Higher-twist B-meson Distribution Amplitudes in HQET

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based on: VB, Yao Ji, A. Manashov, JHEP 1705 (2017) 022

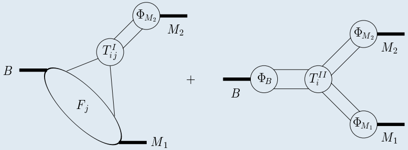
LmC, Siegen, 19.09.2017

Exclusive B–Decays

- Heavy quark expansion methods ($m_b \gg \Lambda_{\rm QCD}$)
- Soft-collinear factorization (final state particle energies $\gg \Lambda_{\rm QCD}$)

BBNS approach:

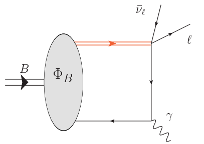
$$\langle M_1 M_2|O_i|B\rangle = F^{B\to M_1}(0)\int_0^1 du\; T^{(1)}(u)\Phi_{M_2}(u)$$
$$+\int_0^\infty d\omega\int_0^1 du\,dv\; T^{(2)}(\omega,u,v)\Phi_+(\omega)\Phi_{M_1}(u)\Phi_{M_2}(v)$$



u, v — *momentum fractions*
ω — *light quark energy in B-meson*
$\Phi_{M,B}$ — *distribution amplitudes*

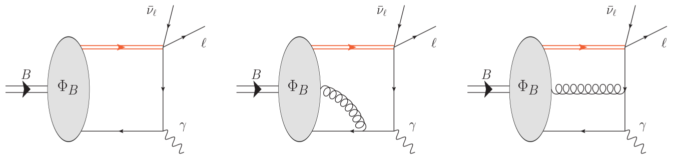
? power-suppressed contributions

- Simplest exclusive B decay: $B \to \ell \nu \gamma$



$$= \int_0^\infty \frac{d\omega}{\omega} T(\omega, E_\gamma; \mu_F) \Phi_+(\omega, \mu_F)$$

- virtuality, transverse momentum, higher Fock states



— A multifaceted problem

- QCD factorization, soft contributions,. . .
- Nonperturbative input: Multiparton (higher-twist) Distribution Amplitudes

Leading-twist B–Meson Distribution Amplitude

Definition

Grozin, Neubert '97

$$\langle 0| \left[\bar{q}(zn) \not{n}[zn,0]\gamma_5 h_v(0)\right]_R |\bar{B}(v)\rangle = i f_B^{\rm stat}(\mu)\, \Phi_+(z,\mu)$$

- v_μ is the heavy quark velocity
- n_μ is the light-like vector, $n^2 = 0$, such that $n \cdot v = 1$
- $\Phi_+(z - i0, \mu)$ is analytic function of z in the lower half-plane

In momentum space

$$\Phi_+(\omega,\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dz\, e^{i\omega z}\Phi_+(z-i0,\mu)$$

- $\omega > 0$ is the $(2\times)$ light quark energy in the b–quark rest frame

Three-particle distribution amplitudes

- **Eight independent Lorentz structures** [Geyer:2005fb]

$$\langle 0|\bar q(nz_1) gG_{\mu\nu}(nz_2)\Gamma h_v(0)|\bar B(v)\rangle =$$
$$= \frac{1}{2}F_B(\mu)\,\mathrm{Tr}\Big\{\gamma_5\Gamma P_+\Big[(v_\mu\gamma_\nu - v_\nu\gamma_\mu)\big[\Psi_A-\Psi_V\big] - i\sigma_{\mu\nu}\Psi_V - (n_\mu v_\nu - n_\nu v_\mu)X_A$$
$$+ (n_\mu\gamma_\nu - n_\nu\gamma_\mu)\big[W + Y_A\big] - i\epsilon_{\mu\nu\alpha\beta}n^\alpha v^\beta\gamma_5\widetilde{X}_A + i\epsilon_{\mu\nu\alpha\beta}n^\alpha\gamma^\beta\gamma_5\widetilde{Y}_A$$
$$- (n_\mu v_\nu - n_\nu v_\mu)\not n\, W + (n_\mu\gamma_\nu - n_\nu\gamma_\mu)\not n\, Z\Big]\Big\}(z_1,z_2;\mu)$$

blue: [Kawamura:2001jm]
red: this work

$$\Psi_A(z_1,z_2) = \int_0^\infty d\omega_1\int_0^\infty d\omega_2\; e^{-i\omega_1 z_1 - i\omega_2 z_2}\,\psi_A(\omega_1,\omega_2)\,, \qquad \text{etc.}$$

Collinear twist decomposition (1)

- **Twist-three**

$$2F_B(\mu)\Phi_3(z_1,z_2;\mu) = \langle 0|\bar q(z_1) gG_{\mu\nu}(z_2) n^\nu \not{n}\gamma_\perp^\mu\gamma_5 h_v(0)|\bar B(v)\rangle$$

$$\Phi_3 = \Psi_A - \Psi_V\,,$$

- **Twist-four**

$$2F_B(\mu)\Phi_4(z_1,z_2;\mu) = \langle 0|\bar q(z_1) gG_{\mu\nu}(z_2) n^\nu \not{\bar n}\gamma_\perp^\mu\gamma_5 h_v(0)|\bar B(v)\rangle$$
$$2F_B(\mu)\Psi_4(z_1,z_2;\mu) = \langle 0|\bar q(z_1) gG_{\mu\nu}(z_2) \bar n^\mu n^\nu \not{n}\gamma_5 h_v(0)|\bar B(v)\rangle$$
$$2F_B(\mu)\widetilde\Psi_4(z_1,z_2;\mu) = \langle 0|\bar q(z_1) ig\widetilde G_{\mu\nu}(z_2) \bar n^\mu n^\nu \not{n} h_v(0)|\bar B(v)\rangle$$

$$\Phi_4 = \Psi_A + \Psi_V\,,$$
$$\Psi_4 = \Psi_A + X_A\,,$$
$$\widetilde\Psi_4 = \Psi_V - \widetilde X_A$$

Collinear twist decomposition (2)

- **Twist-five**

$$2F_B(\mu)\widetilde{\Phi}_5(z_1,z_2;\mu) = \langle 0|\bar q(z_1) g G_{\mu\nu}(z_2)\bar n^\nu \not{n}\gamma_\perp^\mu\gamma_5 h_v(0)|\bar B(v)\rangle$$
$$2F_B(\mu)\Psi_5(z_1,z_2;\mu) = \langle 0|\bar q(z_1) g G_{\mu\nu}(z_2)\bar n^\mu n^\nu \not{\bar n}\gamma_5 h_v(0)|\bar B(v)\rangle$$
$$2F_B(\mu)\widetilde{\Psi}_5(z_1,z_2;\mu) = \langle 0|\bar q(z_1) ig \widetilde{G}_{\mu\nu}(z_2)\bar n^\mu n^\nu \not{\bar n} h_v(0)|\bar B(v)\rangle$$

$$\widetilde{\Phi}_5 = \Psi_A + \Psi_V + 2Y_A - 2\widetilde{Y}_A + 2W\,,$$
$$\Psi_5 = -\Psi_A + X_A - 2Y_A\,,$$
$$\widetilde{\Psi}_5 = -\Psi_V - \widetilde{X}_A + 2\widetilde{Y}_A\,,$$

- **Twist-six**

$$2F_B(\mu)\widetilde{\Phi}_6(z_1,z_2;\mu) = \langle 0|\bar q(nz_1) g G_{\mu\nu}(nz_2)\bar n^\nu \not{\bar n}\gamma_\perp^\mu\gamma_5 h_v(0)|\bar B(v)\rangle$$

$$\Phi_6 = \Psi_A - \Psi_V + 2Y_A + 2W + 2\widetilde{Y}_A - 4Z$$

Conformal spin and helicity assignment

	Φ_3	Φ_4	$\Psi_4 + \widetilde{\Psi}_4$	$\Psi_4 - \widetilde{\Psi}_4$	Φ_5	$\Psi_5 + \widetilde{\Psi}_5$	$\Psi_5 - \widetilde{\Psi}_5$	Φ_6
twist	3	4	4	4	5	5	5	6
j_q	1	1/2	1	1	1	1/2	1/2	1/2
j_g	3/2	3/2	1	1	1/2	1	1	1/2
chirality	$\uparrow\downarrow(\downarrow\uparrow)$	$\upuparrows(\downdownarrows)$	$\upuparrows(\downdownarrows)$	$\uparrow\downarrow(\downarrow\uparrow)$	$\upuparrows(\downdownarrows)$	$\upuparrows(\downdownarrows)$	$\uparrow\downarrow(\downarrow\uparrow)$	$\uparrow\downarrow(\downarrow\uparrow)$

Table: *The twist, conformal spins j_q, j_g of the constituent fields and chirality [same or opposite] of the three-particle B-meson DAs.*

- **Asymptotic behavior at small momenta** [Braun:1989iv]

$$f(\omega_1, \omega_2) \sim \omega_1^{2j_1 - 1} \omega_2^{2j_2 - 1} . \qquad f \in \{\phi_3, \phi_4, \psi_4, \tilde{\psi}_4 \ \ldots\}$$

$$\phi_3(\omega_1, \omega_2) \sim \omega_1 \omega_2^2 , \qquad \phi_4(\omega_1, \omega_2) \sim \omega_2^2 , \qquad \psi_4(\omega_1, \omega_2) \sim \widetilde{\psi}_4(\omega_1, \omega_2) \sim \omega_1 \omega_2$$

— agrees with [Khodjamirian:2006st]

useful relations for twist-four...

- **Neglecting four-particle contributions** $qGGh_v$, $q\bar{q}qh_v$

$$2\partial_1 z_1 \Phi_4(\underline{z}) = \Big(z_2\partial_{z_2} + 2\Big)\Big[\Psi_4(\underline{z}) + \widetilde{\Psi}_4(\underline{z})\Big] \qquad \boxed{\underline{z} = \{z_1, z_2\}}$$

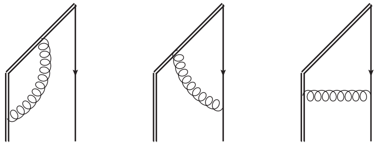
$$z_1 \langle 0|\bar{q}(z_1)\overleftarrow{D}_\alpha\, gG^{\alpha\beta}(z_2) n_\beta \not{n}\gamma_5 h_v(0)|\bar{B}(v)\rangle = F_B\Big[2 + z_1\partial_1 + z_2\partial_2\Big]\Psi_4(\underline{z}) - F_B\Big[\Phi_3 + \Phi_4\Big](\underline{z})$$

$$z_1 \langle 0|\bar{q}(z_1)\overleftarrow{D}_\alpha\, ig\widetilde{G}^{\alpha\beta}(z_2) n_\beta \not{n} h_v(0)|\bar{B}(v)\rangle = F_B\Big[2 + z_1\partial_1 + z_2\partial_2\Big]\widetilde{\Psi}_4(\underline{z}) + F_B\Big[\Phi_3 - \Phi_4\Big](\underline{z})$$

Lange-Neubert evolution equation

light quark with attached cusped Wilson line *Lange, Neubert '03*

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \frac{\alpha_s C_F}{\pi} H_{LN}\right)\Phi_+(\omega,\mu) = 0$$



$$[H_{LN}f](\omega) \quad = \quad -\int_0^\infty d\omega' \left[\frac{\omega}{\omega'}\frac{\theta(\omega'-\omega)}{\omega'-\omega} + \frac{\theta(\omega-\omega')}{\omega-\omega'}\right]_+ + \left[\ln\frac{\mu}{\omega} - \frac{5}{4}\right] f(\omega)$$

- Explicit solution found only recently *G. Bell et. al., '13*

Collinear conformal transformations

- one-loop evolution equations for light quarks (and gluons) are $SL(2)$ invariant

$$S_+ = z^2\partial_z + 2jz \qquad S_0 = z\partial_z + j \qquad S_- = -\partial_z \qquad j = 1 \text{ for quarks}$$

$\Longrightarrow$ evolution kernels can be written in terms of the two-particle quadratic Casimir operator

$$\mathcal{H}_{qq} = \frac{1}{2}\Big[\psi(\widehat{J}+1) + \psi(\widehat{J}-1) - 2\gamma_E - \frac{3}{2}\Big], \qquad S_{12}^2 = \widehat{J}(\widehat{J}-1)$$

- for heavy quark, rescale the coordinates

$$S_+^{(h)} \mapsto \lambda^{-1} S_+^{(h)}, \qquad S_-^{(h)} \mapsto \lambda S_-^{(h)}, \qquad S_0^{(h)} \mapsto S_0^{(h)}, \qquad \boxed{\lambda \sim m_b \to \infty}$$

- for heavy-light operators

$$S_+^{(lh)} = S_+^{(l)} + \lambda^{-1} S_+^{(h)} \mapsto S_+^{(l)} \qquad S_-^{(lh)} = S_-^{(l)} + \lambda S_-^{(h)} \mapsto S_h^{(l)} \mapsto \mu$$

$$S_{lh}^2 = S_+^{(lh)} S_-^{(lh)} + S_0^{(lh)}(S_0^{(lh)} - 1) \mapsto S_+^{(l)} S_-^{(h)} \mapsto \mu S_+^{(l)}$$

- $\Longrightarrow$ Lange-Neubert kernel

$$\mathcal{H}_{lh} = \mathcal{H}_{LN} = \ln(i\mu S_+^{(l)}) + \text{const}$$

Conformal symmetry of the LN equation

$$\mathcal{H}_{LN} = \ln(i\mu\, S^+) - \psi(1) - \frac{5}{4}$$ [Braun:2014owa]

$\mathcal{H}_{LN}$ and S_+ share the same eigenfunctions

$$iS_+\, Q_s(z) = s\, Q_s(z)$$

in position space

$$S_+ \;=\; z^2\partial_z + 2z \qquad \mapsto \qquad Q_s(z) = -\frac{1}{z^2}\exp\left\{\frac{is}{z}\right\}$$

in momentum space

$$\mathcal{S}_+ = i\Big[\omega\partial_\omega^2 + 2\partial_\omega\Big] \qquad \mapsto \qquad Q_s(\omega) = \sqrt{\omega s}\, J_1(2\sqrt{\omega s})$$

$$\phi_+(\omega,\mu) = \int_0^\infty ds\; Q_s(\omega)\; R(s;\mu,\mu_0)\eta_+(s,\mu_0) \qquad s_0 = e^{5/4-\gamma_E}$$

$$R(s;\mu,\mu_0) = L^{3C_F/(2\beta_0)}\exp\left[-\int_{\mu_0}^{\mu}\frac{d\tau}{\tau}\,\Gamma_{cusp}(\alpha_s(\tau))\,\ln(\tau s/s_0)\right], \qquad L = \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}$$

—in agreement with [Bell:2013tfa]

Complete integrability

- One light degree of freedom:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \widehat{H}\right)\phi(\omega,\mu) = 0 \qquad \widehat{H} = h(\widehat{Q})\,, \quad [\widehat{H},\widehat{Q}] = 0$$

☺

- Two light degrees of freedom, $\bar{q}G\bar{b}$ or qqb:

? $$\widehat{H} = h(\widehat{Q}_1,\widehat{Q}_2)\,, \qquad [\widehat{H},\widehat{Q}_1] = [\widehat{H},\widehat{Q}_2] = [\widehat{Q}_1,\widehat{Q}_2] = 0$$

☺☺

?? Extra conserved charge $\mapsto$ Hidden symmetry

?? How to construct explicit expressions for Q_1, Q_2?

?? How to construct complete system of eigenfunctions for Q_1, Q_2?

- Quantum Inverse Scattering Method

VB, Derkachov, Manashov, PLB 738(2014)334
VB, Manashov, Offen, PRD92 (2015) 074044
VB, Yao Ji, Manashov, JHEP1705(2017)022

Twist-three

[Braun:2015pha]

$$\phi_-(\omega,\mu) = \int_\omega^\infty \frac{d\omega'}{\omega'}\phi_+(\omega',\mu) + \int_0^\infty ds\, J_0(2\sqrt{\omega s})\, \eta_3^{(0)}(s,\mu)$$

$$\phi_3(\underline{\omega},\mu) = \int_0^\infty ds\Big[\eta_3^{(0)}(s,\mu)\; Y_3^{(0)}(s\,|\,\underline{\omega}) + \frac{1}{2}\int_{-\infty}^\infty dx\, \eta_3(s,x,\mu)\; Y_3(s,x\,|\,\underline{\omega})\Big],$$

where up to $1/N_c^2$ corrections

$$\eta_3(s,x,\mu) = L^{\gamma_3(x)/\beta_0} R(s;\mu,\mu_0)\,\eta_3(s,x,\mu_0)$$

$$\eta_3^{(0)}(s,\mu) = L^{N_c/\beta_0} R(s;\mu,\mu_0)\eta_3^{(0)}(s,\mu_0)$$

$$Y_3(s,x\,|\,\underline{\omega}) = -\int_0^1 du\,\sqrt{us\omega_1}\; J_1(2\sqrt{us\omega_1})\,\omega_2\; J_2(2\sqrt{\bar{u}s\omega_2})\;{}_2F_1\left(\begin{matrix}-\frac{1}{2}-ix,\,-\frac{1}{2}+ix\\ 2\end{matrix}\,\Big|\,-\frac{u}{\bar{u}}\right)$$

Twist-four

[Braun:2017liq]

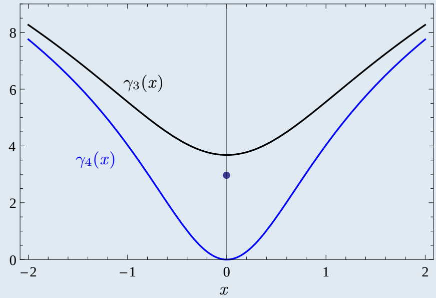
$$
\begin{aligned}
\Phi_4(\underline{\omega}) &= \frac{1}{2}\int_0^\infty ds \int_{-\infty}^\infty dx\, \eta_4^{(+)}(s,x,\mu)\; Y_{4;1}^{(+)}(s,x\,|\underline{\omega})\,,\\
(\Psi_4+\widetilde{\Psi}_4)(\underline{\omega}) &= -\int_0^\infty ds \int_{-\infty}^\infty dx\, \eta_4^{(+)}(s,x,\mu)\; Y_{4;2}^{(+)}(s,x\,|\underline{\omega})\,,\\
(\Psi_4-\widetilde{\Psi}_4)(\underline{\omega}) &= 2\int_0^\infty \frac{ds}{s}\left(-\frac{\partial}{\partial\omega_2}\right)\left\{\eta_3^{(0)}(s,\mu)\; Y_3^{(0)}(s\,|\underline{\omega}) + \frac{1}{2}\int_{-\infty}^{\infty} dx\, \eta_3(s,x,\mu)\; Y_3(s,x\,|\underline{\omega})\right\}\\
&\quad -\int_0^\infty ds \int_{-\infty}^\infty dx\, \varkappa_4^{(-)}(s,x,\mu)\, \omega_{4;2}^{(-)}(s,x\,|\underline{\omega})\,,
\end{aligned}
$$

where up to $1/N_c^2$ corrections

$$
\begin{aligned}
\eta_4^{(+)}(s,x,\mu) &= L^{\gamma_4(x)/\beta_0}\, R(s;\mu,\mu_0)\, \eta_4^{(+)}(s,x,\mu_0)\\
\varkappa_4^{(-)}(s,x,\mu) &= L^{\gamma_4(x)/\beta_0}\, R(s;\mu,\mu_0)\, \varkappa_4^{(-)}(s,x,\mu_0)
\end{aligned}
$$

Anomalous dimensions

[Braun:2017liq]



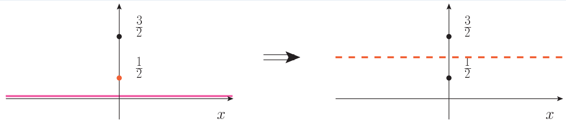
$$\gamma_3(x) = N_c\Big[\psi\Big(3/2+ix\Big)+\psi\Big(3/2-ix\Big)+2\gamma_E\Big]$$
$$\gamma_3^{(0)} = \gamma_3(x=i/2) = N_c\,.$$
$$\gamma_4(x) = N_c\Big[\psi\Big(ix\Big)+\psi\Big(-ix\Big)+2\gamma_E\Big]$$

Large energy limit

- Hierarchy of contributions with rising anomalous dimensions breaks down at $\omega > \mu$

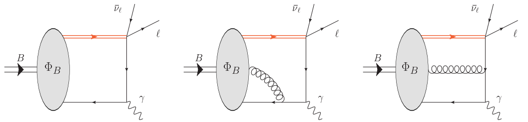
$$\Phi_3(\omega_1,\omega_2,\mu) = \int_0^\infty ds\Big[\eta_0(s,\mu)\,\widetilde{Y}_s^{(0)}(\omega_1,\omega_2) + \frac{1}{2}\int_{-\infty}^\infty dx\,\eta(s,x,\mu)\,\widetilde{Y}_{s,x}(\omega_1,\omega_2)\Big]$$

One finds $\widetilde{Y}_s^{(0)}(\omega_1,\omega_2) \sim$ const, $\widetilde{Y}_{s,x}(\omega_1,\omega_2) \sim w_2^{1/2\pm ix}$ for $\omega_2 \to \infty$, but ...



... $\Phi_3(\omega_1,\omega_2,\mu) \sim 1/\omega_2$ in the sum (integral) over all terms, unless the initial condition at $\mu = \mu_0$ is more singilar

Off-light-cone contributions



contribution of the second diagram encoded in

$$\langle 0|\bar q(x)\Gamma[x,0]h_v(0)|\bar B(v)\rangle = -\frac{i}{2}F_B\,\mathrm{Tr}\Big[\gamma_5\Gamma P_+\Big]\int_0^\infty d\omega\; e^{-i\omega(vx)}\Big\{\phi_+(\omega)+x^2 g_+(\omega)\Big\}$$
$$+\frac{i}{4}F_B\,\mathrm{Tr}\Big[\gamma_5\Gamma P_+\not{x}\Big]\frac{1}{vx}\int_0^\infty d\omega\; e^{-i\omega(vx)}\Big\{[\phi_+-\phi_-](\omega)+x^2[g_+-g_-](\omega)\Big\}$$

$$\mathrm{G}_\pm(z,\mu) = \int_0^\infty d\omega\; e^{-i\omega z} g_\pm(\omega,\mu)$$

Equations of motion (EOM)

[Kawamura:2001jm]

$$\frac{\partial}{\partial x^\mu}\bar q(x)\gamma^\mu\Gamma[x,0]h_v(0) = -i\int_0^1 udu\,\bar q(x)[x,ux]x^\rho gG_{\rho\mu}(ux)[ux,0]\gamma^\mu\Gamma h_v(0)\,,$$

$$v^\mu\frac{\partial}{\partial x^\mu}\bar q(x)\Gamma[x,0]h_v(0) = i\int_0^1 \bar u du\,\bar q(x)[x,ux]x^\rho gG_{\rho\mu}(ux)[ux,0]v^\mu\Gamma h_v(0) + (v\cdot\partial)\bar q(x)\Gamma[x,0]h_v(0)$$

leading to

$$(1)\quad \Big[z\frac{d}{dz}+1\Big]\Phi_-(z) = \Phi_+(z) + 2z^2\int_0^1 udu\,\Phi_3(z,uz)$$

$$(2)\quad 2z^2\mathrm{G}_+(z) = -\Big[z\frac{d}{dz}-\frac{1}{2}+iz\bar\Lambda\Big]\Phi_+(z) - \frac{1}{2}\Phi_-(z) - z^2\int_0^1 \bar udu\,\Psi_4(z,uz)$$

$$(3)\quad 2z^2\mathrm{G}_-(z) = -\Big[z\frac{d}{dz}-\frac{1}{2}+iz\bar\Lambda\Big]\Phi_-(z) - \frac{1}{2}\Phi_+(z) - z^2\int_0^1 \bar udu\,\Psi_5(z,uz)\,,$$

$$(4)\quad \Phi_-(z) = \Big(z\frac{d}{dz}+1+2iz\bar\Lambda\Big)\Phi_+(z) + 2z^2\int_0^1 du\,\Big[u\Phi_4(z,uz)+\Psi_4(z,uz)\Big]$$

[Braun:2017liq]

Equations of motion (EOM) —*continued*

(1) — a relation between twist-three DAs, well known

(2) — expression for G_+ in terms of WW and twist-for DAs

(3) — expression for G_- in terms of WW and twist-five DAs

(4) — a relation between different DAs, nontrivial

[Braun:2017liq]

$$\boxed{\Big[1-(\partial_s s)^2 - 2s\bar{\Lambda}\Big]\,\eta_+(s,\mu) = \pi\sqrt{s}\varkappa_4^{(-)}(s,0,\mu) - \pi\sqrt{s}\eta_4^{(+)}(s,0,\mu)}$$

— a nonlocal generalization of the Grozin-Neubert identities

$$\int_0^\infty d\omega\,\omega\,\phi_+(\omega) = \frac{4}{3}\bar{\Lambda}\,, \qquad \int_0^\infty d\omega\,\omega^2\phi_+(\omega) = 2\bar{\Lambda}^2 + \frac{2}{3}\lambda_E^2 + \frac{1}{3}\lambda_H^2$$

— has to be amended beyond tree level

Models

require

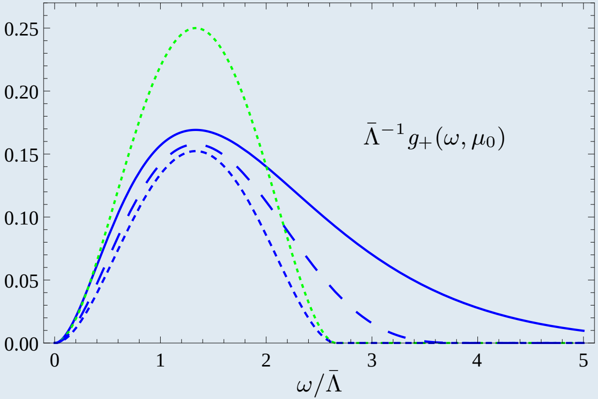
— correct low-momentum behavior
— satisfy EOM (at tree level)
— $\bar{\Lambda}$ from inclusive decays (?), ratio λ_E^2/λ_H^2 from QCD sum rules

possible general ansatz

$$\begin{aligned}
\phi_+(\omega) &= \omega\, f(\omega) \\
\phi_3(\omega_1,\omega_2) &= -\frac{1}{2}\varkappa(\lambda_E^2-\lambda_H^2)\,\omega_1\omega_2^2\,\partial_{\omega_2} f(\omega_1+\omega_2) \\
\psi_4(\omega_1,\omega_2) &= \varkappa\lambda_E^2\,\omega_1\omega_2\, f(\omega_1+\omega_2) \\
\widetilde{\psi}_4(\omega_1,\omega_2) &= \varkappa\lambda_H^2\,\omega_1\omega_2\, f(\omega_1+\omega_2) \\
2\omega_1\,\phi_4(\omega_1,\omega_2) &= \omega_2\left[\psi_4(\omega_1,\omega_2)+\widetilde{\psi}_4(\omega_1,\omega_2)\right]
\end{aligned}$$

where $\varkappa =$ const fixed by (EOM) Grozin-Neubert relations

two-particle twist-four DA



- at small momenta

$$g_+(\omega) \sim \omega^2$$

Summary

- Complete classification of three-particle DAs

  one twist-three DA, two independent twist-4 DAs

- Scale dependence to $\mathcal{O}(1/N_c^2)$

  Complete integrability; anomalous dimensions form a continuum spectrum; $\phi_-(\omega)$ decouples from quark-gluon operators; hierarchy of terms with rising anomalous dimensions breaks down for large energies

- Two-particle higher-twist DAs and EOM at tree level

  further work needed, depends on the factorization scheme

- Models consistent with EOM

  $g_+(\omega)$

- Applications

  work in progress