Search and Discovery Statistics in HEP Lecture 4

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This presentation would have not been possible without the tremendous help of
the following people throughout many years

Louis Lyons, Alex Read, Bob Cousins Glen Cowan ,Kyle Cranmer
Ofer Vitells & Jonathan Shlomi

What can you expect from the Lectures

- Lecture 1-2: Basic Concepts
  Histograms, PDF, Testing Hypotheses,
  LR as a Test Statistics, p-value, POWER, CLs
  Measurements
- Lecture 3: Wald Theorem, Asymptotic Formalism, Asimov Data Set, Feldman-Cousins, PL & CLs
- Lecture 4: Asimov Significance
  Look Elsewhere Effect
  1D LEE the non-intuitive thumb rule
  (upcrossings, trial #~Z)
  2D LEE (Euler Characteristic)

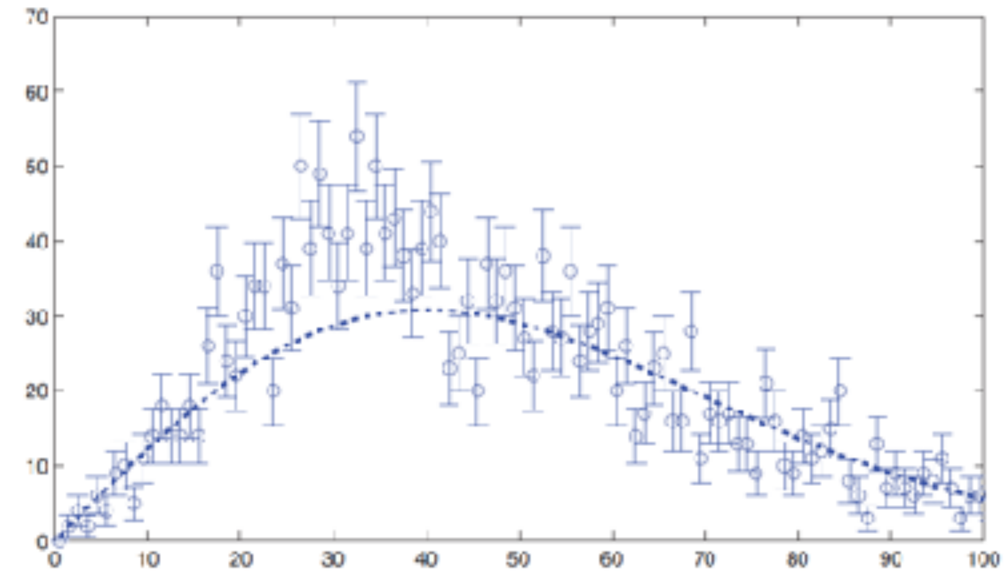
Look Elsewhere Effect

E.G., O. Vitells “Trial factors for the look elsewhere effect in high energy physics”,
Eur. Phys. J. C 70 (2010) 525

O. Vitells and E. G., Estimating the significance of a signal in a multi-dimensional search,
1669 Astropart. Phys. 35 (2011) 230, arXiv:1105.4355

Look Elsewhere Effect

- Is there a signal here?

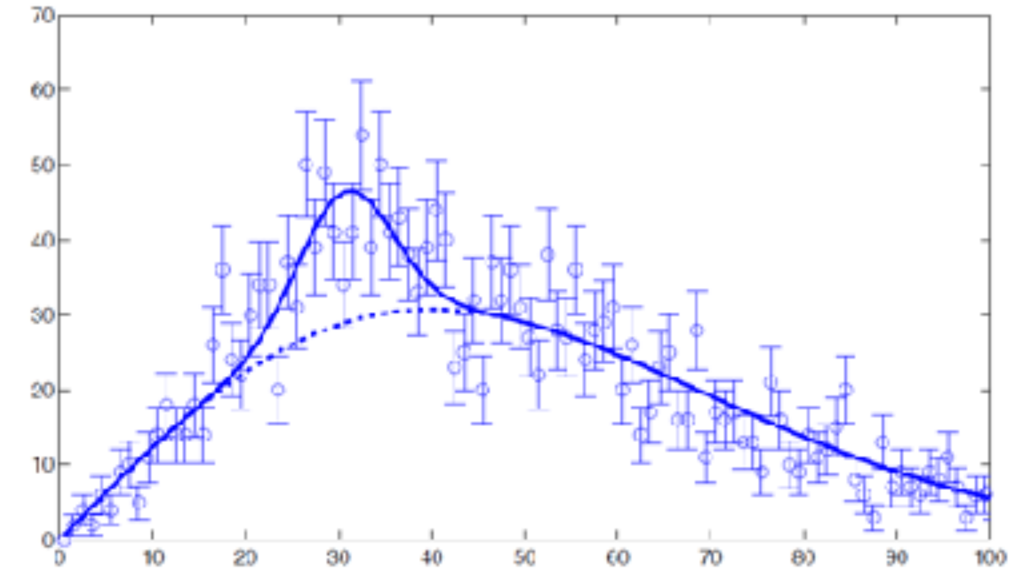


Look Elsewhere Effect

- Looks like a signal at m=30
- What is its significance?

Test the BG hypothesis
At m=30

$$q_0(\theta) = \begin{cases} -2\log\dfrac{\mathcal{L}(\mu=0)}{\mathcal{L}(\hat{\mu},\theta)} \\ 0 \end{cases}$$



$$q_{fix,obs} = -2\ln\frac{L(b)}{L(\hat{\mu}s(m=30)+b)}$$

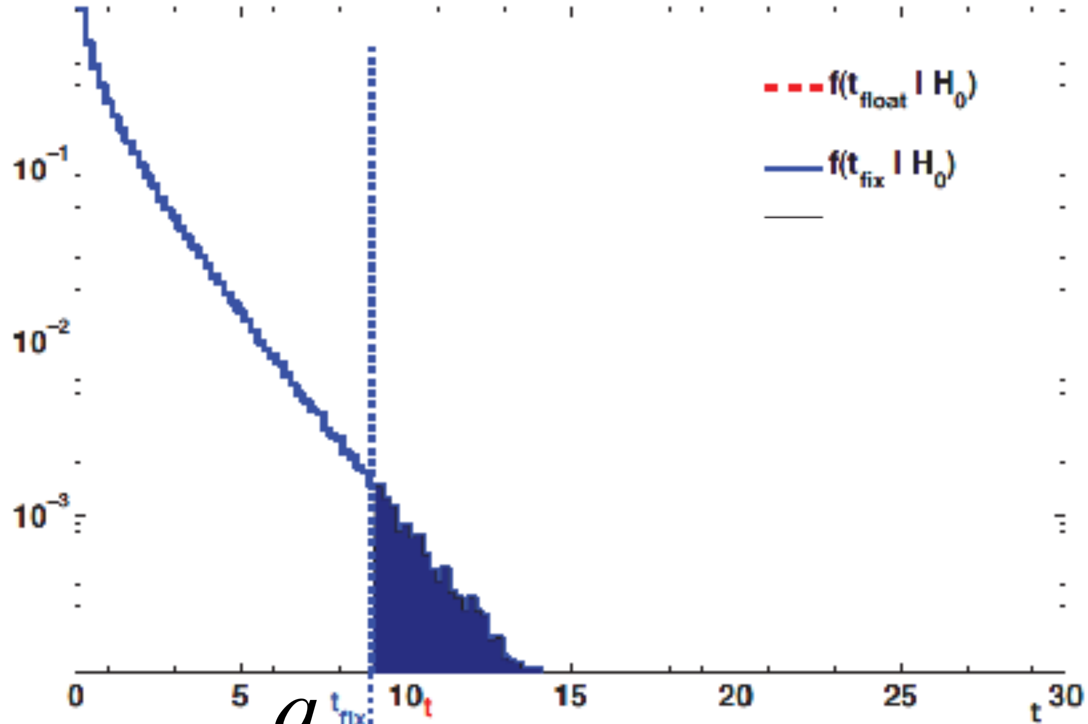
$$Z = \sqrt{q_{0,fix,obs}}$$

Look Elsewhere Effect

$$q_{0,fix} = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(30)+b)}$$

$$f(q_{0,fix} \mid H_0) \sim \chi^2$$

$$p_{fix} = \int_{q_{fix,obs}}^{\infty} f(q_0 \mid H_0)dq_0$$



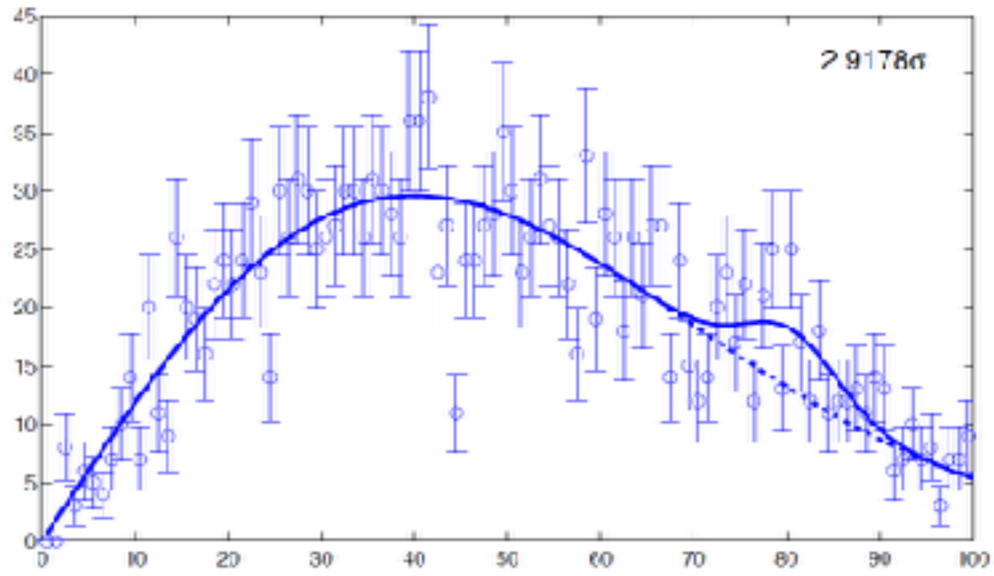
$q_{0,fix,obs}$

p_{fix} answers the question:

What is the probability to have a fluctuation as or bigger than the observed one?

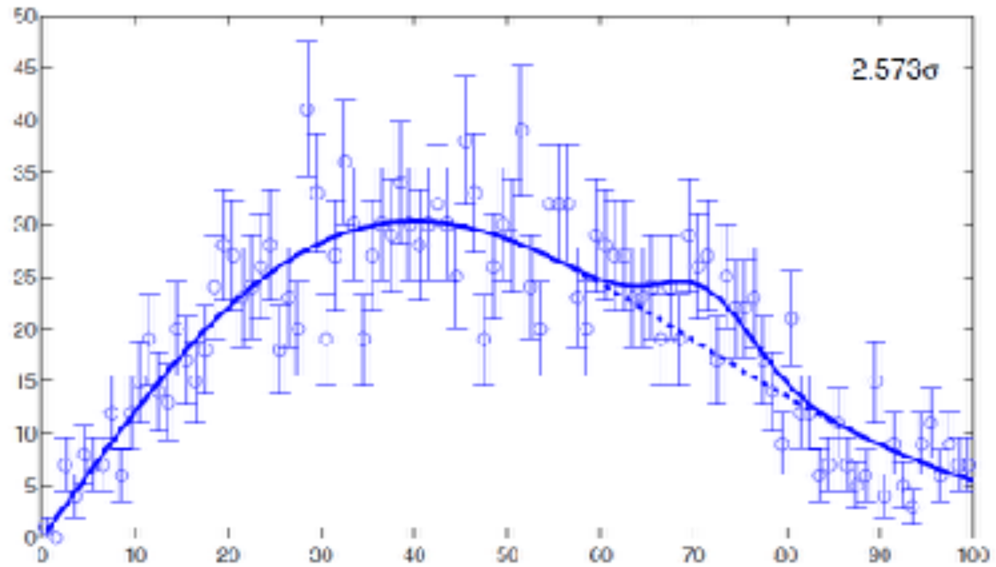
Look Elsewhere Effect

- Would you ignore this signal, had you seen it?



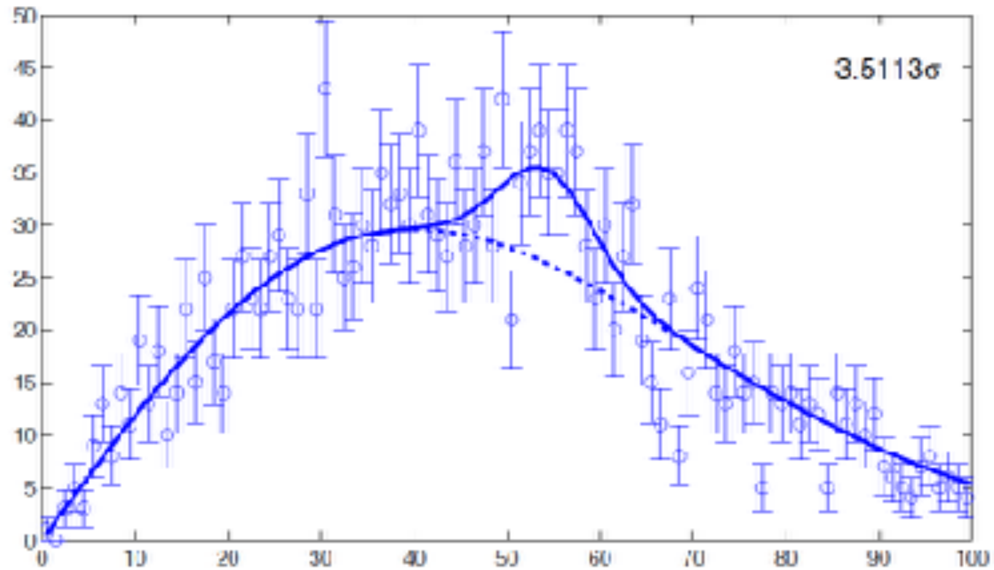
Look Elsewhere Effect

- Or this?



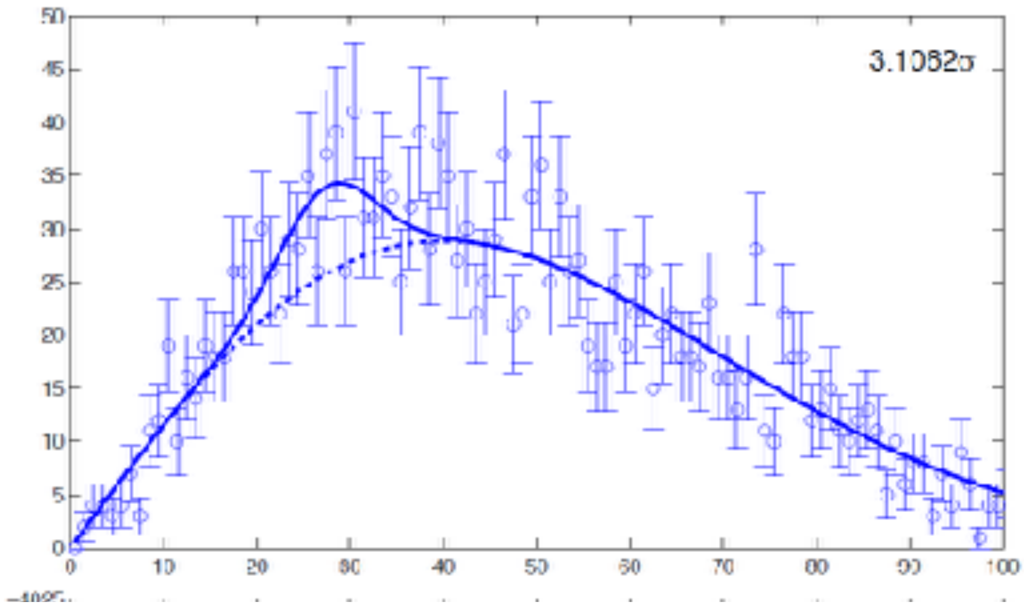
Look Elsewhere Effect

- Or this?



Look Elsewhere Effect

- Or this?
- Obviously NOT!
- ALL THESE "SIGNALS" ARE BG FLUCTUATIONS



The right question :
What is the probability to have a fluctuation as or bigger than the observed one ***ANYWHERE in the mass search range****?*

Look Elsewhere Effect

- Having no idea where the signal might be there are two equivalent options

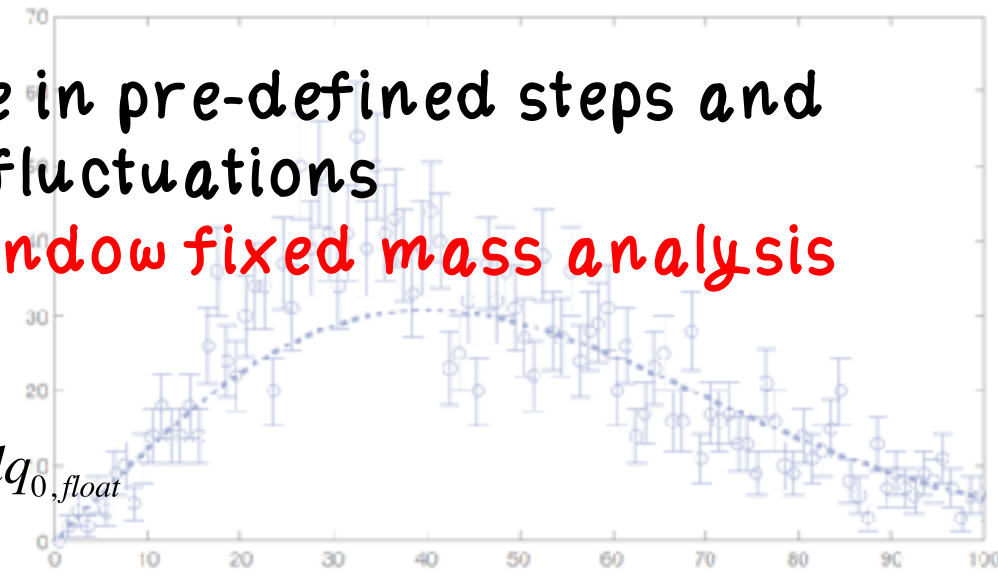
- OPTION I:

scan the mass range in pre-defined steps and test any disturbing fluctuations

Perform a sliding window fixed mass analysis

$$q_{0,float} = \max_m (q_0(m))$$

$$p_{float} = \int_{q_{float,obs}}^{\infty} f(q_{0,float} \mid H_0)\, dq_{0,float}$$



- OPTION II:

Perform a floating mass analysis

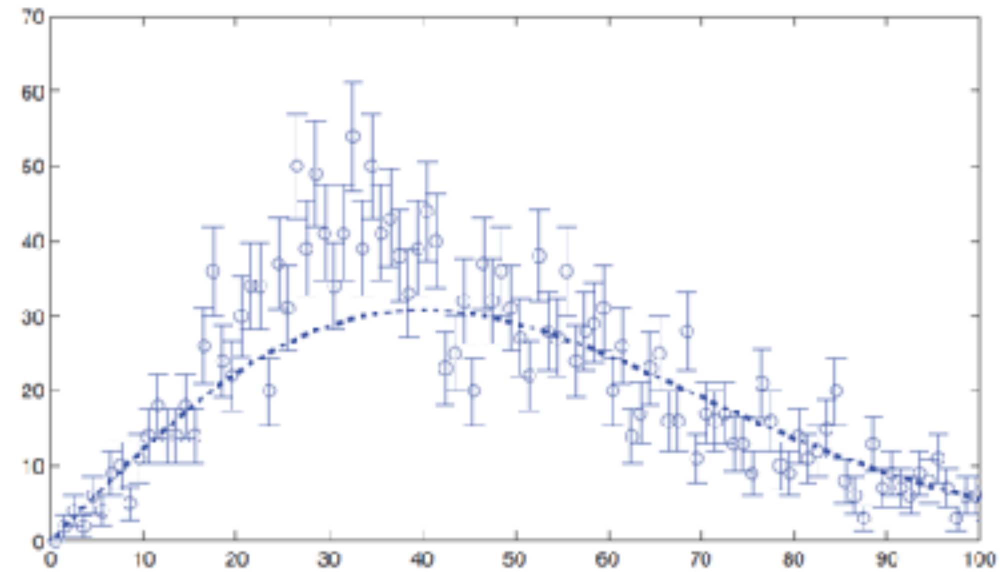
$$q_{0,float} = q_0(\hat{m}) = -2\ln \frac{L(b)}{L(\hat{\mu}s(\hat{m})+b)}$$

$$p_{float} = \int_{q_{float,obs}}^{\infty} f(q_{0,float} \mid H_0)\, dq_{0,float}$$

Sliding Window

- Scan and perform a fixed mass analysis at each point

$$q_0 = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(m)+b)}$$



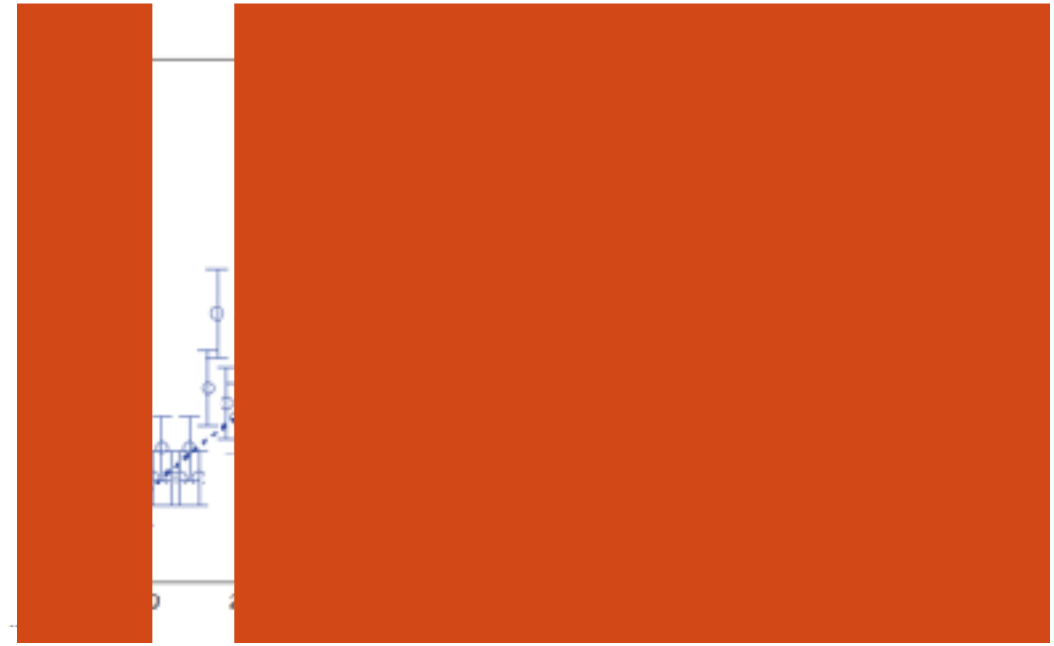
- The scan resolution must be less than the signal mass resolution

Sliding Window

$$q_0 = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(m)+b)}$$

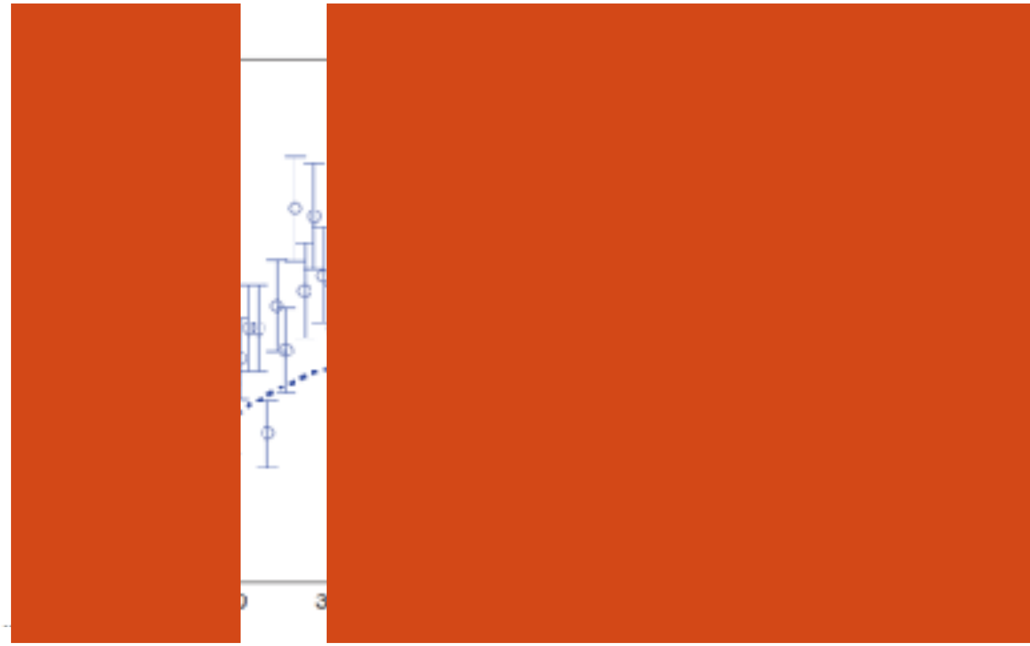
Sliding Window

$$q_0 = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(m)+b)}$$



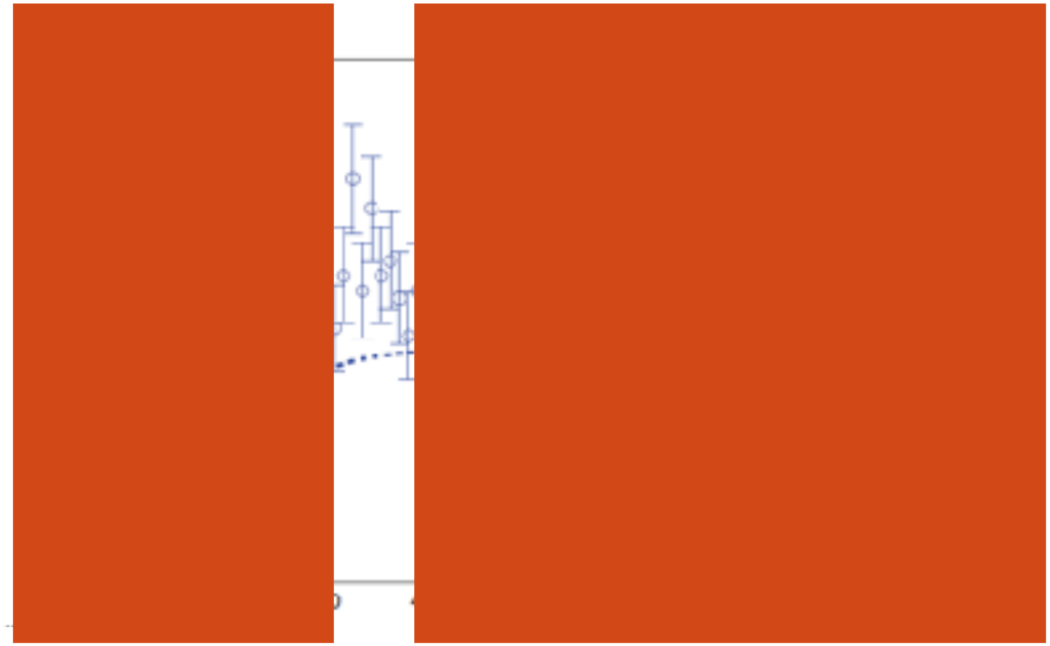
Sliding Window

$$q_0 = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(m)+b)}$$



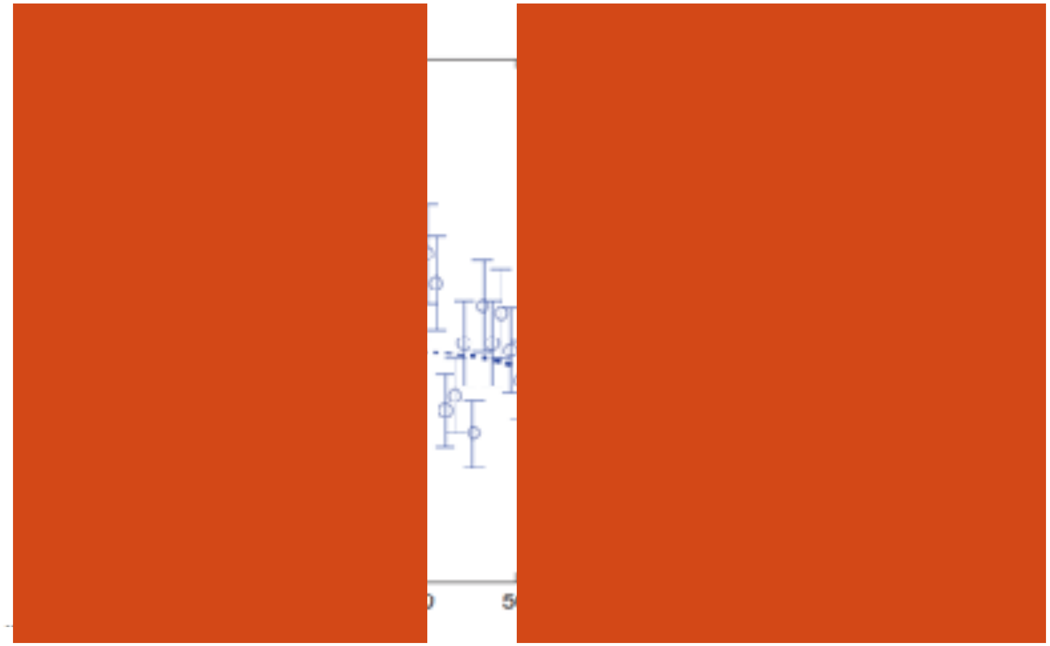
Sliding Window

$$q_0 = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(m)+b)}$$



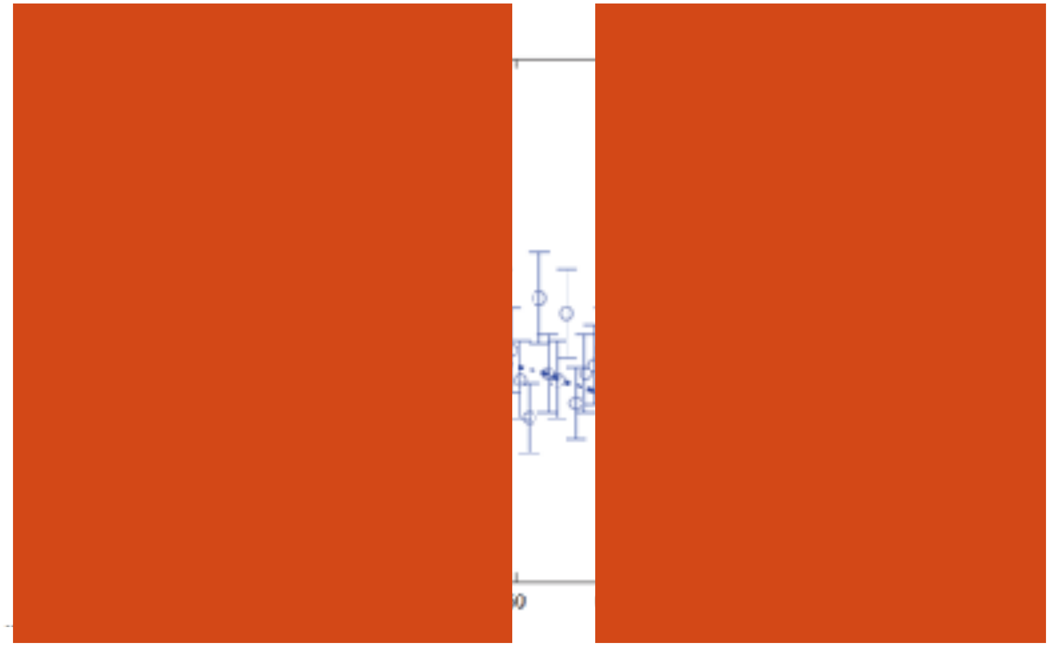
Sliding Window

$$q_0 = -2 ln \frac{L(\mu = 0)}{L(\hat{\mu} s(m) + b)}$$



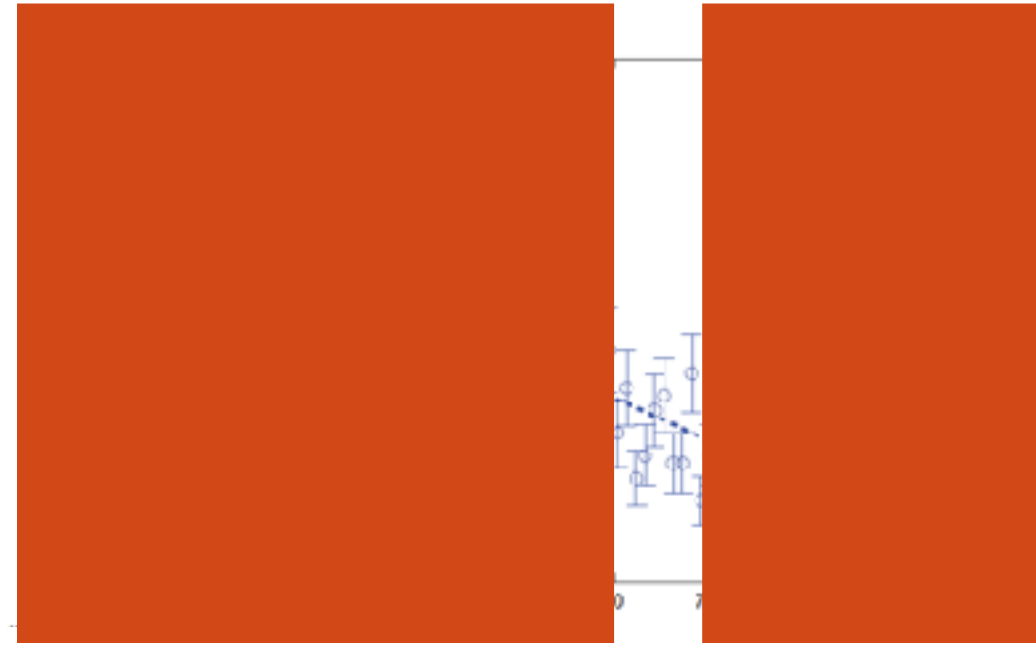
Sliding Window

$$q_0 = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(m)+b)}$$



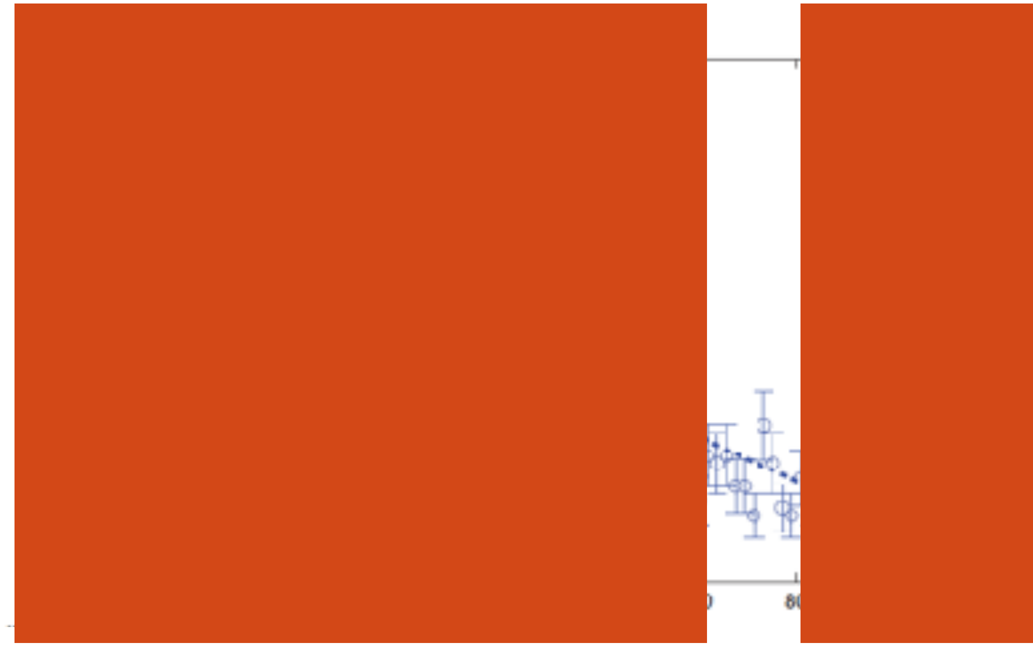
Sliding Window

$$q_0 = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(m)+b)}$$



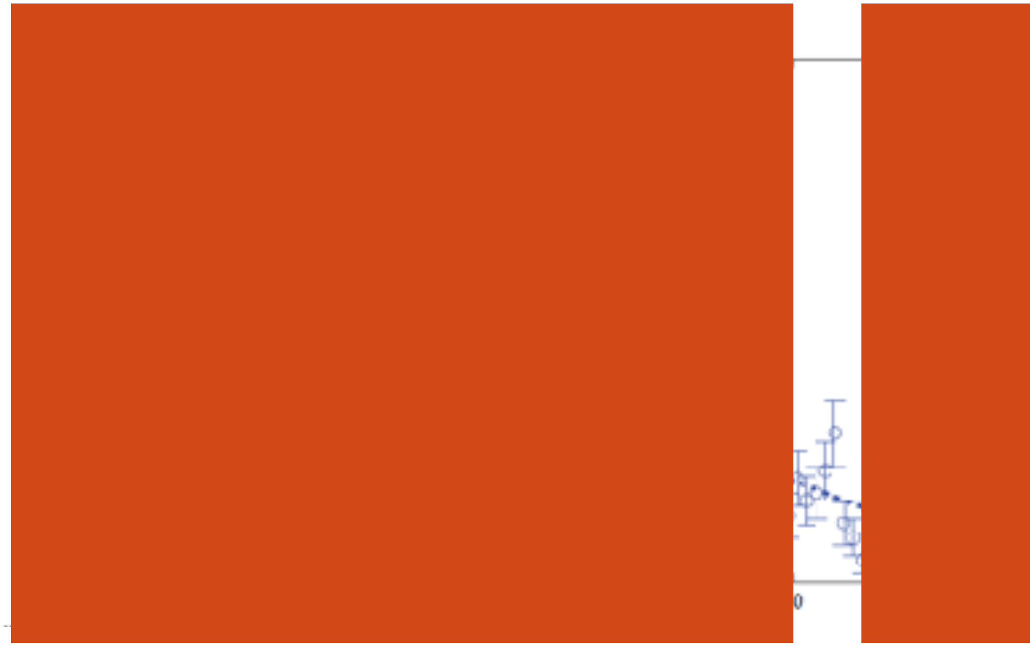
Sliding Window

$$q_0 = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(m)+b)}$$



Sliding Window

$$q_0 = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(m)+b)}$$



Sliding Window

- Assuming the signal can be only at one place
- pick the one with the MAXIMUM SIGNIFICANCE

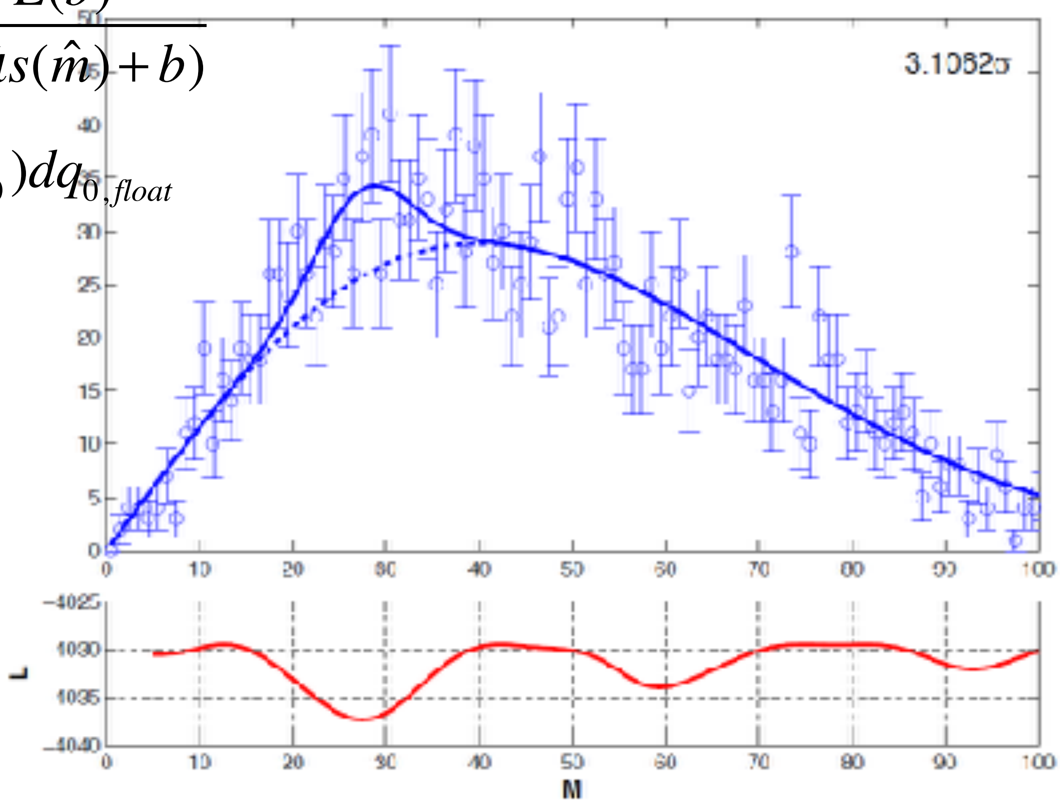
$$q_{0,float} = \max_{m} \left(q_0(m) \right)$$

Look Elsewhere Effect: Floating Mass

- **OPTION II**

$$q_{0,float} = q_0(\hat{m}) = -2\ln\frac{L(b)}{L(\hat{\mu}s(\hat{m})+b)}$$

$$p_{float} = \int_{q_{float,obs}}^{\infty} f(q_{0,float} \mid H_0)\, dq_{0,float}$$

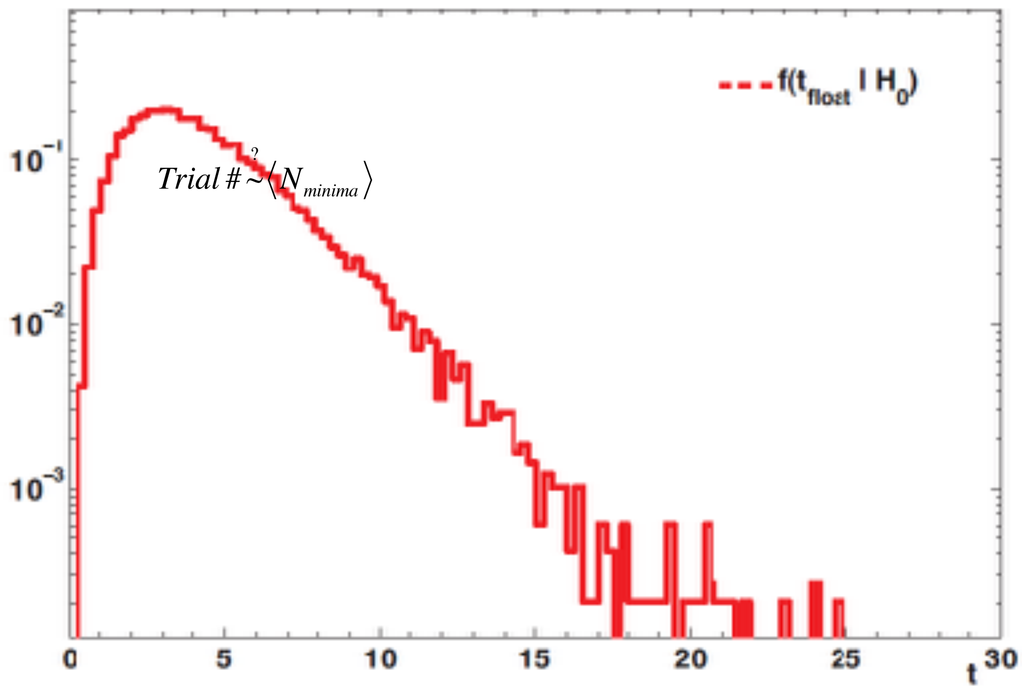


Look Elsewhere Effect

- The distribution $f(q_{float}|H_0)$ does not follow a chi-squared with 2dof because the mass parameter is not defined under the null hypothesis

$$\exists m_{fix} \quad q_0(\hat{m}) \geq q_0(m_{fix})$$

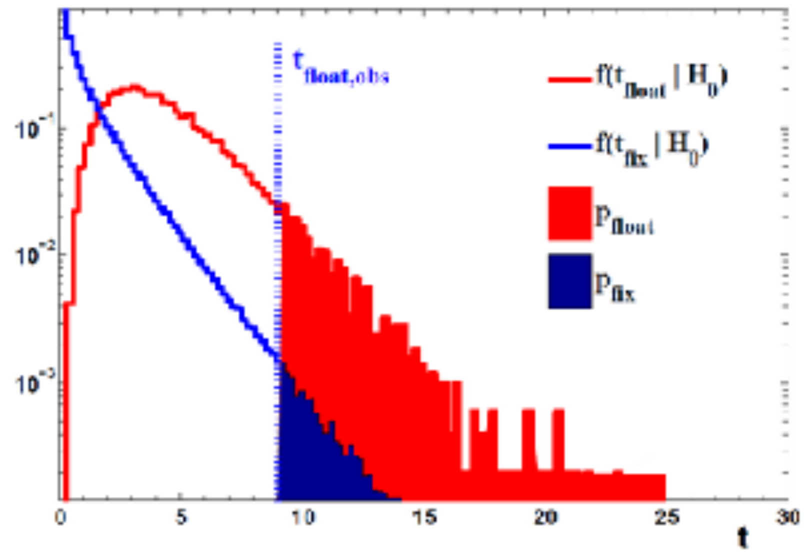
The χ_1^2 *distribution is pushed to the right*



trial#

- Assume a maximal local fluctuation at mass $\hat{m} = 30$
- The observed q_0 is given by

$$q_{0,obs} = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(m)+b)}$$



$$p_{fix} = \int_{q_{0,obs}}^{\infty} f(q_{0,fix} | H_0) dq_{0,fix}$$

$$p_{float} = \int_{q_{0,obs}}^{\infty} f(q_{0,float} | H_0) dq_{0,float}$$

$$\boxed{trial\,\# = \frac{p_{float}}{p_{fix}}}$$

Can we calculate analytically the floating mass p-value

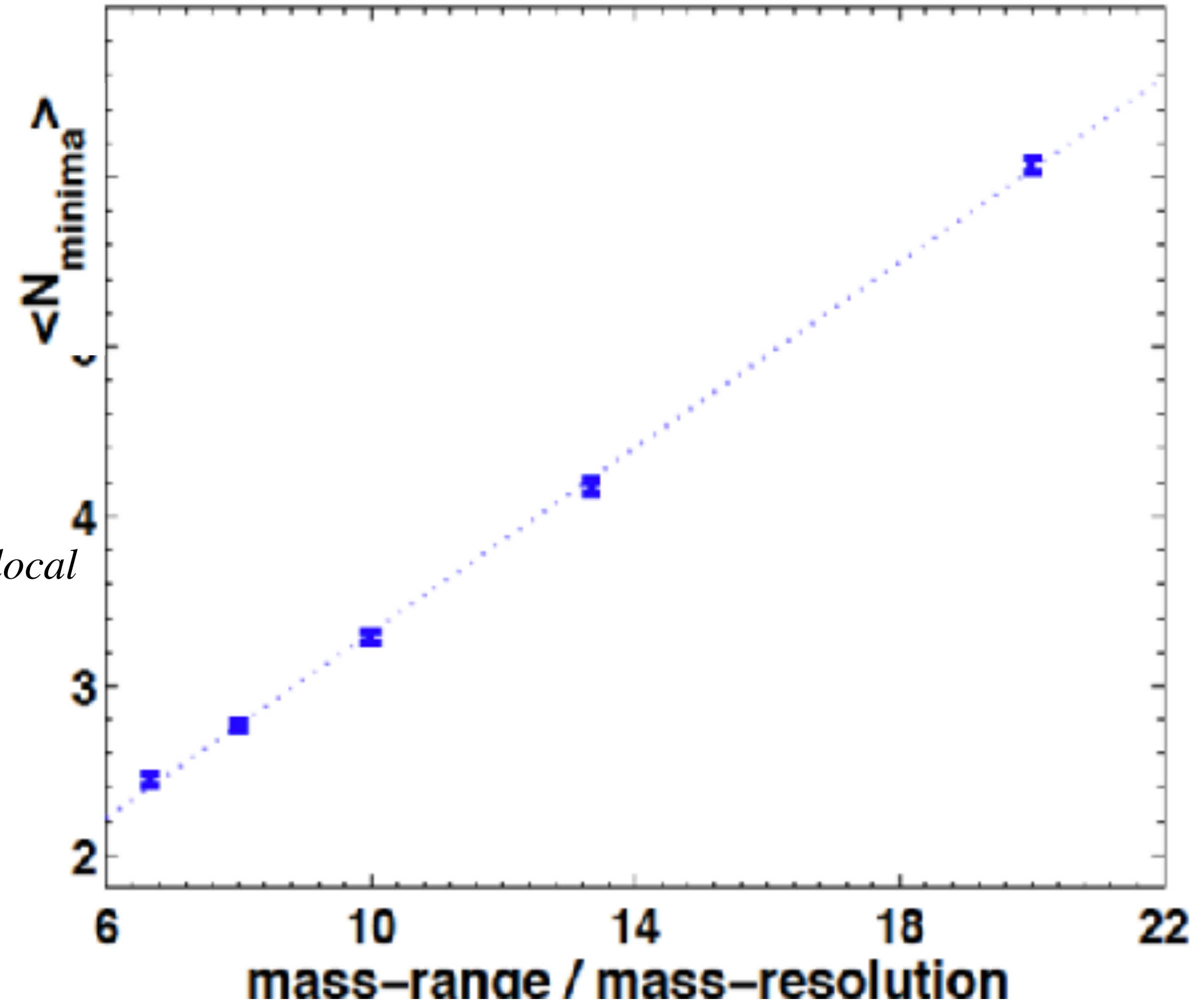
(Wrong) Thumb Rule

$$\boxed{\langle N_{minima} \rangle \sim \frac{Mass\ Range}{Mass\ Resolution}}$$

$$Trial\,\# \sim \langle N_{minima} \rangle$$

$$Trial\,\# \overset{?}{=} \langle N_{minima} \rangle\, p_{local}$$

The answer is NO



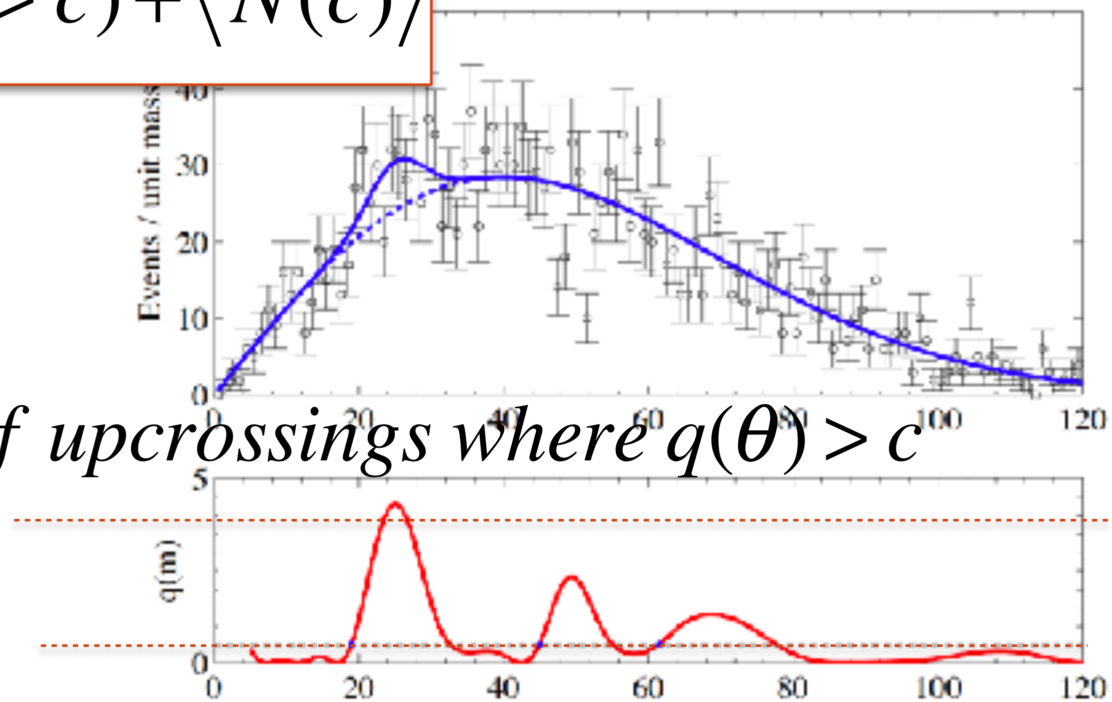
$$The\ right\ question:$$
$$What\ is\ the\ probability\ to\ have\ a\ fluctuation$$
$$as\ or\ bigger\ than\ the\ observed\ one$$
$$ANYWHERE\ in\ the\ mass\ search\ range?$$

$$Let\ \theta\ be\ a\ nuisance\ parameter$$
$$undefined\ under\ the\ null\ hypothesis.$$
$$Define\ q(\hat{\theta}) = max_\theta(q(\theta))$$
$$Davies\ (1987)\ finds,\ for\ c >> 1$$
$$P\left(q(\hat{\theta}) > c\right) \sim P(\chi_1^2 > c) + \langle N(c)\rangle$$

$$\langle N(c)\rangle = Number\ of$$
$$upcrossings\ \ q(\theta) > c$$

Davies Formula

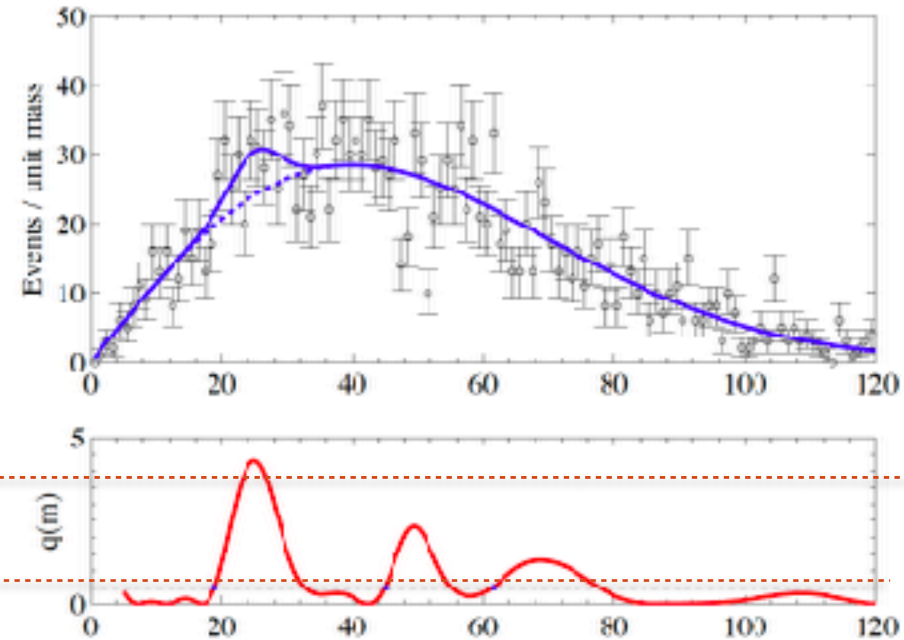
$$\boxed{P\left(q(\hat{\theta})>c\right)\sim P(\chi_1^2>c)+\langle N(c)\rangle}$$



$$\langle N(c)\rangle = Number\ of\ upcrossings\ where\ q(\theta)>c$$

$$or\ c>>1\rightarrow\langle N(c)\rangle<<1$$

Making Davies Formula Accessible



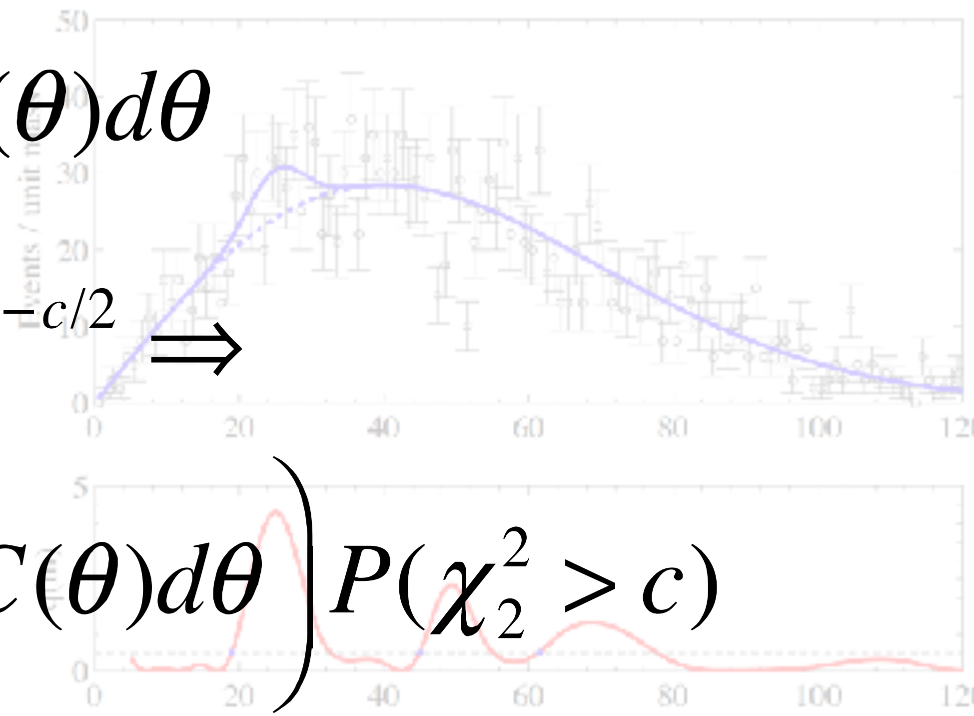
$$\langle N(c) \rangle \ll 1$$

$$\langle N(c) \rangle \sim e^{-c/2}$$

$$P(q(\hat{\theta}) > c) \sim P(\chi_1^2 > c) + \langle N(c_0) \rangle \frac{\langle N(c) \rangle}{\langle N(c_0) \rangle}$$

$$\boxed{P(q(\hat{\theta}) > c) \sim P(\chi_1^2 > c) + \langle N(c_0) \rangle e^{-(c-c_0)/2}}$$

Davies Formula



$$\langle N(c)\rangle \approx \frac{e^{-c/2}}{\sqrt{2\pi}}\int_\theta C(\theta)d\theta$$

$$P(\chi^2_2 > c)\xrightarrow[c\to\infty]{} e^{-c/2} \Rightarrow$$

$$\langle N(c)\rangle \approx \left(\frac{1}{\sqrt{2\pi}}\int_\theta C(\theta)d\theta\right)P(\chi^2_2 > c)$$

$$\langle N(c)\rangle = \mathcal{N}P(\chi^2_2 > c)$$

$$\boxed{P(q(\hat{\theta}) > c) \sim P(\chi^2_1 > c) + \mathcal{N}P(\chi^2_2 > c)}$$

Trial

$$\boxed{\begin{aligned} P(\chi_1^2 > c) &\xrightarrow{c>>1} \sqrt{\frac{2}{c}}\,\frac{e^{-c/2}}{\Gamma\left(\frac{1}{2}\right)} \\ P(\chi_2^2 > c) &\xrightarrow{c>>1} e^{-c/2} \end{aligned}}$$

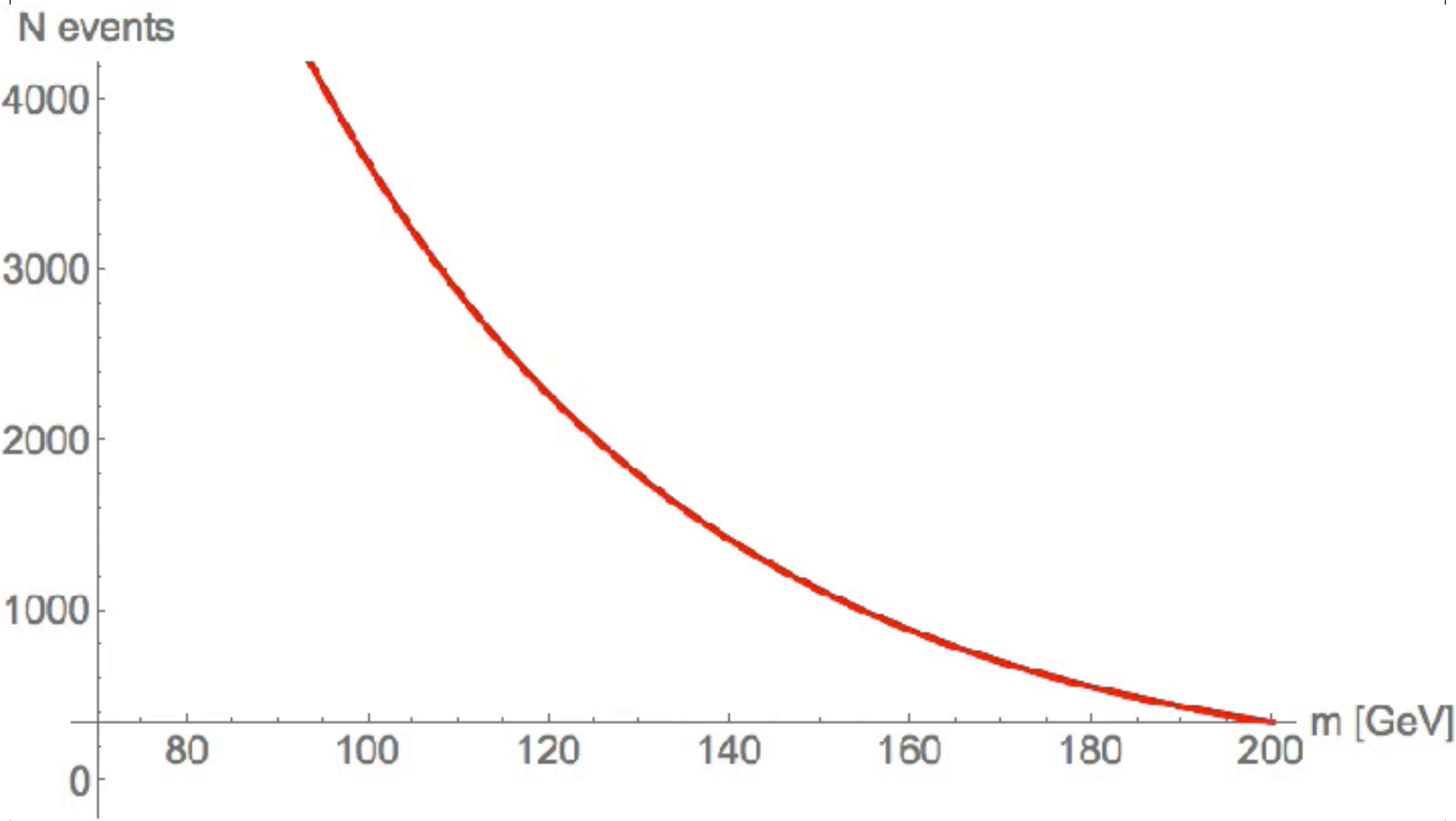
$$trial\,\# = \frac{P(q(\hat{\theta}) > c)}{P(q(\theta) > c)} \approx$$

$$\approx 1 + \mathcal{N}\,\frac{P(\chi_2^2 > c)}{P(\chi_1^2 > c)} \Rightarrow$$

$$trial\,\# \approx 1 + \mathcal{N}\sqrt{\frac{c}{2}}\,\Gamma(1/2) \Rightarrow$$

$$trial\,\# \approx 1 + \sqrt{\frac{\pi}{2}}\,\mathcal{N} Z_{fix}$$

What is Going On?


N events
4000
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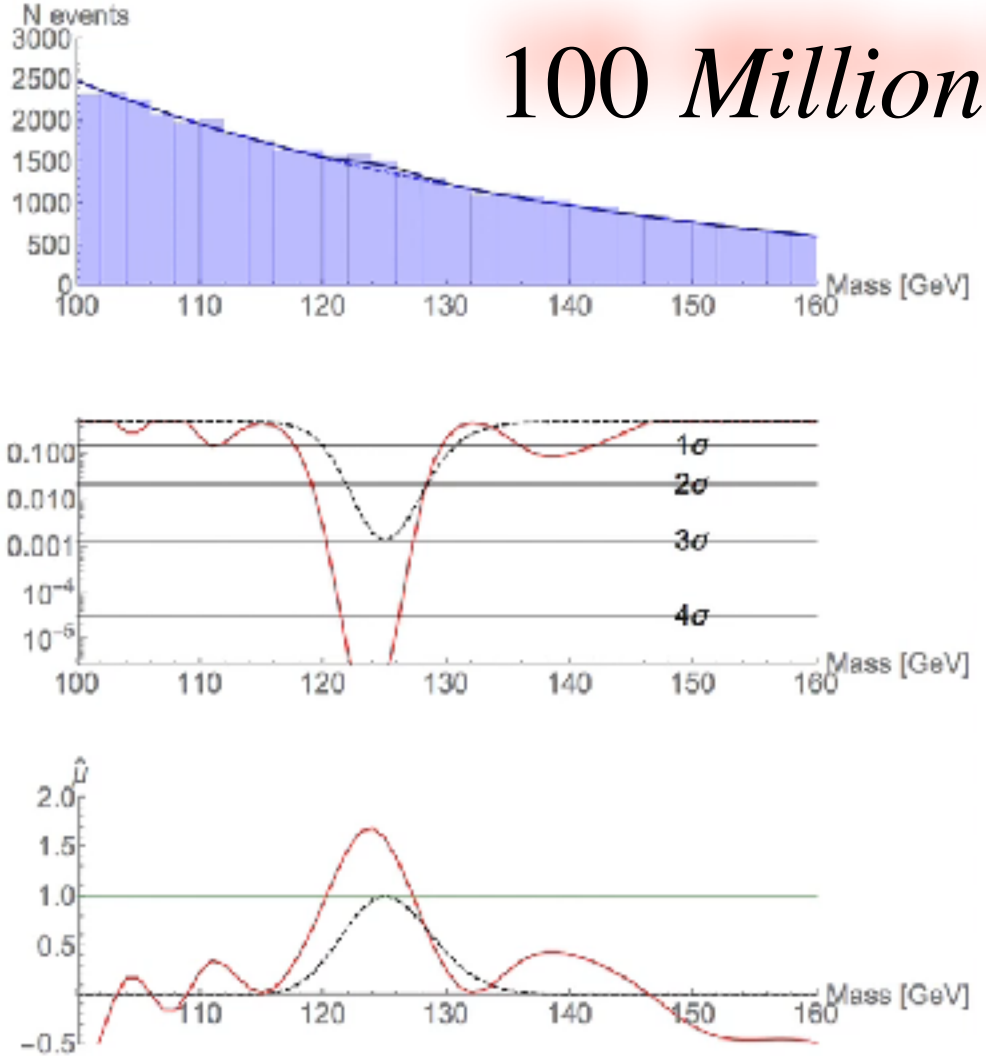
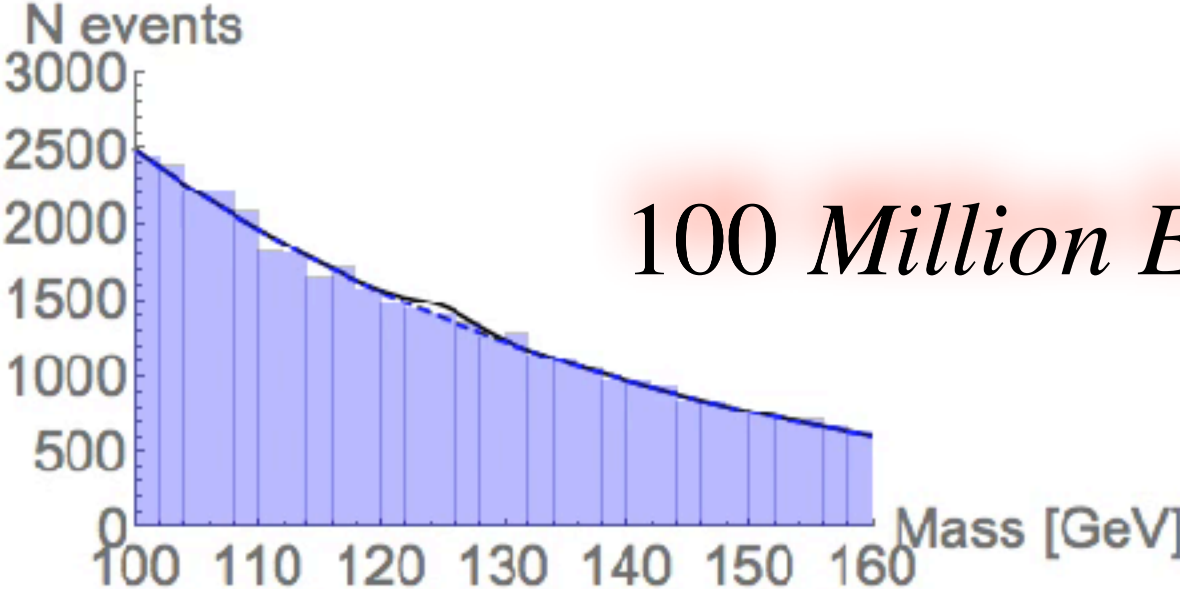
m [GeV]

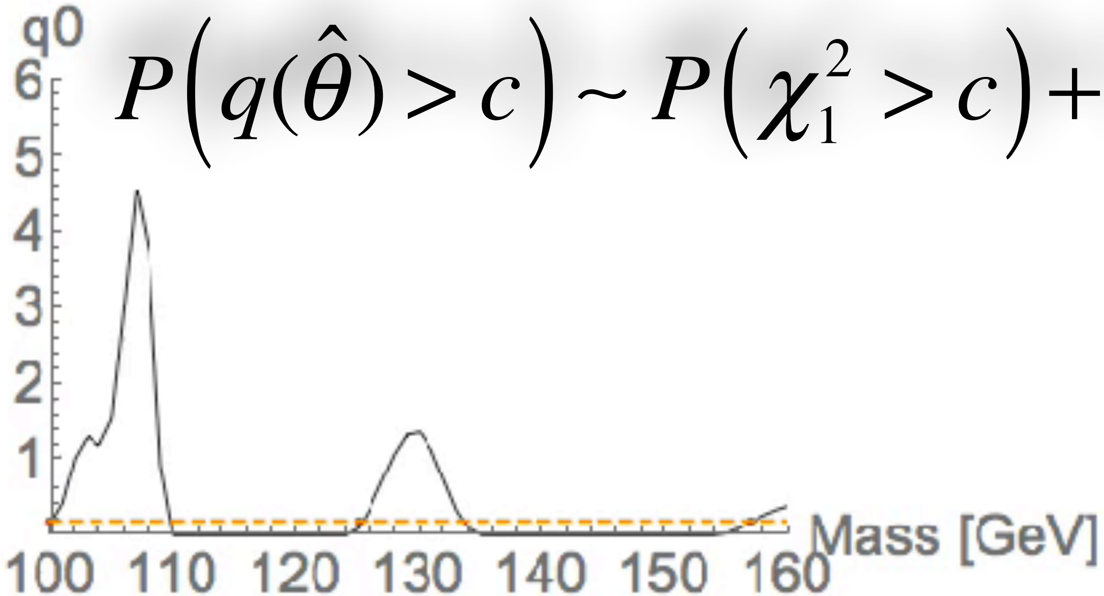
100 *Million Experiments*

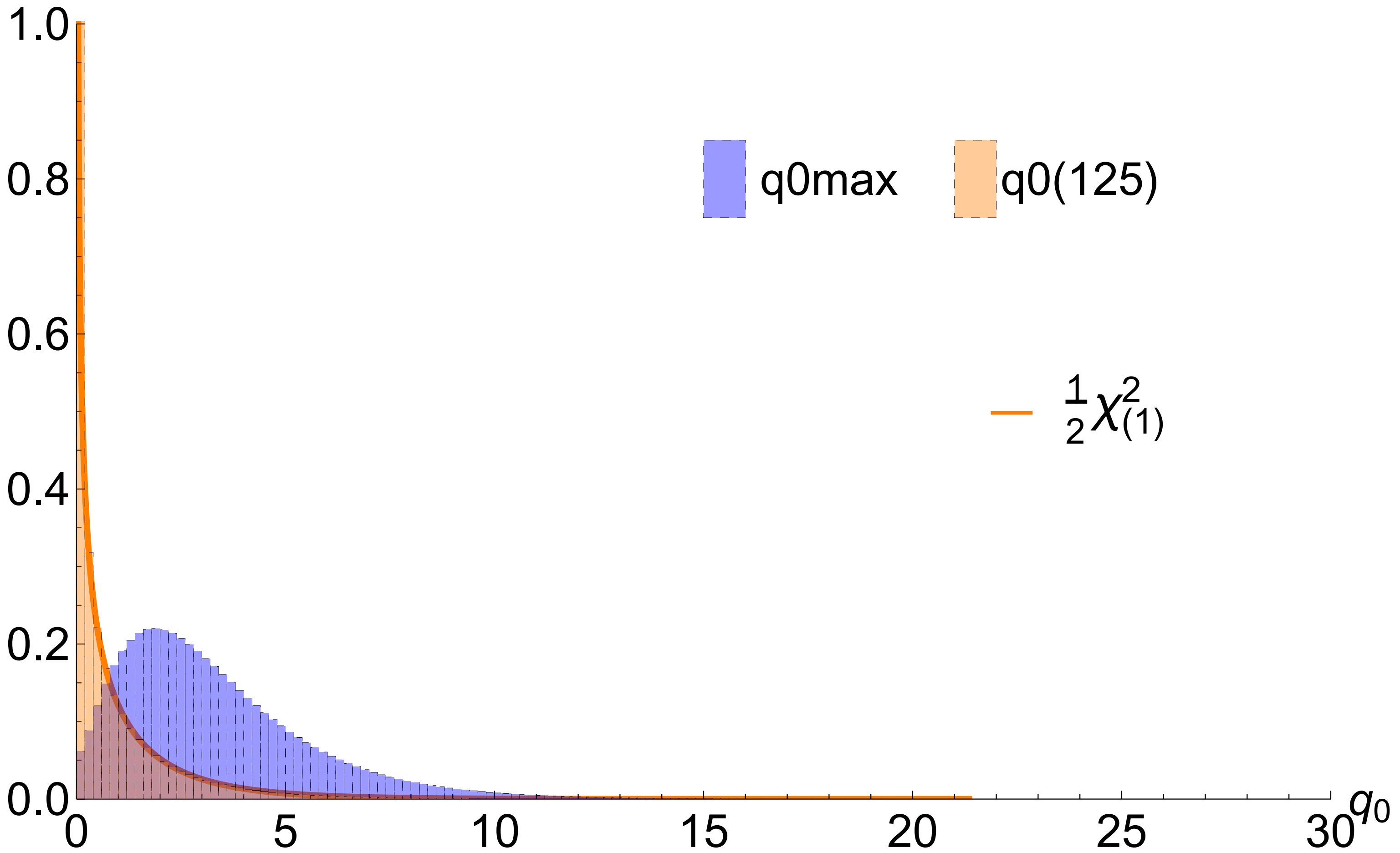




100 Million Experiments

$$P\left(q(\hat{\theta}) > c\right) \sim P\left(\chi_1^2 > c\right) + \langle N(c_0) \rangle e^{-(c-c_0)/2}$$




1.0
0.8
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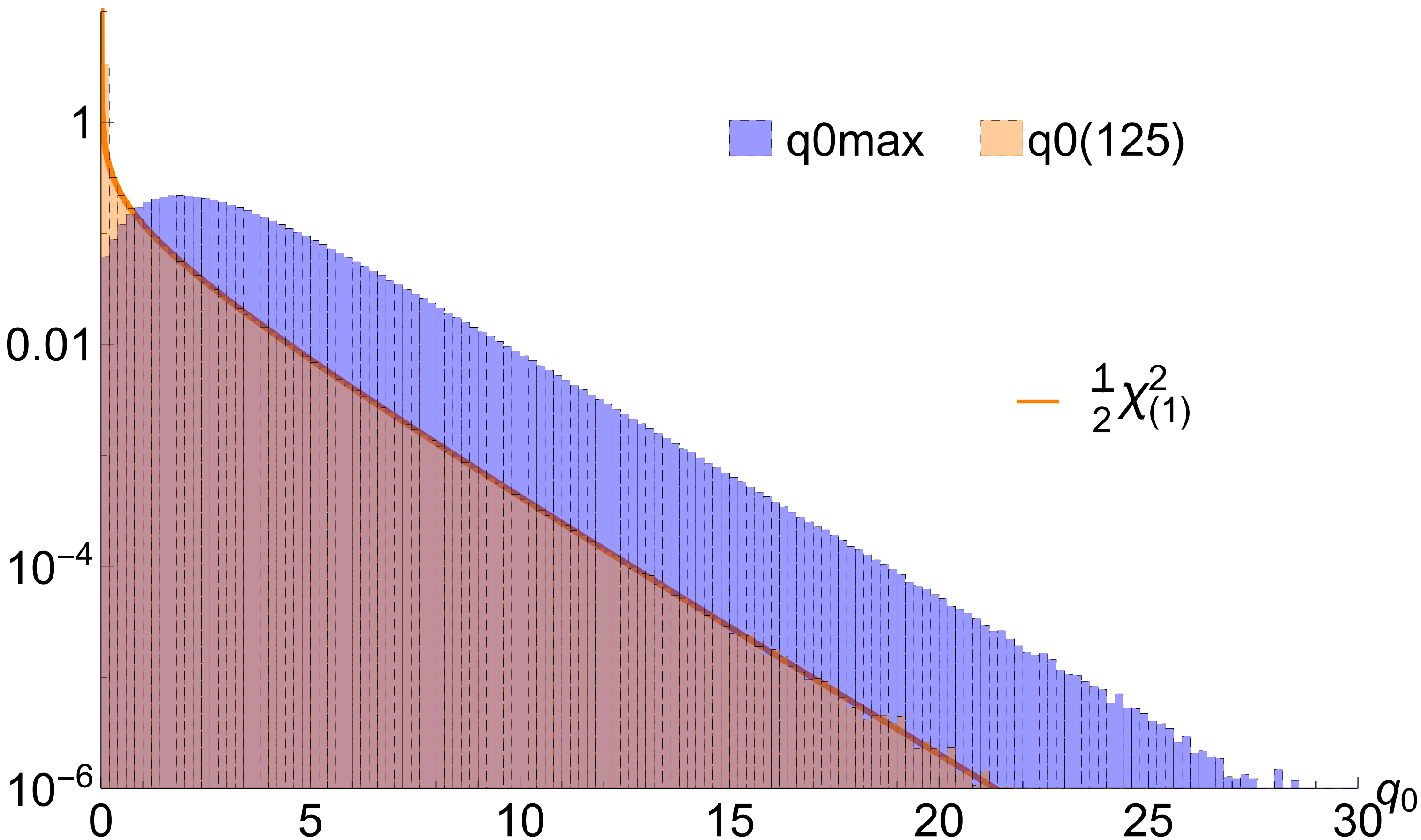
30
q_0
q0max
q0(125)
$\frac{1}{2}\chi^2_{(1)}$



q0max
q0(125)
$\frac{1}{2}\chi^2_{(1)}$
1
0.01
10^{-4}
10^{-6}
0
5
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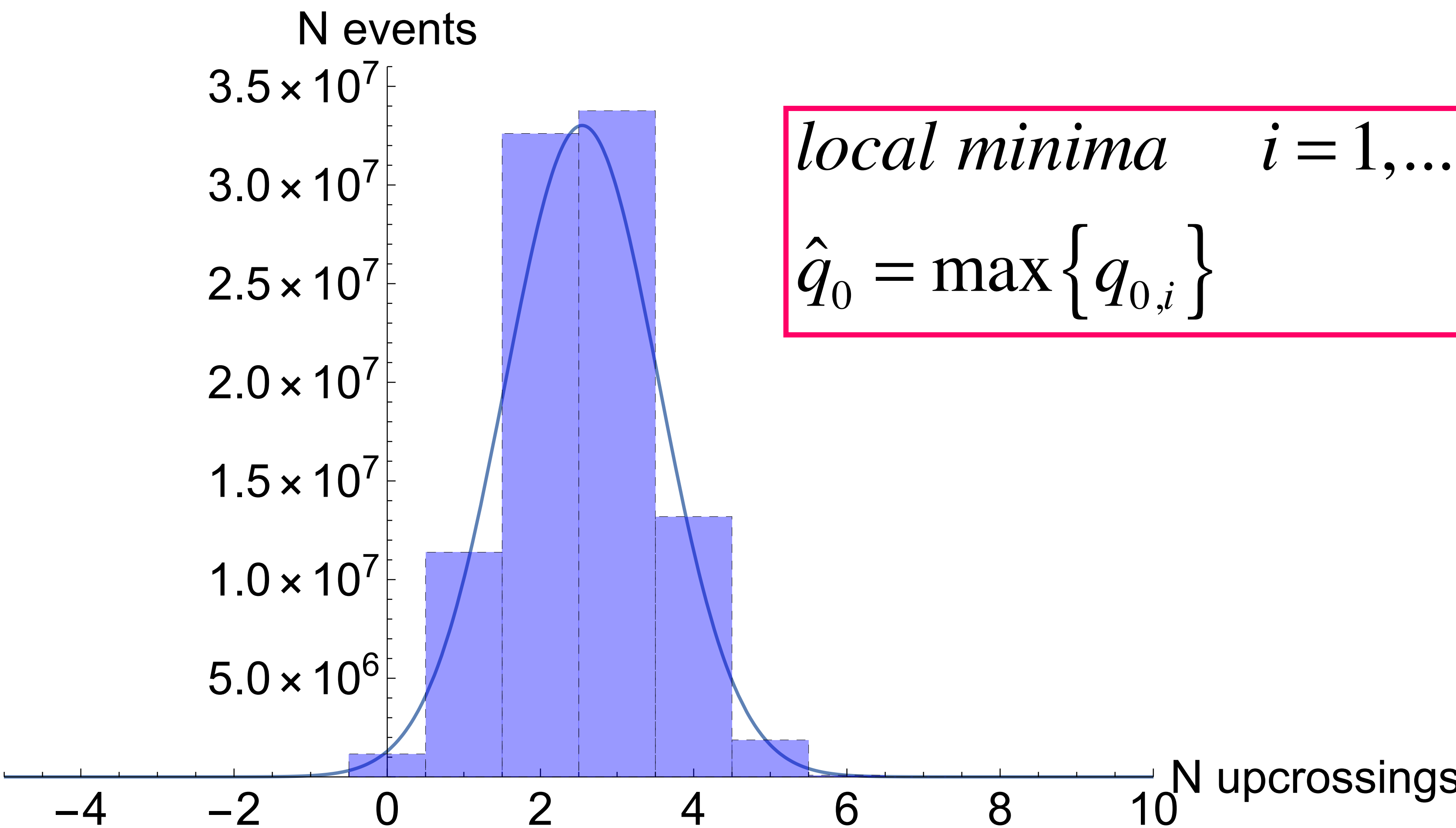
q_0

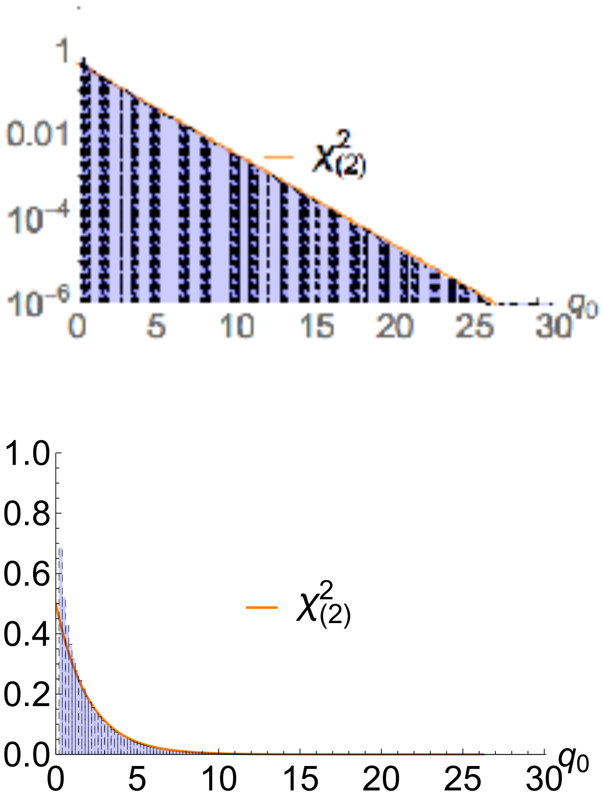


$$\textit{local minima} \qquad i = 1, \ldots\ldots, n$$

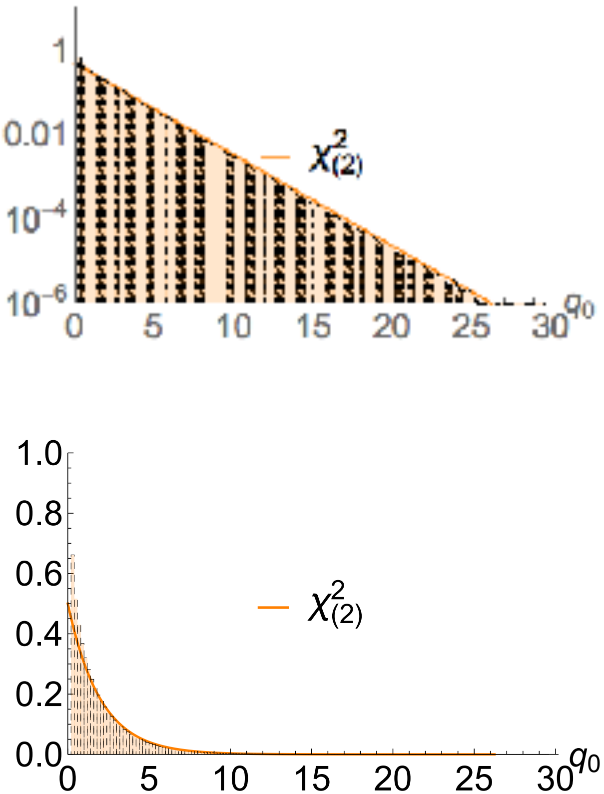
$$\hat{q}_0 = \max\left\{ q_{0,i} \right\}$$

$$\forall i \quad q_{0,i} \sim \chi^2_2$$

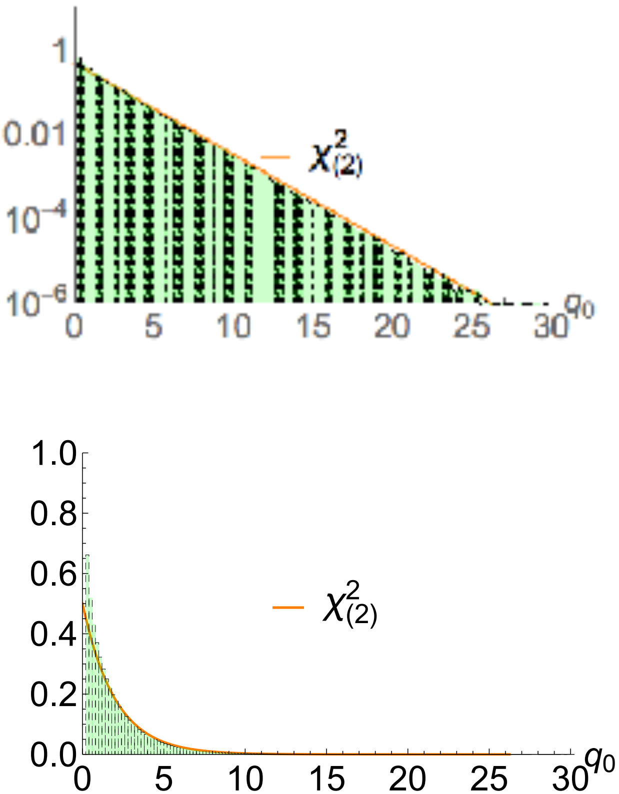
$q_{0,1}$

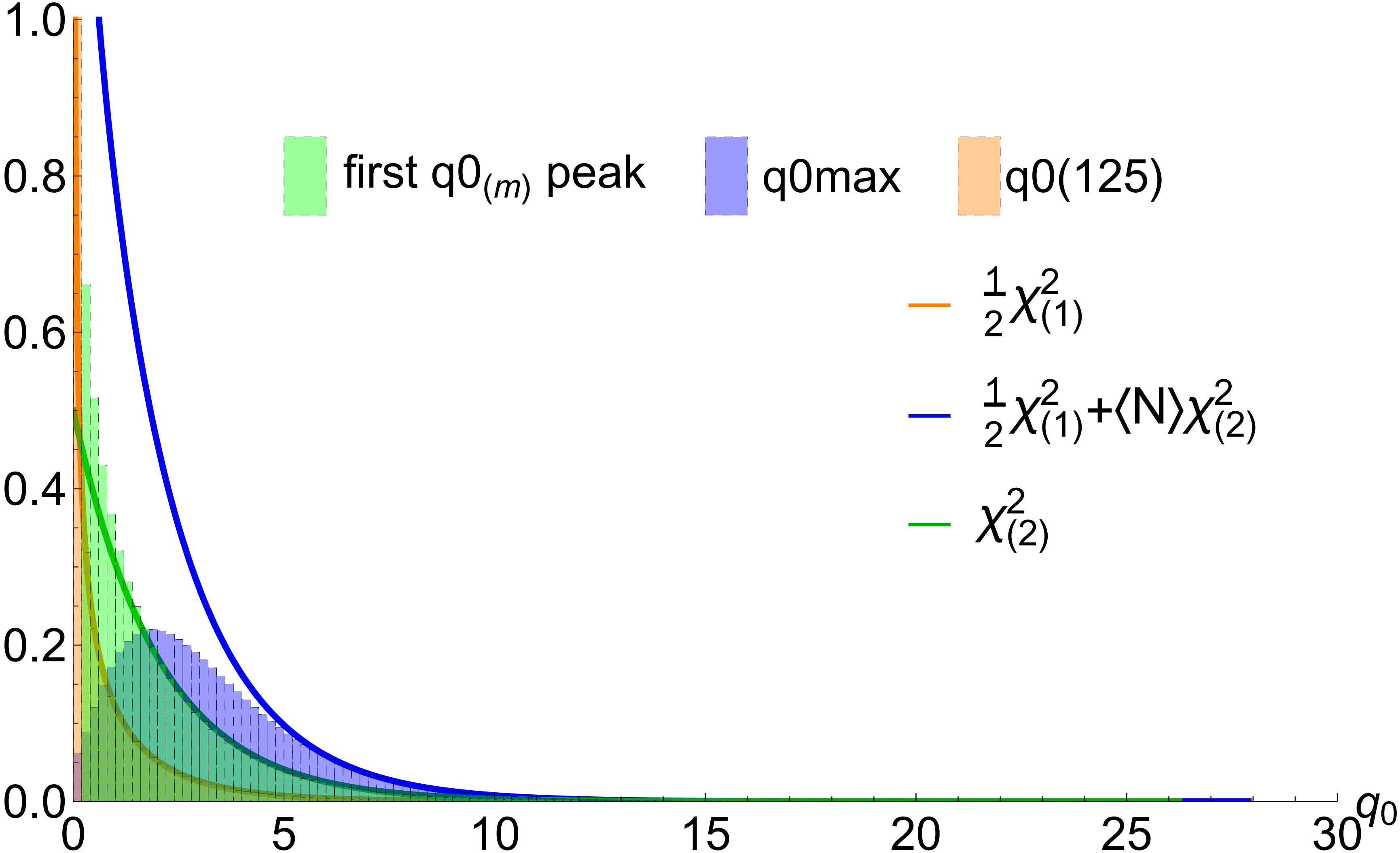


$q_{0,2}$



$q_{0,3}$





1.0
0.8
0.6
0.4
0.2
0.0
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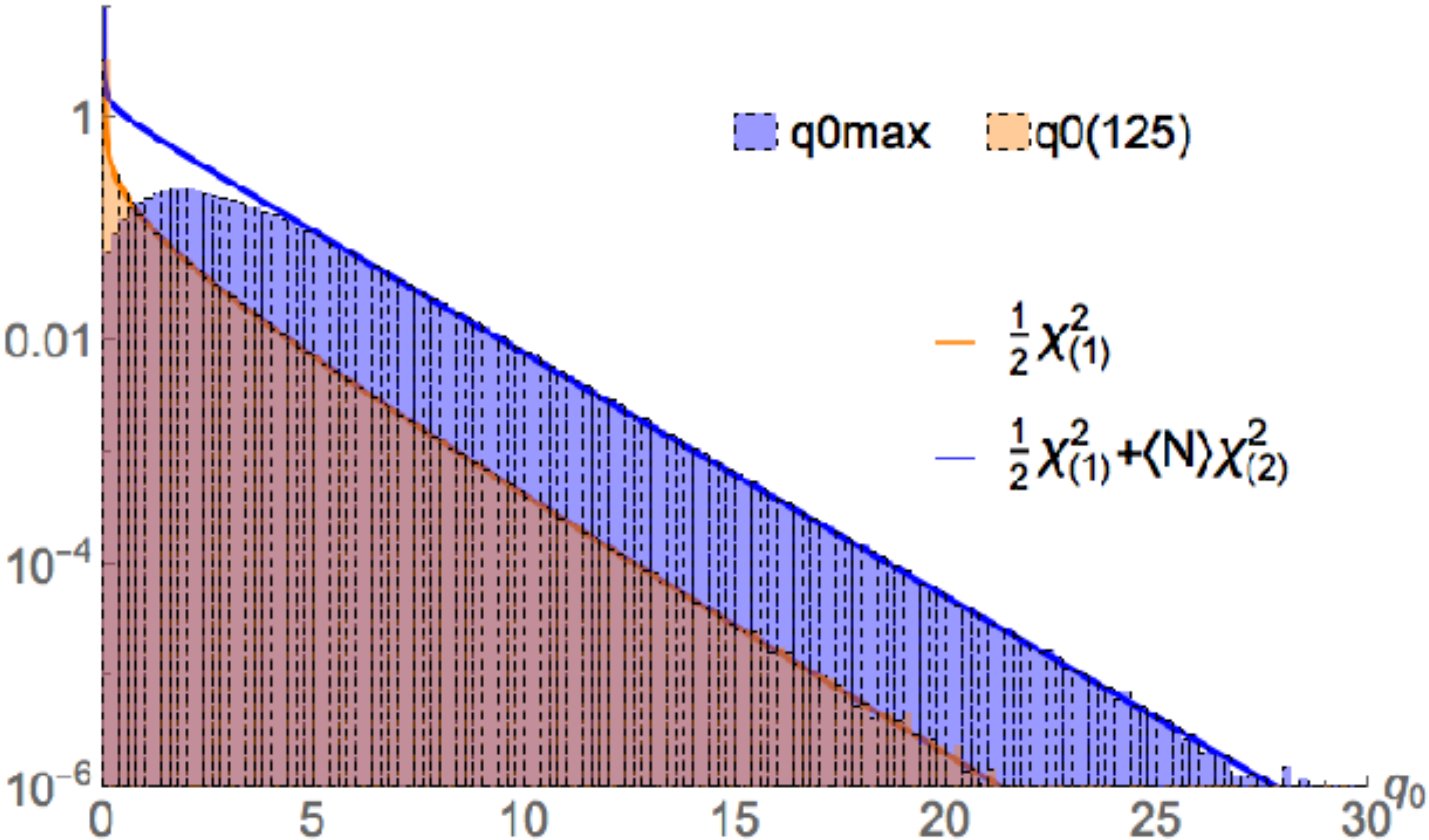
5
10
15
20
25
30
q_0
first $q0_{(m)}$ peak
q0max
q0(125)
$\frac{1}{2}\chi^2_{(1)}$
$\frac{1}{2}\chi^2_{(1)}+\langle N\rangle\chi^2_{(2)}$
$\chi^2_{(2)}$



q0max
q0(125)
$\frac{1}{2}\chi^2_{(1)}$
$\frac{1}{2}\chi^2_{(1)}+\langle N\rangle\chi^2_{(2)}$
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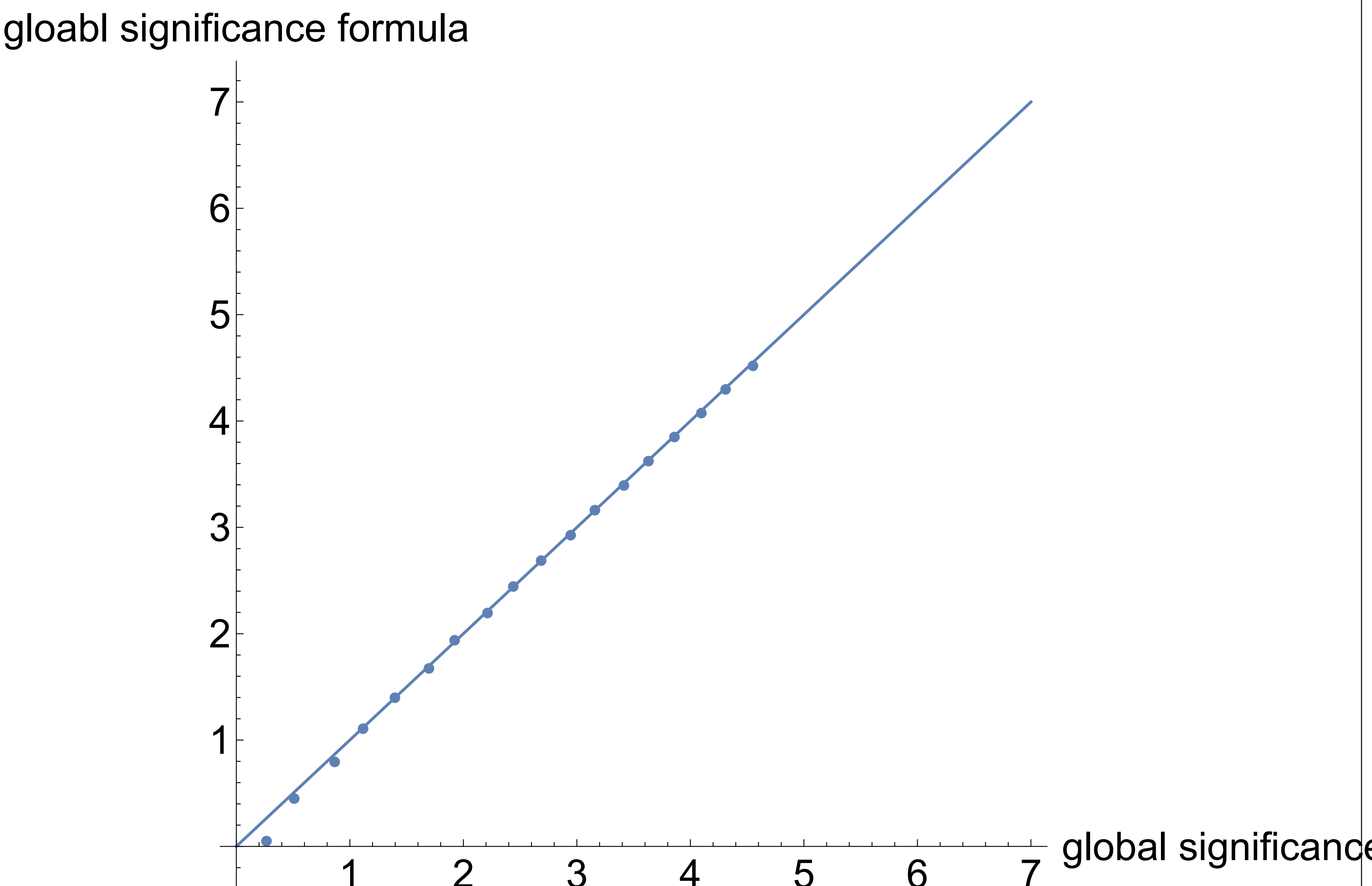
q_0


gloabl significance formula
7
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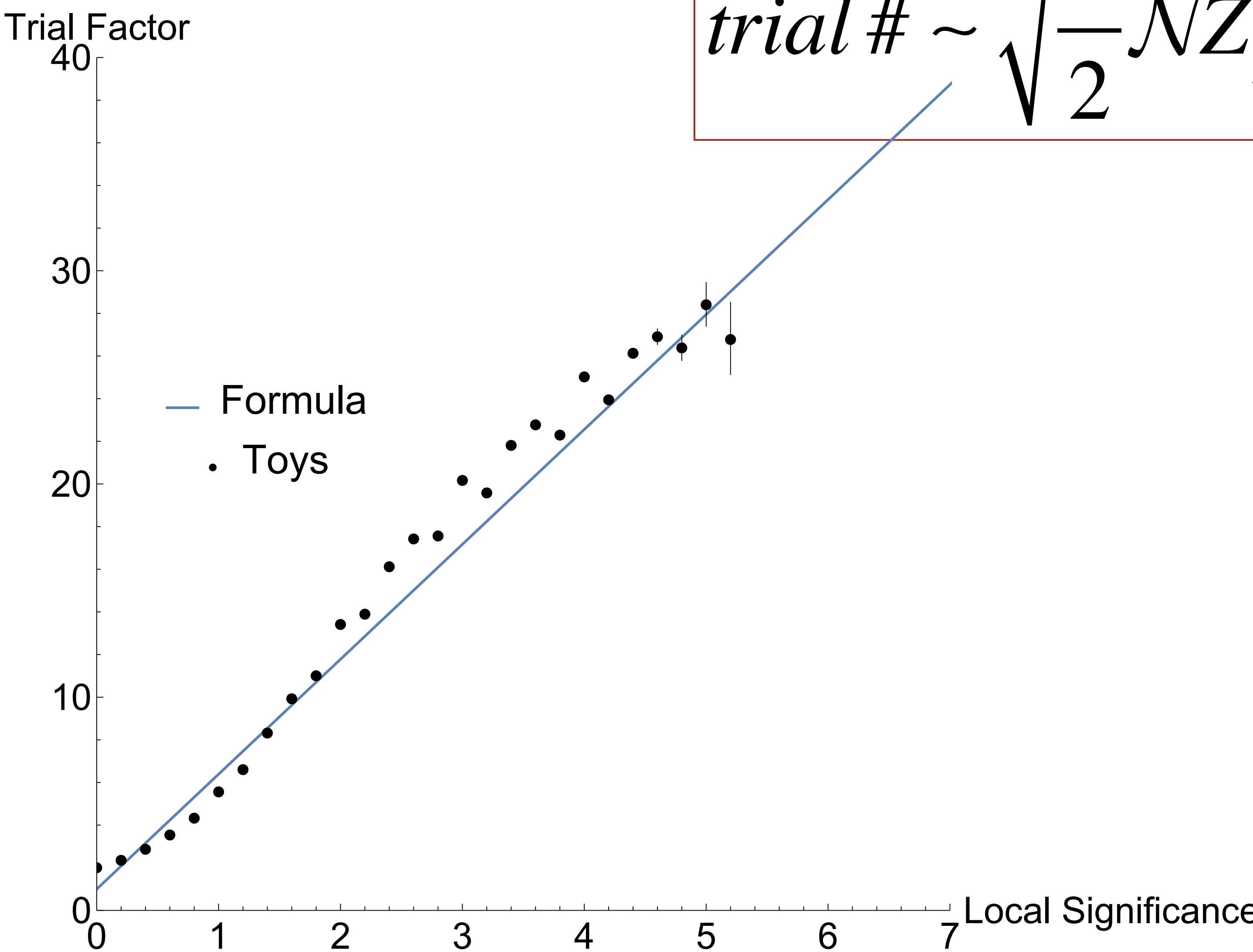
global significance

$$trial\,\# \sim \sqrt{\frac{\pi}{2}}\mathcal{N}Z_{fix}$$



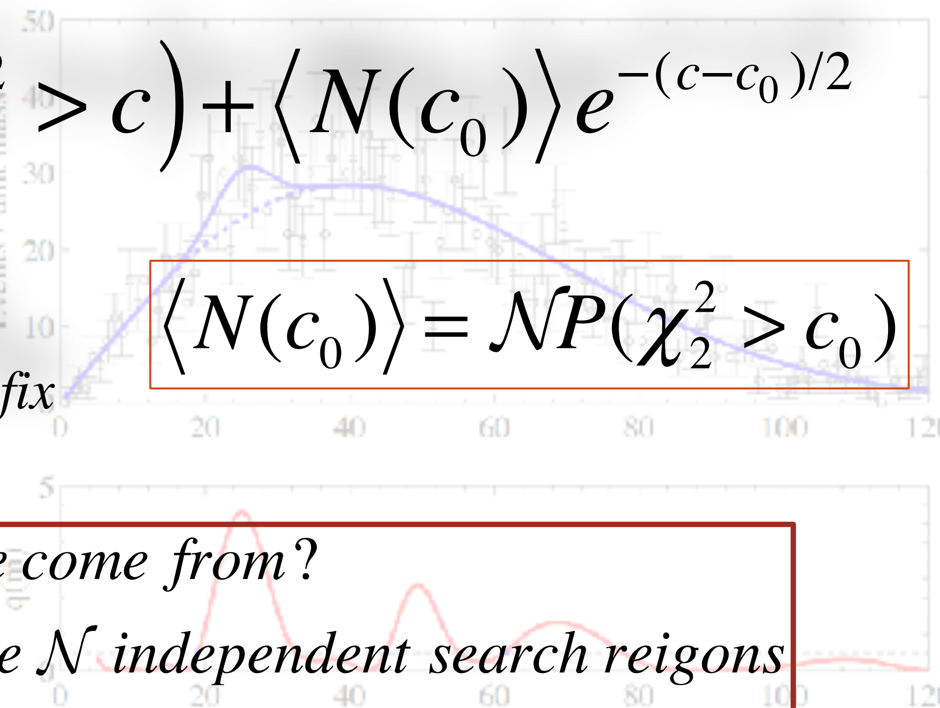
Why Trial#~Z_{fix}?

Solution of the LEE problem

$$P\left(q(\hat{\theta})>c\right)\sim P\left(\chi_1^2>c\right)+\left\langle N(c_0)\right\rangle e^{-(c-c_0)/2}$$

$$trial\,\# \sim 1+\sqrt{\frac{\pi}{2}}\mathcal{N}Z_{fix}$$

$$\boxed{\left\langle N(c_0)\right\rangle=\mathcal{N}P(\chi_2^2>c_0)}$$



Where does the Z dependence come from?

View the results as if there are $\mathcal{N}$ *independent search reigons*

In each one there is a χ_2^2 *distribution of* $q_0(\mu,m)$

The mass is a dof even though it is undefined under the null

$$\sigma_{\hat{m}}\sim\frac{1}{Z}\Rightarrow\frac{\Delta m}{\sigma_{\hat{m}}}\sim Z$$

Why trial#~Z

$$Var(m)=\left[-E\left(\frac{\partial^2\log\mathcal{L}}{\partial m^2}\right)\right]^{-1}$$

$$n\sim Poiss\left(\mu s(m)+b\right)\approx e^{-(\mu s(m)+b)}\left(\mu s(m)+b\right)^n$$

$$\log\mathcal{L}=-\mu s(m)-b+n\log\left(\mu s(m)+b\right)$$

$$\frac{\partial\log\mathcal{L}}{\partial m}=-\mu\frac{\partial s(m)}{\partial m}+n\frac{\mu}{\mu s(m)+b}\frac{\partial s(m)}{\partial m}$$

$$\frac{\partial^2\log\mathcal{L}}{\partial m^2}=-\mu\frac{\partial^2 s(m)}{\partial m^2}+n\frac{\mu}{\mu s(m)+b}\frac{\partial^2 s(m)}{\partial m^2}-n\frac{\mu^2}{\left(\mu s(m)+b\right)^2}\left(\frac{\partial s(m)}{\partial m}\right)^2$$

$$E[n]=\mu s(m)+b$$

$$E\left[\frac{\partial^2\log\mathcal{L}}{\partial m^2}\right]=-\frac{\mu^2}{\mu s(m)+b}\left(\frac{\partial s(m)}{\partial m}\right)^2$$

$$Var[m]\sim\frac{1}{\mu}\sim\frac{1}{Z}\Rightarrow\sigma_{\hat{m}}\sim Z\Rightarrow trial\#\sim\frac{range}{\sigma_{\hat{m}}}\sim Z$$

A real life example

$$P(q_0 > u) \le E[N_u] + P(q_0(0) > u)$$

$$E[N_u] = \mathcal{N}_1 e^{-u/2}$$

$$\mathcal{N}_1 \cong \langle N_{u_0} \rangle e^{u_0/2}$$

$$P(q_0 > u) = \mathcal{N}_1 e^{-u/2} + \frac{1}{2} P(\chi_1^2 > u)$$

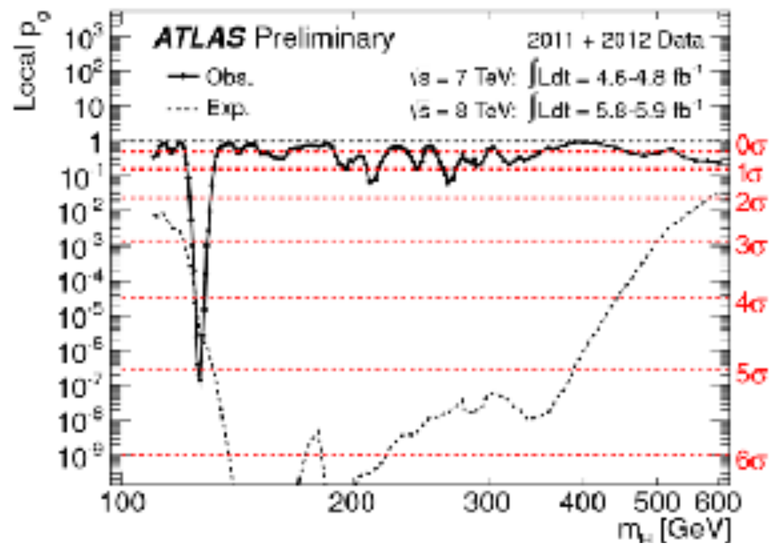
$$p_{global} = \mathcal{N}_1 e^{-u/2} + p_{local}$$

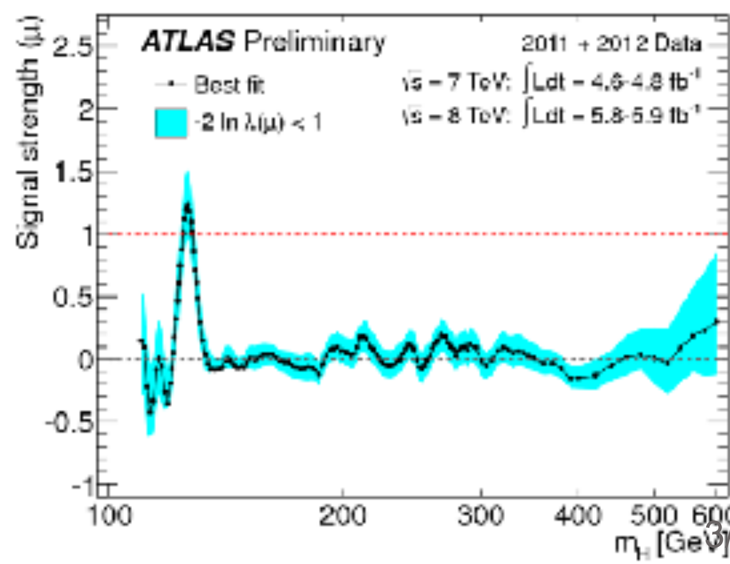
$$p_{global} = \langle N_{u_0} \rangle e^{\frac{u_0 - u}{2}} + p_{local}$$

$$N_{u_0=0} = 9 \pm 3$$

$$p_{global} = 9 \cdot e^{-25/2} + O(10^{-7}) = 3.3 \cdot 10^{-5}$$

$5\sigma \rightarrow 4\sigma$ trial#~100





Example: The 750 GeV Resonance

Spin 0 2015

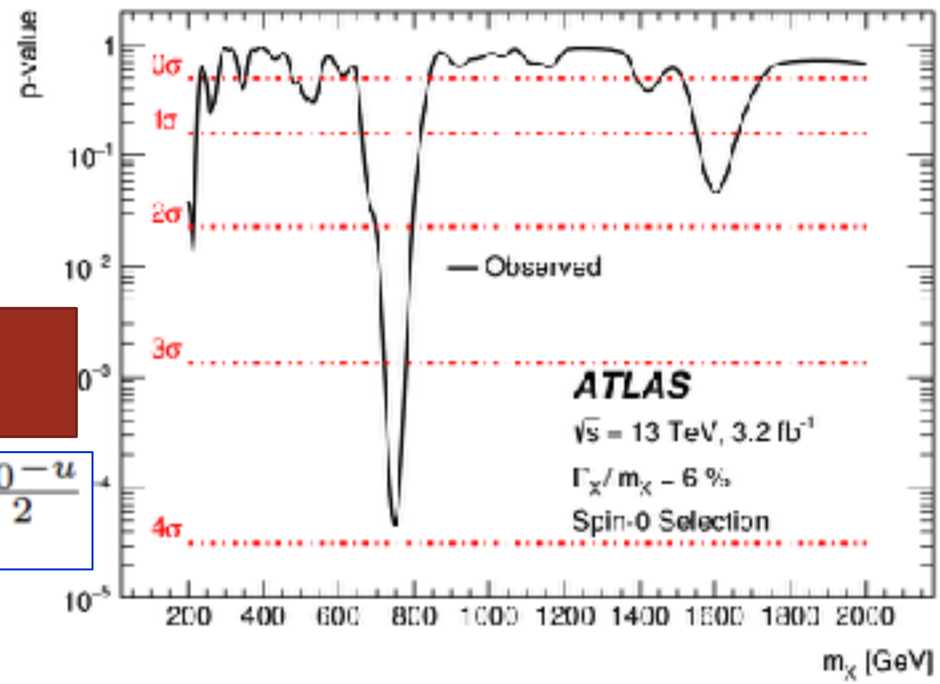
Largest significance

$m_x \sim 750\text{GeV}, \Gamma_x \sim 45\text{GeV}(6$

Local Z = **3.9σ**

Any peak with Z>3.8σ
with m=500-2000 will draw our attention

$$P_{global}(u) \approx p_{local}(u) + E(n_{u_0})e^{\frac{u_0-u}{2}}$$



$$p_{local} = 5 \cdot 10^{-5}$$

$$u_0 = 0$$

$$n_{u_0} = 7 \pm 2.6$$

$$u = Z^2 = 3.9^2 = 15.2$$

$$p_{global} = 5 \cdot 10^{-5} + (7 \pm 2.6)e^{-15.2/2} = (2.2 - 4.8)10^{-3}$$

$$Z_{global} \sim 2.7 \pm 0.1\sigma$$

The LEE is even stronger when you consider another dimension (the width range (0-10%m) should also be taken into account)

End of Lectures
Thank You

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