

Three-particle dynamics on the lattice

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- Introduction
- Non-relativistic EFT and dimer picture
- Finite-volume spectrum
- Independence from the off-shell effects
- Example: three-particle bound state
- Conclusions, outlook

Extraction of the observables on the lattice

Motivation:

- ↪ Decays into the three-particle final states
(examples: $\eta \rightarrow 3\pi$, $\omega \rightarrow 3\pi$, Roper resonance, etc.)
- ↪ Nuclear physics on the lattice

Three-particle sector: continuum

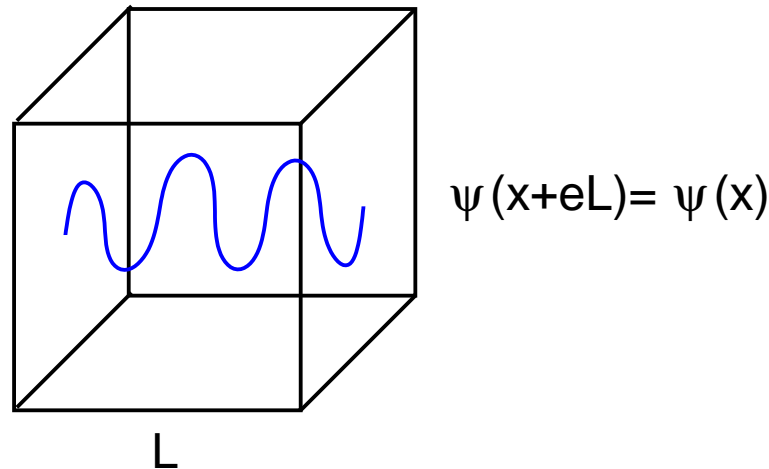
- bound states
- elastic scattering, rearrangement reactions
- breakup...

Three-particle sector: finite volume

- energy levels below and above three-particle threshold

How does one extract infinite-volume observables from lattice data?

Elastic scattering: Lüscher formalism

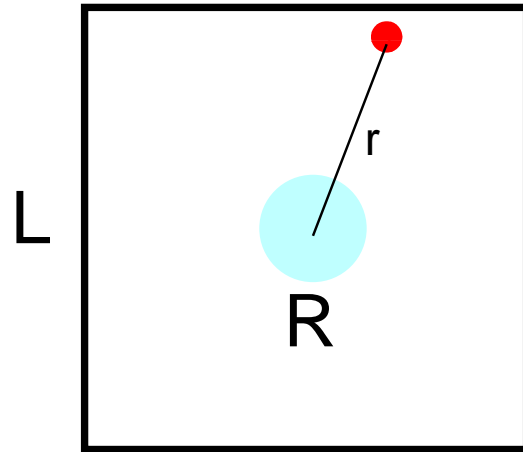


- Lattice calculations always done in a finite box: no asymptotic states & scattering
- Discrete energy levels determined from the boundary conditions
- If $L \gg R$, then the infinite-volume phase determined through the Lüscher equation ($\phi(q)$ is a known function)

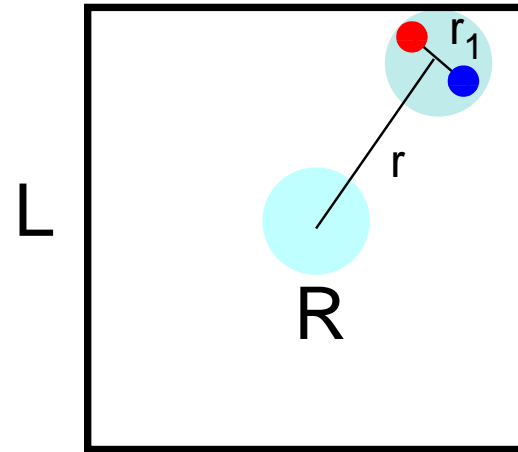
$$\tan \delta(p) = -\tan \phi(q), \quad q = pL/(2\pi)$$

- Exponentially suppressed corrections, partial-wave mixing, . . .

Three particles in a finite volume: the problem



2 particles



3 particles

- The three-particle wave function near the box walls is not always described by the asymptotic wave function
- Is the three-particle spectrum determined solely in terms of the S -matrix?

The history

K. Polejaeva and AR, EPJA 48 (2012) 67

Finite volume energy levels determined solely by the S -matrix

M. Hansen and S. Sharpe, PRD 90 (2014) 116003; PRD 92 (2015) 114509

Quantization condition

R. Briceno and Z. Davoudi, PRD 87 (2013) 094507

Dimer formalism, quantization condition

P. Guo, PRD 95 (2017) 054508

Quantization condition in the 1+1-dimensional case

S. Kreuzer and H.-W. Hammer, PLB 694 (2011) 424; EPJA 43 (2010) 229; PLB 673 (2009) 260; S. Kreuzer and H. W. Grießhammer, EPJA 48 (2012) 93

Dimer formalism, numerical solution

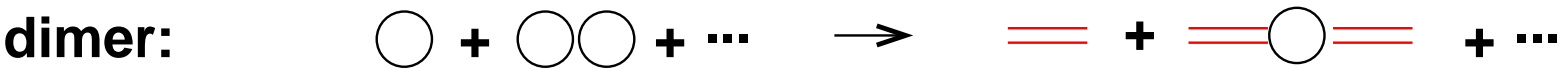
- ↪ The quantization condition complicated, not well suited for the analysis of the lattice data
- ↪ What is the convenient set of observables to be extracted from data?

NREFT: dimer picture in the two-particle sector

$$\mathcal{L} = \psi^\dagger \left(i\partial_0 - \frac{\nabla^2}{2m} \right) \psi + \mathcal{L}_2$$

$$\mathcal{L}_2 = -\frac{C_0}{2} \psi^\dagger \psi^\dagger \psi \psi - \frac{C_2}{2} (\psi^\dagger \nabla^2 \psi^\dagger \psi \psi + \text{h.c.}) + \dots$$

C_0, C_2, \dots matched to $p \cot \delta(p) = -\frac{1}{a} + \frac{r}{2} p^2 + \dots$



$$\mathcal{L}_2 \rightarrow \mathcal{L}_2^{\text{dimer}} = \sigma T^\dagger T + \left(T^\dagger [f_0 \psi \psi + f_1 \psi \nabla^2 \psi + \dots] + \text{h.c.} \right)$$

- Two frameworks algebraically equivalent
- Higher partial waves can be included: dimers with arbitrary spin $T_{i_1 \dots i_{2k}}$
- Can be generalized to the non-rest frames

NREFT: three-particle sector

$$\begin{aligned}\mathcal{L}_3 &= -\frac{D_0}{6} \psi^\dagger \psi^\dagger \psi^\dagger \psi \psi \psi - \frac{D_2}{12} (\psi^\dagger \psi^\dagger \nabla^2 \psi^\dagger \psi \psi \psi + \text{h.c.}) \\ &+ \frac{D_4}{12} (\psi^\dagger \psi^\dagger \nabla^4 \psi^\dagger \psi \psi \psi + \text{h.c.}) \\ &+ \frac{D'_4}{12} (\psi^\dagger \psi^\dagger \nabla^4 \psi^\dagger \psi \psi \psi + 2\psi^\dagger \nabla^2 \psi^\dagger \nabla^2 \psi^\dagger \psi \psi \psi + 3\psi^\dagger \psi^\dagger \nabla^2 \psi^\dagger \psi \psi \nabla^2 \psi + \text{h.c.}) \\ &+ \frac{D''_4}{12} (\psi^\dagger \psi^\dagger \nabla^4 \psi^\dagger \psi \psi \psi + 2\psi^\dagger \nabla^2 \psi^\dagger \nabla^2 \psi^\dagger \psi \psi \psi - 3\psi^\dagger \psi^\dagger \nabla^2 \psi^\dagger \psi \psi \nabla^2 \psi + \text{h.c.}) + \dots\end{aligned}$$

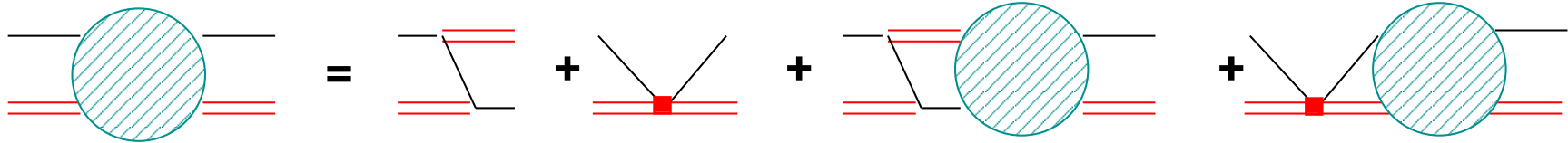
Off-shell term proportional to D_4 – can be eliminated using EOM

$$\begin{aligned}\mathcal{L}_3^{\text{dimer}} &= h_0 T^\dagger T \psi^\dagger \psi + h_2 T^\dagger T (\psi^\dagger \nabla^2 \psi + \text{h.c.}) \\ &+ h_4 T^\dagger T (\psi^\dagger \nabla^4 \psi + \text{h.c.}) + h'_4 T^\dagger T \nabla^2 \psi^\dagger \nabla^2 \psi + \dots\end{aligned}$$

↪ No restriction on angular momentum: dimers with arbitrary spin

↪ Two couplings h_4, h'_4 : off-shell coupling D''_4 can be eliminated!

The scattering equation



$$\mathcal{M}(\mathbf{p}, \mathbf{q}; E) = Z(\mathbf{p}, \mathbf{q}; E) + \int_{\mathbf{k}}^{\Lambda} Z(\mathbf{p}, \mathbf{q}; E) \tau(\mathbf{k}; E) \mathcal{M}(\mathbf{k}, \mathbf{q}; E)$$

$$Z(\mathbf{p}, \mathbf{q}; E) = \frac{1}{\mathbf{p}^2 + \mathbf{q}^2 + \mathbf{p}\mathbf{q} - mE} + H_0 + H_2(\mathbf{p}^2 + \mathbf{q}^2) + \dots$$

H_0, H_2, \dots are related to the couplings h_0, h_2, \dots

$$\tau^{-1}(\mathbf{k}; E) = k^* \cot \delta(k^*) + \underbrace{\sqrt{\frac{3}{4} \mathbf{k}^2 - mE}}_{=k^*}$$

Finite volume

$$\mathbf{k} = \frac{2\pi}{L} \mathbf{n}, \quad \mathbf{n} \in \mathbb{Z}^3, \quad \int_{\mathbf{k}}^{\Lambda} \rightarrow \frac{1}{L^3} \sum_{\mathbf{k}}^{\Lambda}$$

$$\mathcal{M}_L(\mathbf{p}, \mathbf{q}; E) = Z(\mathbf{p}, \mathbf{q}; E) + \frac{8\pi}{L^3} \sum_{\mathbf{k}}^{\Lambda} Z(\mathbf{p}, \mathbf{q}; E) \tau_L(\mathbf{k}; E) \mathcal{M}_L(\mathbf{k}, \mathbf{q}; E)$$

$$\tau_L^{-1}(\mathbf{k}; E) = k^* \cot \delta(k^*) - \frac{4\pi}{L^3} \sum_{\mathbf{l}} \frac{1}{\mathbf{k}^2 + \mathbf{l}^2 + \mathbf{k}\mathbf{l} - mE}$$

- ↪ Poles of \mathcal{M}_L → finite-volume energy spectrum
- ↪ $k^* \cot \delta(k^*)$ fitted in the two-particle sector; H_0, H_2, \dots should be fitted to the three-particle energies
- ↪ S -matrix in the infinite volume → equation with H_0, H_2, \dots

The fate of the off-shell effects

- Off-shell potential in the momentum space:

$$V^{\text{off-shell}} \propto D_4''(E(\mathbf{p}) - E(\mathbf{q}))^2, \quad E(\mathbf{p}) = \frac{1}{2m} (\mathbf{p}_1^2 + \mathbf{p}_2^2 + \mathbf{p}_3^2)$$

- Potential vanishes on-shell $E(\mathbf{p}) = E(\mathbf{q})$
- n^{th} iteration vanishes on-shell: no-scale integrals (no dependence on D_4'')

$$V^{\text{off-shell}} G_0 V^{\text{off-shell}} G_0 \dots G_0 V^{\text{off-shell}} = 0 \quad (\text{on shell})$$

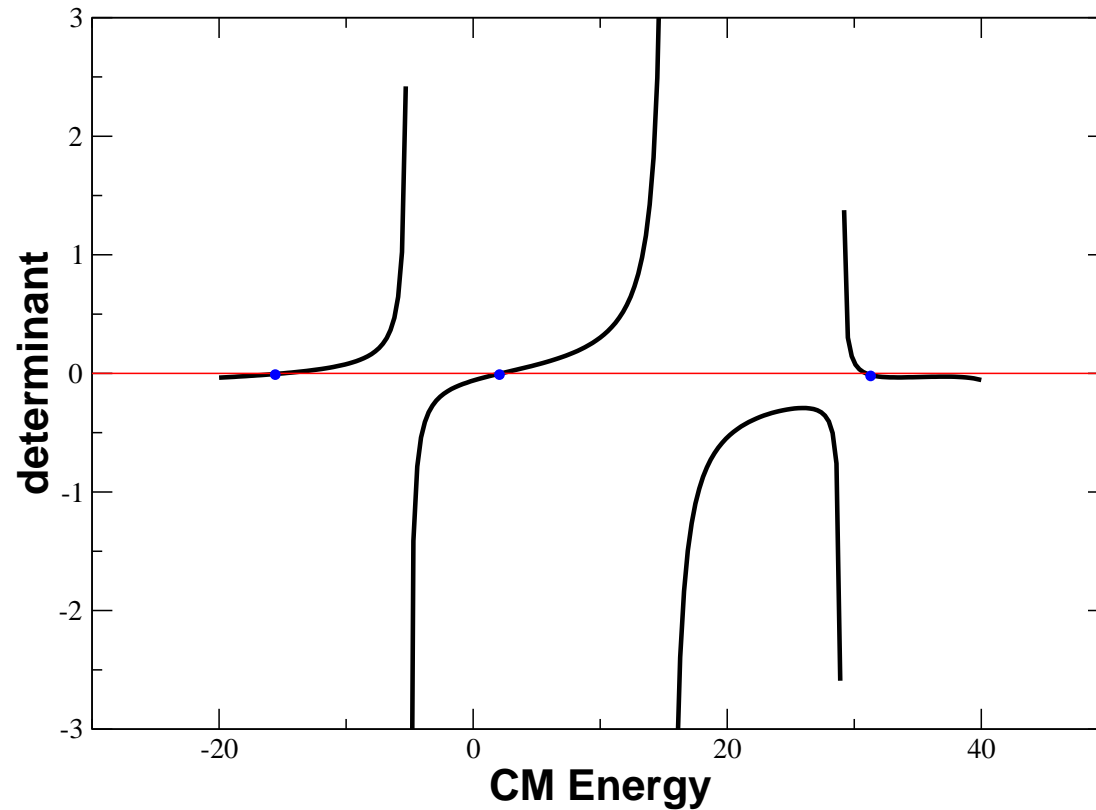
- It can be proven that the full 3-particle T -matrix obeys

$$T^{\text{on-shell}} = T^{\text{on-shell}} \Big|_{D_4''=0}$$

- The no-scale arguments apply in a finite volume as well!

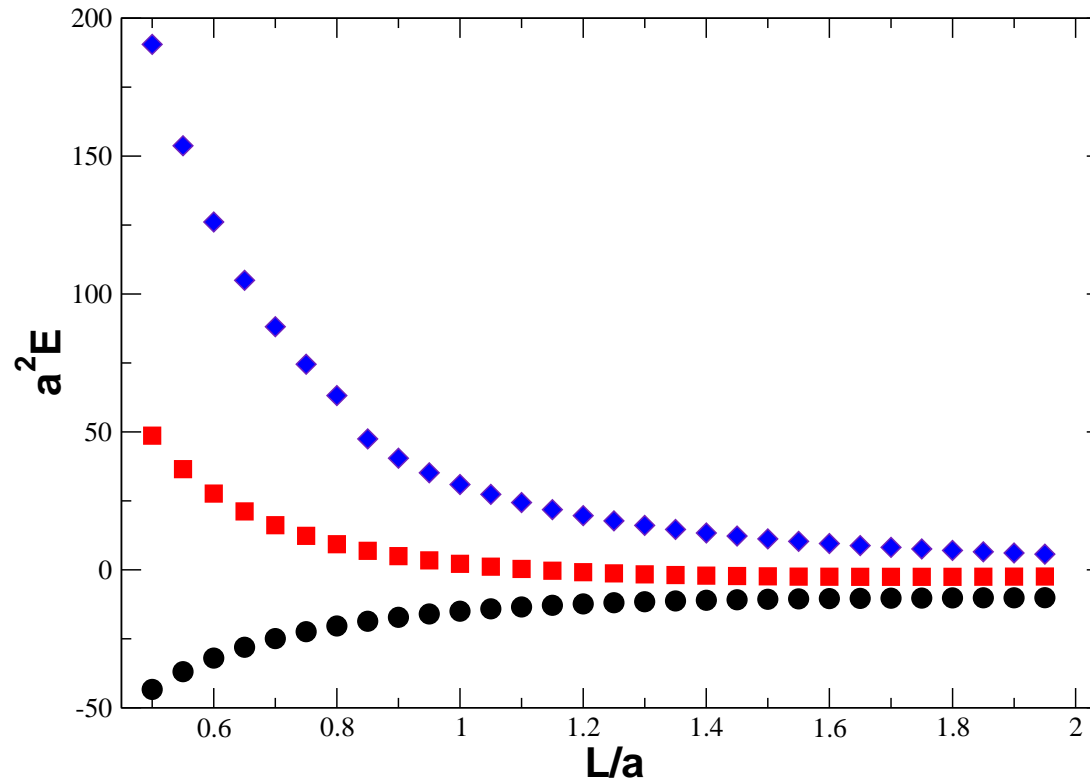
Finite volume energy levels determined solely by the S -matrix

Application 1: solving the equation numerically



- The determinant for a given box size L
- No derivative couplings

The finite-volume spectrum (*preliminary*)



- The spectrum both below and above the three-particle threshold is given

Application 2: three-particle bound state

Unitary limit $a \rightarrow \infty$: U.-G. Meißner, G. Rios and AR, PRL 114 (2015) 091602

See also M. T. Hansen and S. R. Sharpe, PRD 95 (2017) 034501

Using Poisson's formula. . .

$$\mathcal{M}_L(\mathbf{p}, \mathbf{q}; E) = Z(\mathbf{p}, \mathbf{q}; E) + 8\pi \int_{\mathbf{k}}^{\Lambda} Z(\mathbf{p}, \mathbf{q}; E) \hat{\tau}_L(\mathbf{k}; E) \mathcal{M}_L(\mathbf{k}, \mathbf{q}; E)$$

$$\hat{\tau}_L(\mathbf{k}; E) = \frac{1 + \sum_{\mathbf{n} \neq 0} e^{iL\mathbf{n}\mathbf{k}}}{\tau^{-1}(\mathbf{k}; E) + \underbrace{\Delta_L(\mathbf{k}; E)}_{\text{zeta-function}}} = \tau(\mathbf{k}; E) + \sum_{\mathbf{n} \neq 0} e^{iL\mathbf{n}\mathbf{k}} \tau(\mathbf{k}; E) + \dots$$

$$\hookrightarrow \Delta E = 8\pi \int_{\mathbf{k}}^{\Lambda} [\Psi(\mathbf{k})]^2 \sum_{\mathbf{n} \neq 0} e^{iL\mathbf{n}\mathbf{k}} \tau(\mathbf{k}; E) + \dots$$

Normalization condition

$$-8\pi \int_{\mathbf{p}}^{\Lambda} [\Psi(\mathbf{p})]^2 \frac{\partial \tau(\mathbf{p}; E)}{\partial E} - (8\pi)^2 \int_{\mathbf{p}}^{\Lambda} \int_{\mathbf{q}}^{\Lambda} \Psi(\mathbf{p}) \tau(\mathbf{p}; E) \frac{\partial Z(\mathbf{p}, \mathbf{q}; E)}{\partial E} \tau(\mathbf{q}; E) \Psi(\mathbf{q}) = 1$$

Faddeev-Minlos solution: $\Lambda \rightarrow \infty$ and $H(\Lambda) = 0$.

$$\Psi_0(p) = iN_0 \frac{\kappa}{p} \sin(s_0 u), \quad u = \ln \left(\frac{\sqrt{3}}{2} \frac{p}{\kappa} + \sqrt{\frac{3p^2}{4\kappa^2} + 1} \right), \quad E = \frac{\kappa^2}{m}$$

Asymptotic normalization coefficient

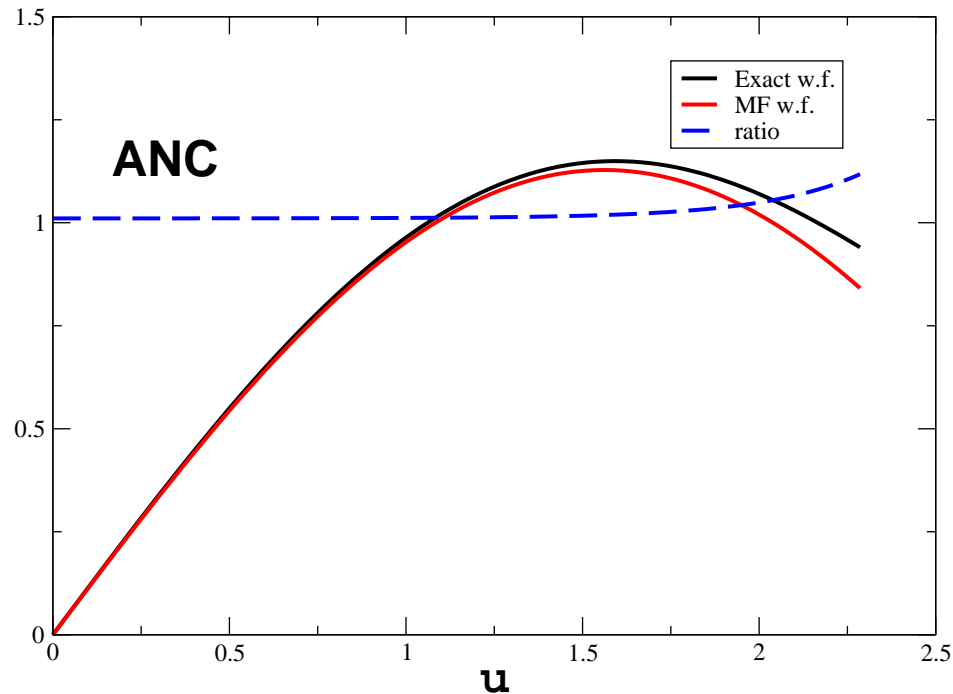
$$\mathcal{A} = \lim_{p \rightarrow 0} \Psi(p) / \Psi_0(p)$$

No derivative couplings: $\mathcal{A} = 1 + O(\kappa/\Lambda)$

Derivative couplings: $\mathcal{A} \neq 1$

Energy shift in the unitary limit

U.-G. Meißner, G. Rios and AR, PRL 114 (2015) 091602



$$\frac{\Delta E}{|E|} = c(\kappa L)^{-3/2} \mathcal{A}^2 \exp\left(-\frac{2\kappa L}{\sqrt{3}}\right)$$

$\mathcal{A} = 1 + O(\kappa/\Lambda)$ in the absence of derivative couplings

Going beyond unitary limit

$$\Delta E \propto \int_{\mathbf{p}}^{\Lambda} \frac{[\Psi(\mathbf{p})]^2 e^{iL\mathbf{n}\mathbf{p}}}{-a^{-1} + \sqrt{\frac{3}{4}\mathbf{p}^2 + \kappa^2}}, \quad |\mathbf{n}| = 1$$

$\Psi(\mathbf{p})$ is only weakly singular in the low-momentum region \rightarrow const.

$$\Delta E = \frac{\#}{aL} \exp\left(-\frac{2}{\sqrt{3}} \sqrt{\kappa^2 - \frac{1}{a^2}L}\right) + \frac{\#}{(\kappa L)^{3/2}} \exp\left(-\frac{2\kappa L}{\sqrt{3}}\right) + \dots$$

1. Lüscher equation, bound state of a particle and a dimer
2. Three-particle bound state in the unitary limit

Conclusions

- An EFT formalism in a finite volume is proposed to analyze the data in the three-particle sector
- The low-energy couplings H_0, H_2, \dots are fitted to the spectrum; S -matrix is obtained through the solution of equations
- A systematic approach: allows the inclusion of higher partial waves, derivative couplings, two \rightarrow three transitions, relativistic kinematics, . . .
- Equivalent to other known approaches, much easier to use!
- Other EFT's in a finite volume could be used (e.g., the chiral EFT for nucleons)

- Three-particle Lellouch-Lüscher formula
- Three-nucleon interactions: inclusion of the long-range forces
- Inclusion of relativistic effects, higher partial waves, spin, partial wave mixing, etc
- Full group-theoretical analysis of the three-particle equation in the rectangular box
- Using three-body quantization condition for the analysis of lattice data

The toy model with ϕ^4 interaction (ongoing)

Study of the three-pion interactions in QCD (planned)