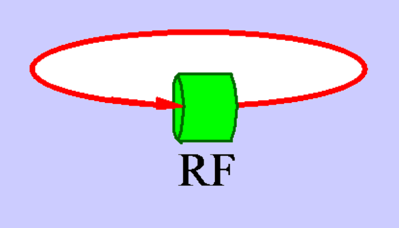
γ $\quad 2\gamma$ $\quad 4\gamma$

$$\varepsilon \propto \frac{1}{\gamma}$$

or

$$\gamma\varepsilon = \text{const.}$$

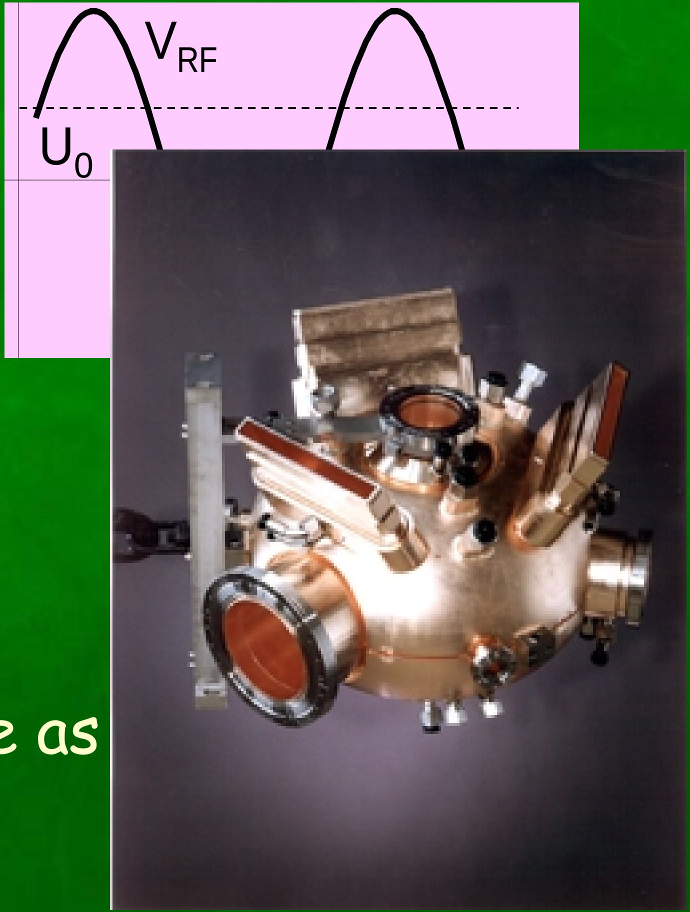
Longitudinal motion: compensating radiation loss U_0



- RF cavity provides accelerating field with frequency

$$f_{RF} = h \cdot f_0$$

  - h – harmonic number

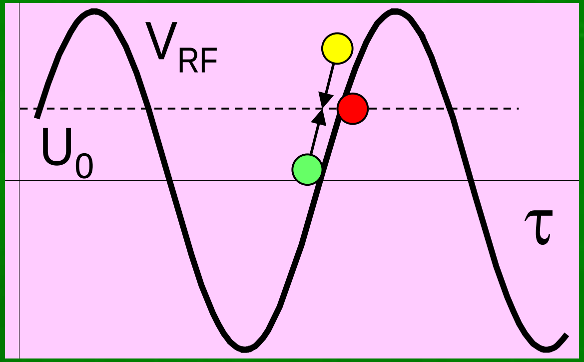


- The energy gain:

$$U_{RF} = eV_{RF}(\tau)$$

- Synchronous particle:
  - has design energy
  - gains from the RF on the average as much as it loses per turn U_0

Longitudinal motion: phase stability

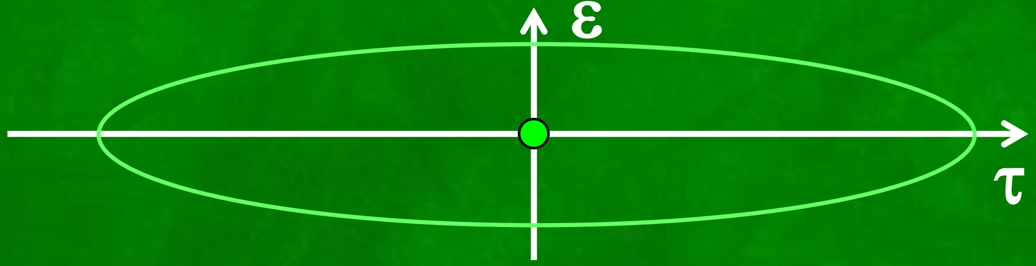


- Particle ahead of synchronous one
  - gets too much energy from the RF
  - goes on a longer orbit (not enough B)
    >> takes longer to go around
  - comes back to the RF cavity closer to synchronous part.

- Particle behind the synchronous one
  - gets too little energy from the RF
  - goes on a shorter orbit (too much B)
  - catches-up with the synchronous particle

Longitudinal motion: energy-time oscillations

energy deviation from the design energy, or the energy of the synchronous particle



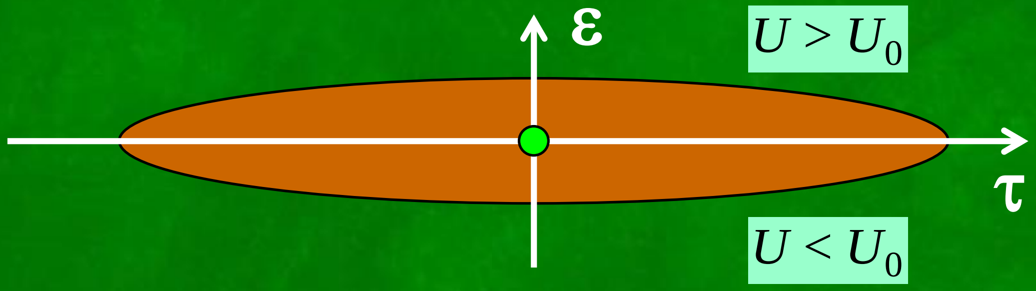
longitudinal coordinate measured from the position of the synchronous electron

Longitudinal motion: damping of synchrotron oscillations

$P_\gamma \propto \boldsymbol{E}^2 B^2$

During one period of synchrotron oscillation:

- when the particle is in the upper half-plane, it loses more energy per turn, its energy gradually reduces



- when the particle is in the lower half-plane, it loses less energy per turn, but receives U_0 on the average, so its energy deviation gradually reduces

The synchrotron motion is damped

- the phase space trajectory is spiraling towards the origin

Equilibrium beam sizes

Radiation effects in electron storage rings

Average radiated power restored by RF $\boxed{U_0 \cong 10^{-3} \text{ of } E_0}$

- Electron loses energy each turn to synchrotron radiation $\boxed{V_{RF} > U_0}$
- RF cavities accelerate electrons back to the nominal energy

Radiation damping

- Average rate of energy loss produces **DAMPING** of electron oscillations in all three degrees of freedom (if properly arranged!)

Quantum fluctuations

- Statistical fluctuations in energy loss (from quantized emission of radiation) produce **RANDOM EXCITATION** of these oscillations

Equilibrium distributions

- The balance between the damping and the excitation of the electron oscillations determines the equilibrium distribution of particles in the beam

Quantum nature of synchrotron radiation

Damping only

- If damping was the whole story, the beam emittance (size) would shrink to microscopic dimensions!*
- Lots of problems! (e.g. coherent radiation)

- How small? On the order of electron wavelength

$$E = \gamma mc^2 = h\nu = \frac{hc}{\lambda_e} \quad \Rightarrow \quad \lambda_e = \frac{1}{\gamma}\frac{h}{mc} = \frac{\lambda_C}{\gamma}$$

$\lambda_C = 2.4 \cdot 10^{-12} m$ – Compton wavelength

Diffraction limited electron emittance

$$\varepsilon \geq \frac{\lambda_C}{4\pi\gamma}\left(\times N^{1/3} - \text{fermions}\right)$$

Quantum nature of synchrotron radiation

Quantum fluctuations

- Because the radiation is emitted in quanta, radiation itself takes care of the problem!
- It is sufficient to use quasi-classical picture:
  - » *Emission time is very short*
  - » *Emission times are statistically independent (each emission - only a small change in electron energy)*

Purely stochastic (Poisson) process

Visible quantum effects

I have always been somewhat amazed that a purely quantum effect can have gross macroscopic effects in large machines;

and, even more,

that Planck's constant has just the right magnitude needed to make practical the construction of large electron storage rings.

A significantly larger or smaller value of $\hbar$

would have posed serious -- perhaps insurmountable -- problems for the realization of large rings.

Mathew Sands

Quantum excitation of energy oscillations

Photons are emitted with typical energy $u_{ph} \approx \hbar\omega_{typ} = \hbar c \frac{\gamma^3}{\rho}$

at the rate (photons/second) $\mathcal{N} = \frac{P_\gamma}{u_{ph}}$

Fluctuations in this rate excite oscillations

During a small interval Δt electron emits photons $N = \mathcal{N} \cdot \Delta t$

losing energy of $N \cdot u_{ph}$

Actually, because of fluctuations, the number is $N \pm \sqrt{N}$

resulting in **spread in energy loss** $\pm\sqrt{N} \cdot u_{ph}$

For large time intervals RF compensates the energy loss, providing damping towards the design energy E_0

Steady state: typical deviations from E_0

≈ typical fluctuations in energy during a damping time τ_ε

Equilibrium energy spread: rough estimate

We then expect the rms energy spread to be $\sigma_\varepsilon \approx \sqrt{N \cdot \tau_\varepsilon \cdot u_{ph}}$

and since $\tau_\varepsilon \approx \frac{E_0}{P_\gamma}$ and $P_\gamma = N \cdot u_{ph}$

$\sigma_\varepsilon \approx \sqrt{E_0 \cdot u_{ph}}$ geometric mean of the electron and photon energies!

Relative energy spread can be written then as:

$$\frac{\sigma_\varepsilon}{E_0} \approx \gamma \sqrt{\frac{\lambda\!\!\!^{-}_e}{\rho}}$$

$$\lambda\!\!\!^{-}_e = \frac{\hbar}{m_e c} \simeq 4 \cdot 10^{-13} m$$

it is roughly constant for all rings

$$\frac{\sigma_\varepsilon}{E_0} \sim const \sim 10^{-3}$$

- typically $\rho \propto E^2$

Equilibrium energy spread

More detailed calculations give

- for the case of an ‘isomagnetic’ lattice

$$\rho(s) = \begin{array}{ll} \rho_0 & \text{in dipoles} \\ \infty & \text{elsewhere} \end{array}$$

$$\boxed{\left(\frac{\sigma_\varepsilon}{E}\right)^2 = \frac{C_q E^2}{J_\varepsilon \rho_0}}$$

with

$$C_q = \frac{55}{32\sqrt{3}} \frac{\hbar c}{(m_e c^2)^3} = 1.468 \cdot 10^{-6} \left[\frac{\mathrm{m}}{\mathrm{GeV}^2}\right]$$

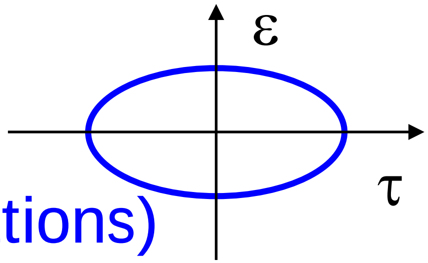
It is difficult to obtain energy spread < 0.1%

- limit on undulator brightness!

Equilibrium bunch length

Bunch length is related to the energy spread

- Energy deviation and time of arrival (or position along the bunch) are conjugate variables (synchrotron oscillations)



$$\hat{\tau} = \frac{\alpha}{\Omega_s}\left(\frac{\hat{\varepsilon}}{E}\right)$$

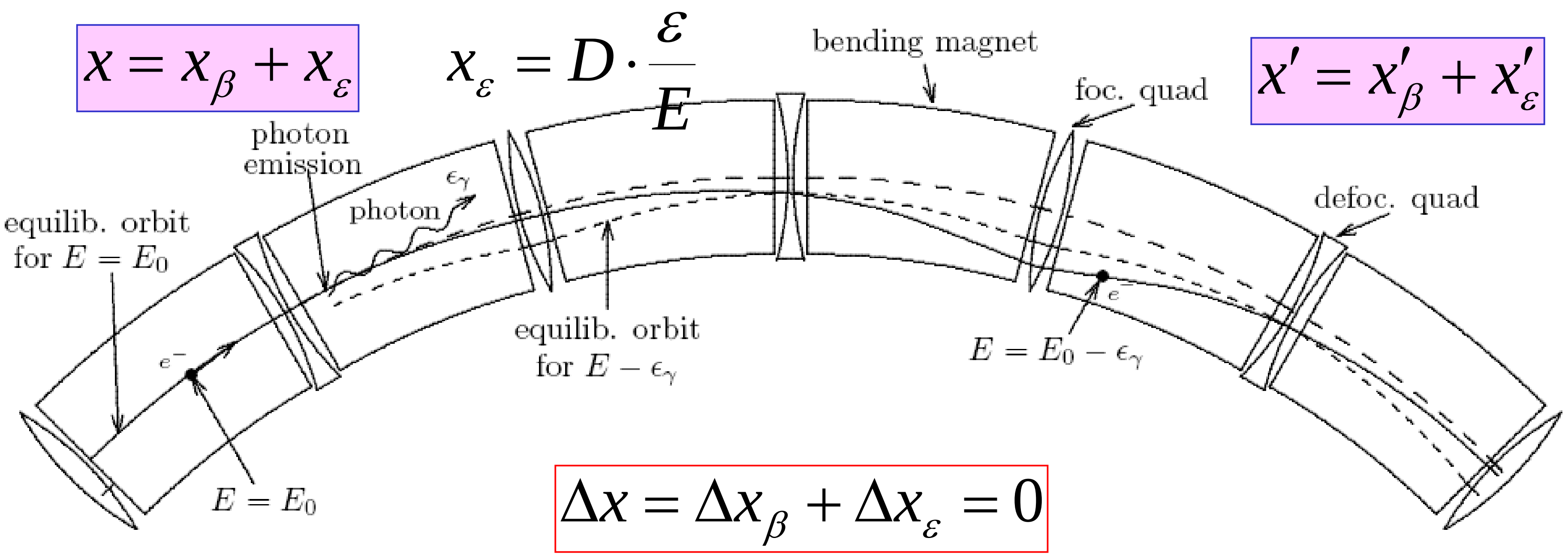
$$\boxed{\sigma_\tau = \frac{\alpha}{\Omega_s}\left(\frac{\sigma_\varepsilon}{E}\right)}$$

- recall that $\boxed{\Omega_s \propto \sqrt{V_{RF}}}$

Two ways to obtain short bunches:

- RF voltage (power!) $\boxed{\sigma_\tau \propto 1/\sqrt{V_{RF}}}$
- Momentum compaction factor in the limit of $\alpha = 0$
  isochronous ring: particle position along the bunch is frozen $\boxed{\sigma_\tau \propto \alpha}$

Excitation of betatron oscillations



$$x = x_\beta + x_\varepsilon$$

$$x_\varepsilon = D \cdot \frac{\varepsilon}{E}$$

$$x' = x'_\beta + x'_\varepsilon$$

$$\Delta x = \Delta x_\beta + \Delta x_\varepsilon = 0$$

$$\Delta x_\beta = -D \cdot \frac{\varepsilon_\gamma}{E}$$

Courant Snyder invariant

$$\Delta x'_\beta = -D' \cdot \frac{\varepsilon_\gamma}{E}$$

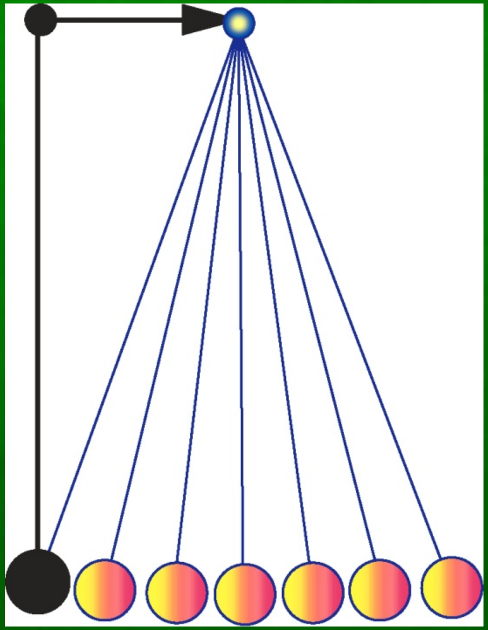
$$\Delta\varepsilon = \gamma \Delta x_\beta^2 + 2\alpha \Delta x_\beta \Delta x'_\beta + \beta \Delta x'^2_\beta = \left[\gamma D^2 + 2\alpha D D' + \beta D'^2\right] \cdot \left(\frac{\varepsilon_\gamma}{E}\right)^2$$

Excitation of betatron oscillations

Electron emitting a photon

- at a place with non-zero dispersion
- starts a betatron oscillation around a new reference orbit

$$\boxed{x_\beta \approx D \cdot \frac{\varepsilon_\gamma}{E}}$$



Horizontal oscillations: equilibrium

Emission of photons is a random process

- Again we have random walk, now in **x**. How far particle will wander away is limited by the radiation damping
- The balance is achieved on the time scale of the damping time $\tau_x = 2\,\tau_\varepsilon$

$$\sigma_{x\beta} \approx \sqrt{\mathcal{N} \cdot \tau_x} \cdot D \cdot \frac{\varepsilon_\gamma}{E} = \sqrt{2} \cdot D \cdot \frac{\sigma_\varepsilon}{E}$$

- Typical horizontal beam size ~ 1 mm

Quantum effect visible to the naked eye!

- Vertical size - determined by coupling

Beam emittance

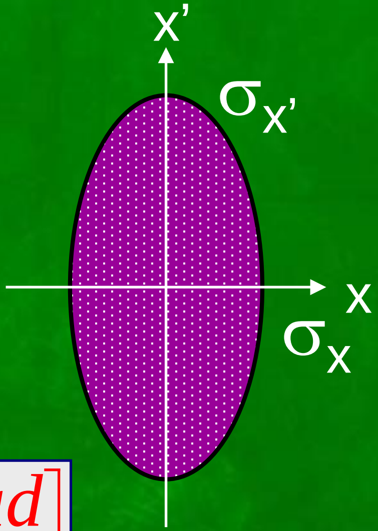
$\text{Area} = \pi \cdot \varepsilon$

Betatron oscillations

- Particles in the beam execute betatron oscillations with different amplitudes.

Transverse beam distribution

- Gaussian (electrons)
- "Typical" particle: 1 - σ ellipse
  (in a place where $\alpha = \beta' = 0$)



Units of ε $[m \cdot rad]$

$$\text{Emittance} \equiv \frac{\sigma_x^2}{\beta}$$

$$\varepsilon = \sigma_x \cdot \sigma_{x'}$$

$$\sigma_x = \sqrt{\varepsilon\,\beta}$$
$$\sigma_{x'} = \sqrt{\varepsilon / \beta}$$

$$\beta = \frac{\sigma_x}{\sigma_{x'}}$$

Equilibrium horizontal emittance

Detailed calculations for isomagnetic lattice

$$\varepsilon_{x0} \equiv \frac{\sigma_{x\beta}^2}{\beta} = \frac{C_q E^2}{J_x} \cdot \frac{\langle \mathcal{H} \rangle_{mag}}{\rho}$$

where

$$\mathcal{H} = \gamma D^2 + 2\alpha D D' + \beta D'^2$$
$$= \frac{1}{\beta}\left[D^2 + (\beta D' + \alpha D)^2\right]$$

and $\langle \mathcal{H} \rangle_{mag}$ is average value in the bending magnets