The $\pi^0 \to \gamma^*\gamma^*$ transition form factor and the hadronic light-by-light contribution to the $(g-2)_\mu$

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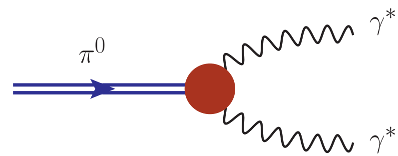
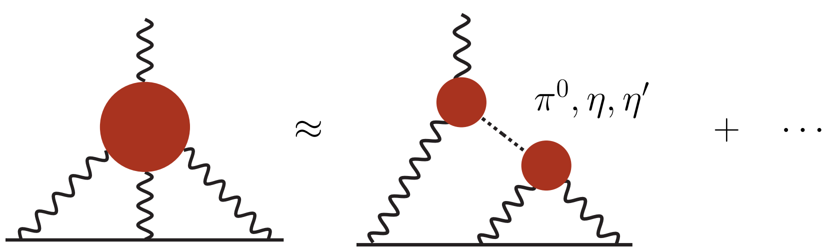
Motivations

The $\pi^0 \to \gamma^* \gamma^*$ transition form factor describe the interaction between a neutral pion and two off-shell photons



- The pion transition form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q_1^2, Q_2^2)$ yields important insights into the dynamics of QCD
  - $\to \mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0)$: Chiral anomaly
  - $\to \mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q^2,0)$: Brodsky-Lepage behaviour, pion distribution amplitude
  - $\to \mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q^2,Q^2)$: Test the operator product expansion (OPE) in the doubly-virtual case

- Hadronic light-by-light contribution to the anomalous magnetic moment of the muon : $(g-2)_\mu$
  - $\to$ pion-pole contribution (dominant contribution)

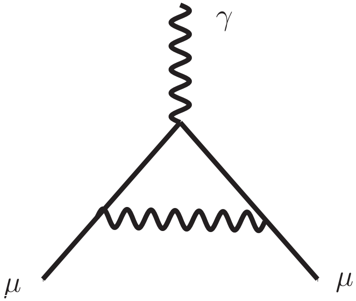


Magnetic moment $\vec{\mu}$:

$$\vec{\mu} = g \left(\frac{Qe}{2m} \right) \vec{s}$$

Anomalous magnetic moment of the muon : $(g-2)_\mu$

- Classical result : $g = 2$ (g-factor) - Quantum field theory : $a_\mu = \frac{g-2}{2} \neq 0$



$$a_\mu^{(1)} = \frac{\alpha}{2\pi}$$

[Schwinger '48]

$(g-2)_\mu$: present status

- Experimental value : $a_\mu = (116\ 592\ 089 \pm 63) \times 10^{-11}$ [Bennett et al. '06]
- Theory :

Contribution	$a_\mu \times 10^{11}$	
- QED (leptons, 5^{th} order)	$116\ 584\ 718.846 \pm 0.037$	[Aoyama et al. '12]
- Electroweak	153.6 ± 1.0	[Gnendiger et al. '13]
- Strong contributions		
HVP (LO)	$6\ 869.9 \pm 42.1$	[Jegerlehner '15]
HVP (NLO)	-98 ± 1	[Hagiwara et al. 11]
HVP (NNLO)	12.4 ± 0.1	[Kurtz et al. '14]
HLbL	102 ± 39	[Jegerlehner '15, Nyffeler '09]
Total (theory)	$116\ 591\ 811 \pm 62$	

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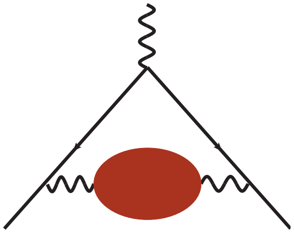
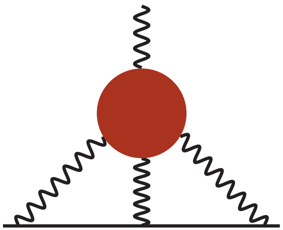
Hadronic Vacuum Polarisation (HVP, α^2) :



Hadronic Light-by-Light (HLbL, α^3) :



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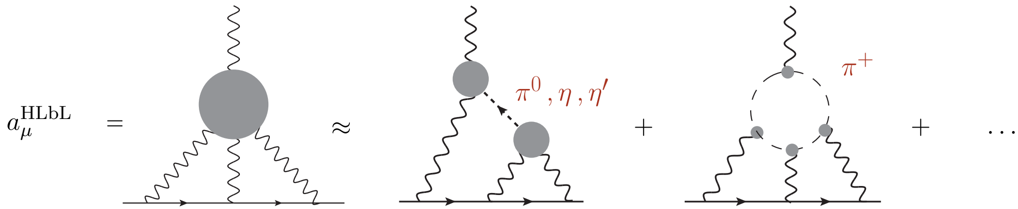
- $\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{th}} = 278 \times 10^{11}$ $\rightarrow$ $\sim 3-4\ \sigma$ discrepancy between experiment and theory
- Future experiment at Fermilab and J-PARC : reduction of the error by a factor of 4 $\rightarrow$ $\delta a_\mu = 16 \times 10^{-11}$
- Theory error is dominated by hadronic contributions
- HVP : more precise measurement of $\sigma(e^+e^- \rightarrow \text{hadrons})$ + Lattice QCD calculations
- HLbL : aim at $\sim 15\ \%$ precision

Hadronic Light-by-Light scattering (HLbL) : model calculations



[Table extracted from A. Nyffeler's slide]

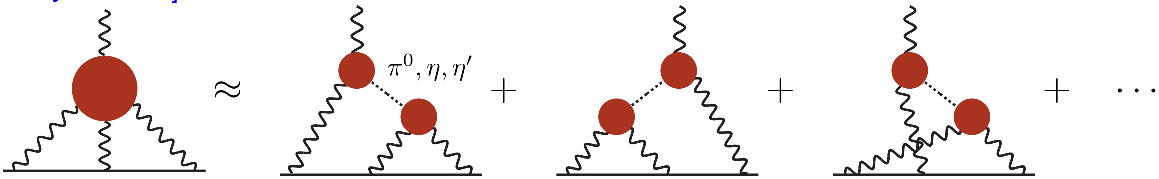
Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99 ± 16
axial vectors	2.5±1.0	1.7±1.7	–	22±5	–	15 ± 10	22 ± 5
scalars	−6.8±2.0	–	–	–	–	−7±7	−7±2
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
π, K loops +subl. N_C	–	–	–	0±10	–	–	–
quark loops	21±3	9.7±11.1	–	–	–	2.3 (c-quark)	21±3
Total	83±32	89.6±15.4	80±40	136±25	110±40	105 ± 26	116 ± 39

BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = AN '09, JN = Jegerlehner, AN '09

- Pseudoscalar-exchanges dominate numerically (but other contributions are not negligible)
- Error are hard to estimate (model calculations)
- Need pseudoscalar transition form factors as input parameters
- In the following, I will focus on the the pion-pole contribution (dominant)

The pion-pole contribution

[Jegerlehner & Nyffeler '09]



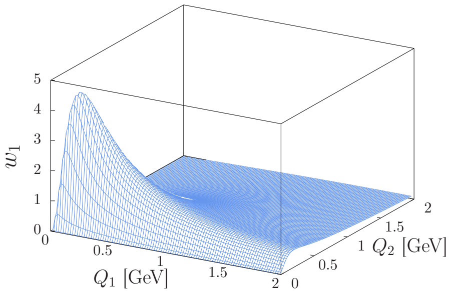
$$a_\mu^{\mathrm{HLbL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \; w_1(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0) + w_2(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$

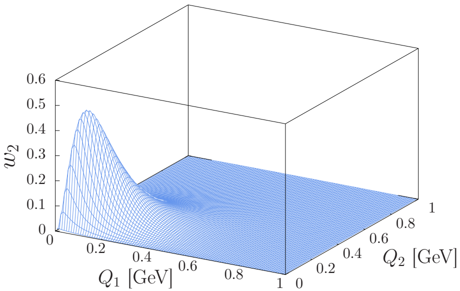
→ $\tau = \cos(\theta)\,,\, Q_1 \cdot Q_2 = Q_1 Q_2 \cos(\theta)$

→ Product of one single-virtual and one double-virtual transition form factors

→ $w_{1,2}(Q_1,Q_2,\tau)$ are known model-independent weight functions

→ Weight functions are concentrated at small momenta below 1 GeV (here for $\tau = -0.5$)





→ The non-perturbative information is encoded into the pseudoscalar TFF

Theoretical constraints on the transition form factor

- **Adler-Bell-Jackiw (ABJ) anomaly :**

↪ Low virtualities ($Q_1^2 \to 0,\ Q_2^2 \to 0$)

↪ Chiral limit

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0) = \frac{1}{4\pi^2 F_\pi}$$

- **Brodsky-Lepage behavior :**

↪ Single-virtual form factor

↪ Off-shell photon : $Q^2 \to \infty$

$$\mathcal{F}_{\pi^0\gamma^*\gamma}(-Q^2,0) \xrightarrow[Q^2\to\infty]{} \frac{2F_\pi}{Q^2}$$

  - The pre-factor depends on the shape of the pion distribution amplitude
  - $\mathcal{O}(\alpha_s)$ corrections are known

- **OPE prediction :** [Nesterenko + Radyushkin '83 ; Novikov et al. '84]

↪ Double-virtual form factor

↪ Large virtualities : $Q^2 \to \infty$

$$\mathcal{F}_{\pi^0\gamma^*\gamma}(-Q^2,-Q^2) \xrightarrow[Q^2\to\infty]{} \frac{2F_\pi}{3Q^2}$$

  - Higher-twist matrix element in the OPE are known : $\frac{2F_\pi}{3Q^2}\left[1-\frac{8}{9}\frac{\delta^2}{Q^2}+\mathcal{O}\left(\frac{1}{Q^4}\right)\right]$
  - $\mathcal{O}(\alpha_s)$ corrections are known

Experimental status

- Decay width : $\Gamma_{\pi^0\gamma\gamma} = 7.82(22)\ \mathrm{eV} \quad \sim 3\%$ [PrimEx '10]

$$\Gamma_{\pi^0\gamma\gamma} = \frac{\pi \alpha_e^2 m_\pi^3}{4} \mathcal{F}^2_{\pi^0\gamma^*\gamma^*}(0,0)$$

$\rightarrow$ Consistent with current theoretical predictions

$\rightarrow$ Experimental test of the chiral anomaly

$\rightarrow$ A further reduction of the error by a factor of two is expected soon

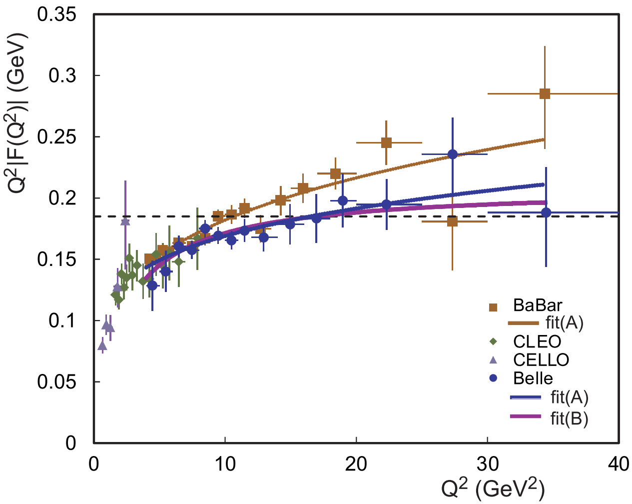
- The single-virtual form factor has been measured (CELLO, CLEO, BaBar, Belle)

[Belle '12]

$\rightarrow$ Belle data seem to confirm the Brodsky-Lepage behavior $\sim 1/Q^2$.

$\rightarrow$ Belle and Babar results are quite different

$\rightarrow$ No measurement at low $Q < 0.8\ \mathrm{GeV}$ (dominant contribution)



- No result yet for the double-virtual form factor

$\hookrightarrow$ measurement planned at BESIII

$\hookrightarrow$ challenging (small cross section)

Motivations

To estimate the pion pole contribution we need :

- The single and double virtual transition form factor for arbitrary space-like virtualities
- In the kinematical range $Q^2 \in [0-2]\ \mathrm{GeV}^2$ (non-perturbative regime of QCD)

Present status

- Experiments give information on the single-virtual form factor only
- Experimental data are available only for relatively large virtualities $Q^2 > 0.6\ \mathrm{GeV}^2$
- The theory imposes strong constraints for the normalisation and the asymptotic behavior of the TFF

$\hookrightarrow$ Most evaluations of the pion-pole contribution are therefore based on phenomenological models

$\hookrightarrow$ Systematic errors are difficult to estimate

$\hookrightarrow$ A model independent approach is highly desirable

Lattice QCD is particularly well suited to compute the form factor in the energy range relevant to $g-2$!

Lattice calculation

Lattice calculation

In Minkowski space-time :

$$\epsilon_{\mu\nu\alpha\beta}\, q_1^\alpha\, q_2^\beta\, \mathcal{F}_{\pi^0\gamma\gamma}(q_1^2, q_2^2) = i \int \mathrm{d}^4x\, e^{iq_1 x}\, \langle \Omega | T\{J_\mu(x) J_\nu(0)\} | \pi^0(p) \rangle = M_{\mu\nu}(q_1^2, q_2^2)$$

- $J_\mu(x)$ hadronic component of the electromagnetic current : $J_\mu(x) = \frac{2}{3}\overline{u}(x)\gamma_\mu u(x) - \frac{1}{3}\overline{d}(x)\gamma_\mu d(x) + \ldots$

In Euclidean space-time : [Ji & Jung '01] [Cohen et al. '08] [Feng et al. '12]

$$M_{\mu\nu}^E(q_1^2, q_2^2) = -\int \mathrm{d}\tau\, e^{\omega_1 \tau} \int \mathrm{d}^3z\, e^{-i\vec{q}_1 \vec{z}}\, \langle 0 | T\left\{ J_\mu(\vec{z}, \tau) J_\nu(\vec{0}, 0) \right\} | \pi^0(p) \rangle$$

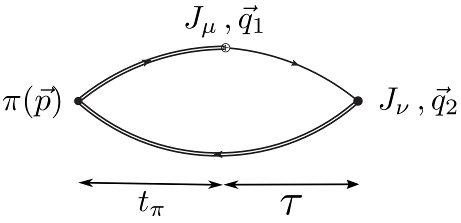
- Analytical continuation (« $\tau = -i\,t$ »)
- We must keep $q_{1,2}^2 < M_V^2 = \min(M_\rho^2, 4m_\pi^2)$ to avoid poles
- $q_1 = (\omega_1, \vec{q}_1)$

Lattice calculation

$$\epsilon_{\mu\nu\alpha\beta}\, q_1^\alpha\, q_2^\beta\, \mathcal{F}_{\pi^0\gamma\gamma}(q_1^2, q_2^2) = -\int \mathrm{d}\tau\, e^{\omega_1 \tau} \int \mathrm{d}^3 z\, e^{-i\vec{q}_1 \vec{z}}\, \langle 0 | T\left\{ J_\mu(\vec{z},\tau) J_\nu(\vec{0},0)\right\} |\pi^0(p)\rangle$$

- Pion at rest : $\vec{p} = \vec{0}$
- We consider the following 3-pt correlation function

$$C^{(3)}_{\mu\nu}(\tau, t_\pi; \vec{q}_1) = \sum_{\vec{x},\vec{z}} \langle T\left\{ J_\nu(\vec{0}, t_f) J_\mu(\vec{z}, t_i) P(\vec{x}, t_0)\right\} \rangle\, e^{-i\vec{q}_1 \vec{z}}$$

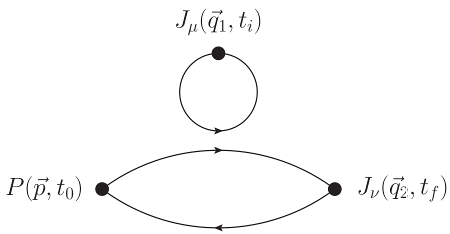


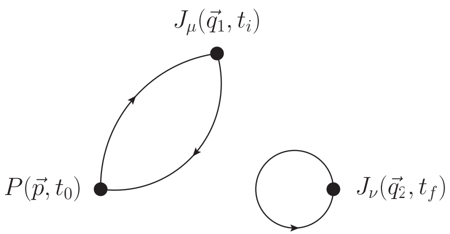
$$\mathcal{F}_{\pi^0\gamma\gamma}(q_1^2, q_2^2) \;\propto\; \frac{2E_\pi}{Z_\pi} \int_{-\infty}^{\infty} \mathrm{d}\tau\, \widetilde{A}_{\mu\nu}(\tau)\, e^{\omega_1 \tau}$$

$$A_{\mu\nu}(\tau) = \lim_{t_\pi \to \infty} C_{\mu\nu}(\tau, t_\pi) e^{E_\pi t_\pi}$$

$$\widetilde{A}_{\mu\nu}(\tau) = \begin{cases} A_{\mu\nu}(\tau) & \tau > 0 \\ A_{\mu\nu}(\tau)\, e^{-E_\pi \tau} & \tau < 0 \end{cases}$$

- There are also (quark) disconnected contributions : will be discussed later





Lattice QCD : sources of systematic error

- Lattice QCD is not a model : specific regularisation of the theory adapted to numerical simulations
- However there are systematic errors that we need to understand :

1) We used $N_f = 2$ simulations (Only u and d quarks are dynamical)
   → $\mathcal{O}(a)$-improved Wilson-Clover Fermions (CLS)

2) Finite lattice spacing : discretisation errors
   → 3 lattice spacings ($a = 0.075, 0.065, 0.048$ fm)
   → Extrapolation to the continuum limit $a = 0$

3) Unphysical quark masses
   → Different simulations with different pion mass in the range [190-440] MeV
   → Extrapolation to $m_\pi = m_\pi^{\text{exp}}$

4) Finite volume
   → Periodic boundary conditions in space, volume effects are $\mathcal{O}(e^{-m_\pi L})$, we use $m_\pi L > 4$
   → Discrete spatial momenta $\vec{p} = 2\pi/L\vec{n}$
   → We average over all possible photons spatial momenta $\vec{q}_1$ and $\vec{q}_2$ to increase the statistic

Kinematic reach in the photon virtualities

Photons virtualities for a pion at rest : $q_1 = (\omega_1, \vec{q}_1)$, $q_2 = (m_\pi - \omega_1, \vec{q}_2)$

$$q_1^2 = \omega_1^2 - |\vec{q}_1|^2$$
$$q_2^2 = (m_\pi - \omega_1)^2 - |\vec{q}_1|^2$$

$\Rightarrow |\vec{q}_1|^2 = (2\pi/L)^2 |\vec{n}|^2$, $|\vec{n}|^2 = 1, 2, 3, 4, 5, \ldots$

$\Rightarrow \omega_1$ is a (real) free parameter

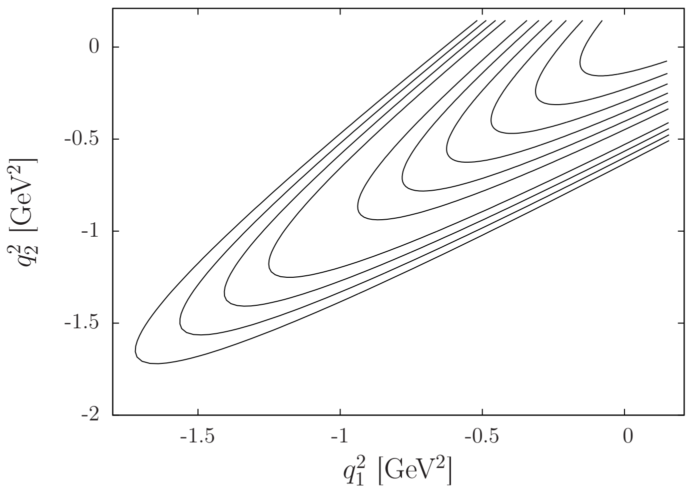


Figure – $L/a = 48$ at $a = 0.065$ fm.

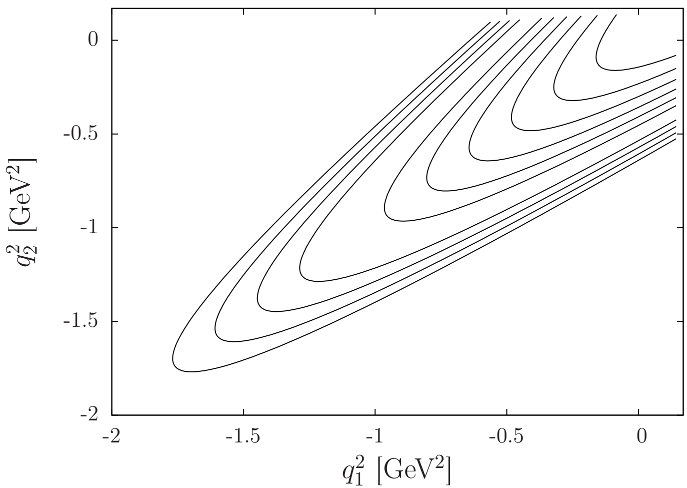


Figure – $L/a = 64$ at $a = 0.048$ fm.

Results

Shape of the integrand for F7 ($a = 0.065$ fm and $m_\pi = 270$ MeV)

$$\mathcal{F}_{\pi^0\gamma\gamma}(q_1^2, q_2^2) \propto \frac{2E_\pi}{Z_\pi} \int_{-\infty}^{\infty} \mathrm{d}\tau\, \widetilde{A}_{\mu\nu}(\tau)\, e^{\omega_1 \tau}$$

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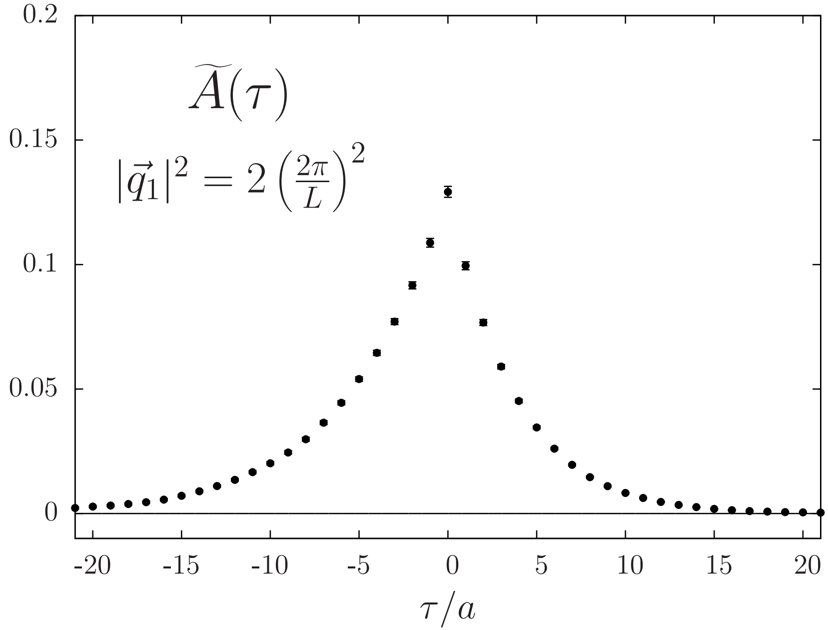
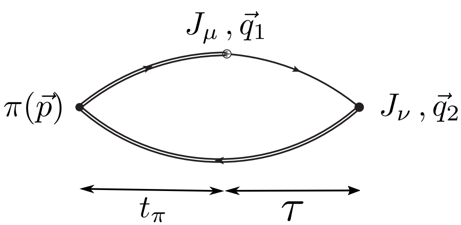




On the lattice :

- Discrete sum over lattice points :

$$\int \mathrm{d}\tau \to a \sum_\tau$$

- Finite size of the box :

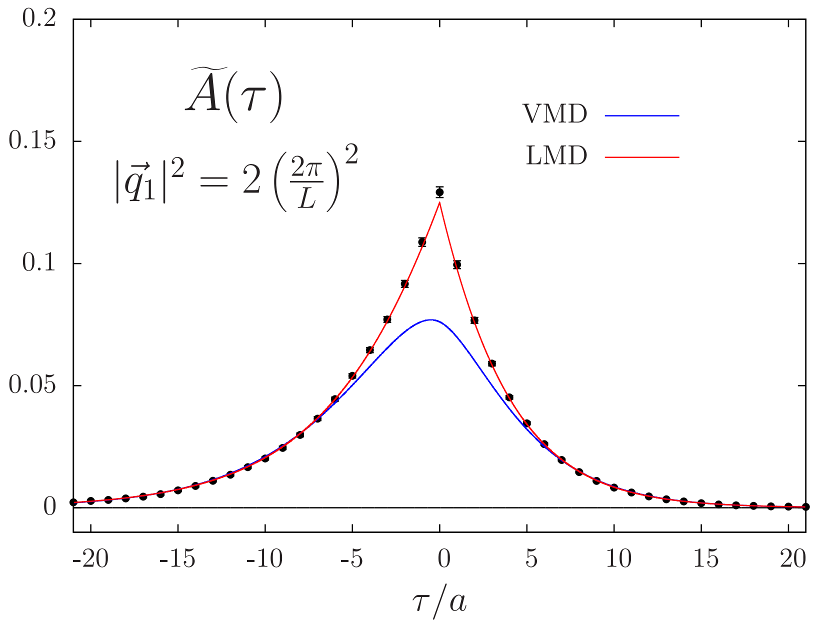
$$|\tau| <= \tau_{\max} \neq \infty$$

Shape of the integrand for F7 ($a = 0.065$ fm and $m_\pi = 270$ MeV)

1) The vector meson dominance (VMD) model is expected to give a good description of the data at large τ

$$\mathcal{F}^{\rm VMD}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{\alpha M_V^4}{(M_V^2-q_1^2)(M_V^2-q_2^2)} \quad \rightarrow \quad \widetilde{A}(\tau) = \ \ldots \quad \text{(known analytical expression)}$$

2) Fit the data at large τ and integrate for $\tau > \tau_c \gtrsim 1.3$ fm using the fit



- Check the dependance on the model using LMD rather than VMD :

$$\mathcal{F}^{\rm LMD}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2+q_2^2)}{(M_V^2-q_1^2)(M_V^2-q_2^2)}$$

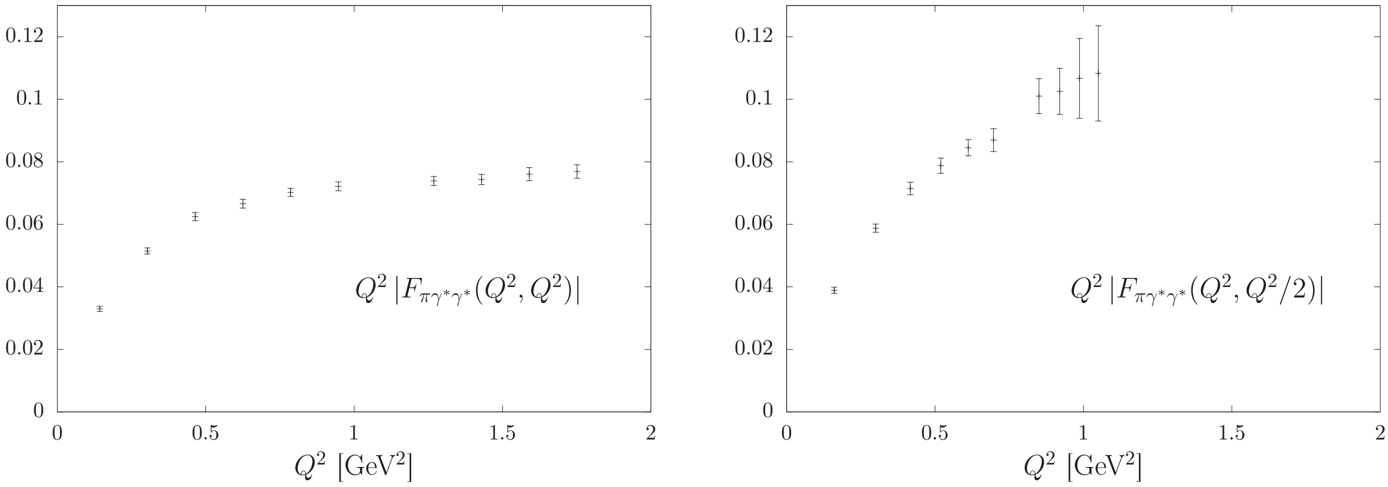
- The difference between the models is included in the systematic error.

- cusp at $\tau = 0$ (related to OPE)

$$\mathcal{F}_{\pi^0\gamma^*\gamma}(-Q^2,-Q^2) \xrightarrow[Q^2\to\infty]{} \frac{2F_\pi}{3Q^2}$$

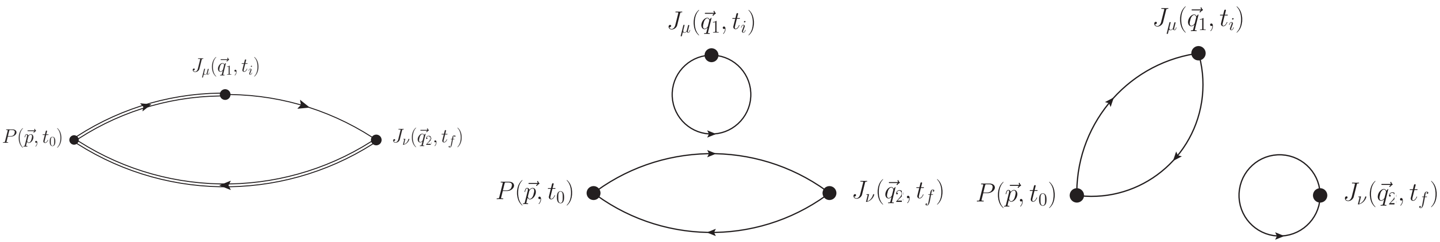
Transition form factor : results

- Results for one of the eight ensembles with $a = 0.048$ fm and $m_\pi = 270$ MeV
- We have access to the single and double-virtual form factor. Two special cases
  - $\rightarrow$ doubly-virtual form factor with $Q_1^2 = Q_2^2$
  - $\rightarrow$ doubly-virtual form factor with $Q_1^2 = 2\,Q_2^2$

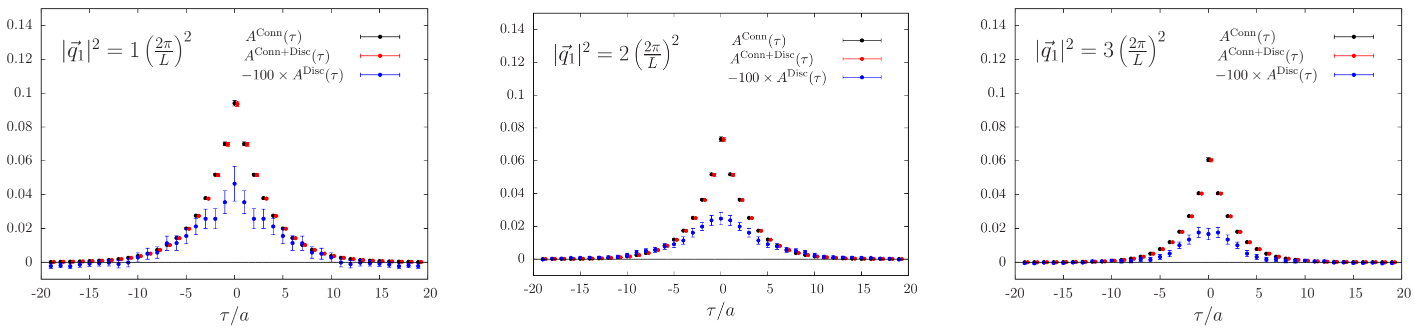


(Quark) Disconnected contributions

- Disconnected contributions were neglected so far



- Disconnected contributions are known to be much more challenging on the lattice
- Computed on E5 only ($a = 0.065$ fm, $m_\pi = 440$ MeV)



→ The disconnected contribution is below 1%
→ But the pion mass dependence could be large ...

Continuum and chiral extrapolation

- **VMD model** (Vector Meson Dominance)

$$\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

- Reproduces the anomaly constraint with $\alpha = 1/4\pi^2 F_\pi$
- And the Brodsky-Lepage in the single-virtual case : $\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(-Q^2, 0) \sim \alpha M_V^2/Q^2$
- But it fails to reproduces the OPE prediction in the double-virtual case

$$\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(-Q^2, -Q^2) \sim \alpha M_V^4/Q^4 \quad , \quad \mathcal{F}^{\mathrm{OPE}}_{\pi^0\gamma^*\gamma^*}(-Q^2, -Q^2) \sim 2F_\pi/(3Q^2)$$

Continuum and chiral extrapolation

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$$\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

- **LMD model** (Lowest Meson Dominance) [Moussallam '94] [Knecht et al. '99]

$$\mathcal{F}^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2 + q_2^2)}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

- Inspired from the large-N_C approximation to QCD
- Reproduces the anomaly constraint with $\alpha = 1/4\pi^2 F_\pi$
- Compatible with the OPE prediction in the double-virtual case with $\beta = -F_\pi/3$
- But this model is not compatible with the Brodsky-Lepage behavior in the single virtual case

$$\mathcal{F}^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma^*}(-Q^2, 0) \sim -\beta/M_V^2 \quad , \quad \mathcal{F}^{\mathrm{BL}}_{\pi^0\gamma^*\gamma^*}(-Q^2, -Q^2) \sim 1/Q^2$$

Continuum and chiral extrapolation

- **VMD model** (Vector Meson Dominance)

$$\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

- **LMD model** (Lowest Meson Dominance) [Moussallam '94] [Knecht et al. '99]

$$\mathcal{F}^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2 + q_2^2)}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

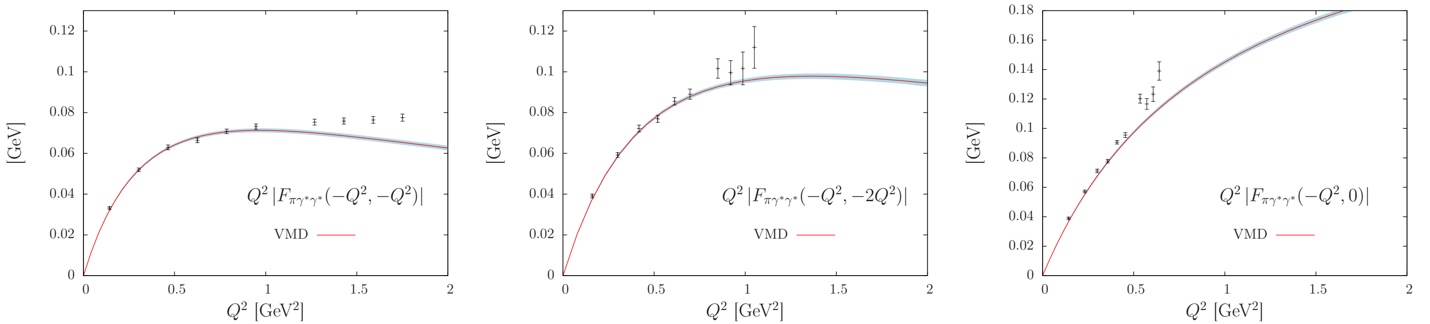
- **LMD+V model** [Knecht & Nyffeler '01]

$$\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\widetilde{h}_0\, q_1^2 q_2^2 (q_1^2 + q_2^2) + \widetilde{h}_1 (q_1^2 + q_2^2)^2 + \widetilde{h}_2\, q_1^2 q_2^2 + \widetilde{h}_5\, M_{V_1}^2 M_{V_2}^2\, (q_1^2 + q_2^2) + \alpha\, M_{V_1}^4 M_{V_2}^4}{(M_{V_1}^2 - q_1^2)(M_{V_2}^2 - q_1^2)(M_{V_1}^2 - q_2^2)(M_{V_2}^2 - q_2^2)}$$

- Refinement of the LMD model (includes a second vector resonance, $\rho' : M_{V_2}$)
- All the theoretical constraints are satisfied (if one sets $\widetilde{h}_1 = 0$)
- Many more fit parameters

Comparison with phenomenological models : VMD

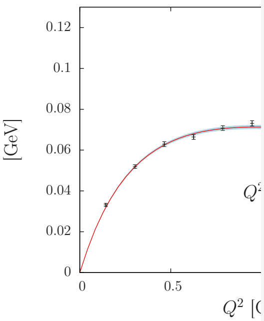
$$\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

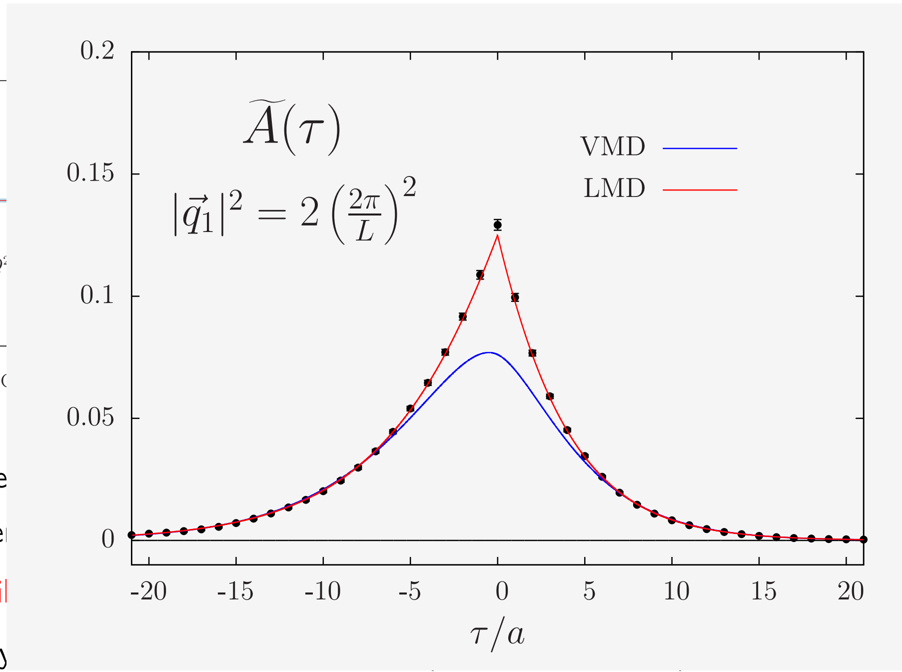


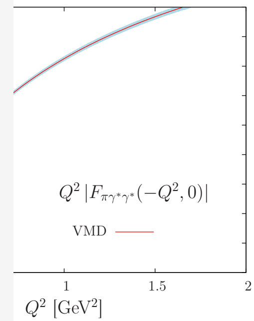
- $\hookrightarrow$ α and M_V are fit parameters
- $\hookrightarrow$ Global fit (8 ensembles + chiral and continuum extrapolation)
- $\hookrightarrow$ The model fails to describe our data ! $(\alpha = 0.243(18)\ \mathrm{GeV}^{-1} \neq \alpha_{\mathrm{th}} = 0.274\ \mathrm{GeV}^{-1})$
- $\hookrightarrow$ The wrong asymptotic behavior of this model (double virtual case) already matters at $Q^2 \sim 1-2\ \mathrm{GeV}^2$
- $\hookrightarrow$ Already seen when fitting $A(\tau)$: the model do not satisfy the OPE constraint !

Comparison with phenomenological models : VMD

$$\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$







$\hookrightarrow$ α and M_V are

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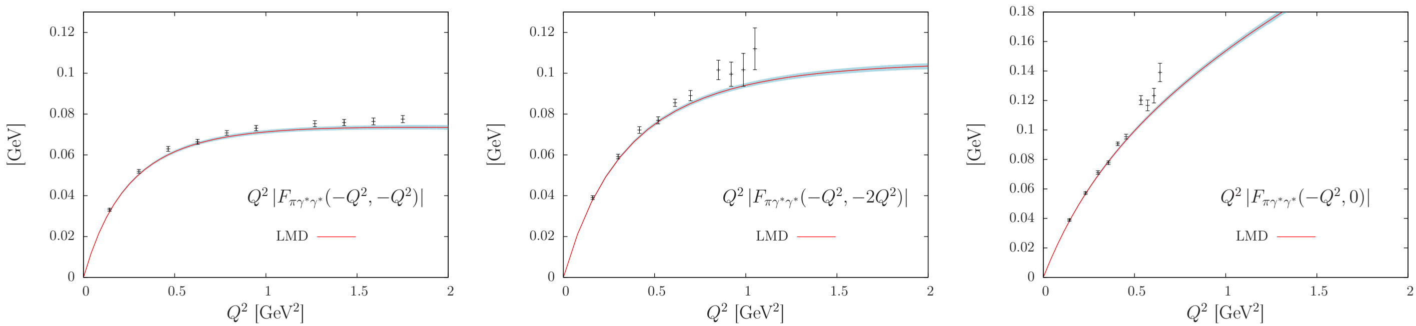
Comparison with phenomenological models : LMD

$$\mathcal{F}^{\text{LMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2 + q_2^2)}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$



$$\alpha^{\text{LMD}} = 0.275(18)\ \text{GeV}^{-1} \quad , \quad \beta = -0.028(4)\ \text{GeV} \quad , \quad M_V^{\text{LMD}} = 0.705(24)\ \text{GeV}\,.$$

- α, β and M_V are fit parameters
- The model gives a good description of our data
- α^{LMD} is compatible with the theoretical prediction $\alpha^{\text{th}} = 1/(4\pi^2 F_\pi) = 0.274\ \text{GeV}^{-1}$ $\rightarrow$ (accuracy 7%)
- β^{LMD} is compatible with the OPE prediction $\beta^{\text{OPE}} = -F_\pi/3 = -0.0308\ \text{GeV}$

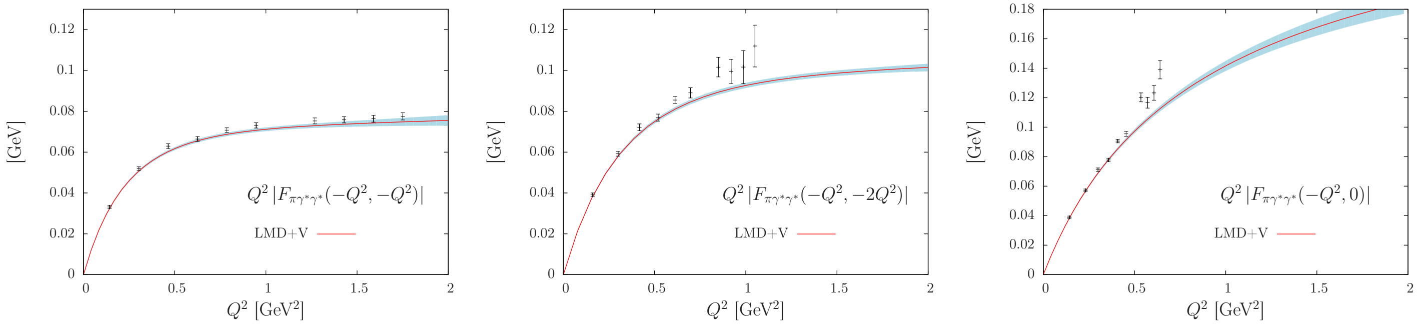
Comparison with phenomenological models : LMD+V

$$\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\widetilde{h}_0\, q_1^2 q_2^2 (q_1^2+q_2^2) + \widetilde{h}_2\, q_1^2 q_2^2 + \widetilde{h}_5\, M_{V_1}^2 M_{V_2}^2\, (q_1^2+q_2^2) + \alpha\, M_{V_1}^4 M_{V_2}^4}{(M_{V_1}^2 - q_1^2)(M_{V_2}^2 - q_1^2)(M_{V_1}^2 - q_2^2)(M_{V_2}^2 - q_2^2)}$$

- Assumptions :
  - $M_{V_1} = m_\rho^{\mathrm{exp}} = 0.775\ \mathrm{GeV}$ in the continuum and chiral limit (but chiral corrections are taken into account in the fit)
  - Constant shift in the spectrum : $M_{V_2}(\widetilde{y}) = m_{\rho'}^{\mathrm{exp}} + M_{V_1}(\widetilde{y}) - m_\rho^{\mathrm{exp}}$

Comparison with phenomenological models : LMD+V

$$\mathcal{F}^{\text{LMD+V}}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{\widetilde{h}_0\, q_1^2 q_2^2 (q_1^2+q_2^2) + \widetilde{h}_2\, q_1^2 q_2^2 + \widetilde{h}_5\, M_{V_1}^2 M_{V_2}^2\,(q_1^2+q_2^2) + \alpha\, M_{V_1}^4 M_{V_2}^4}{(M_{V_1}^2-q_1^2)(M_{V_2}^2-q_1^2)(M_{V_1}^2-q_2^2)(M_{V_2}^2-q_2^2)}$$



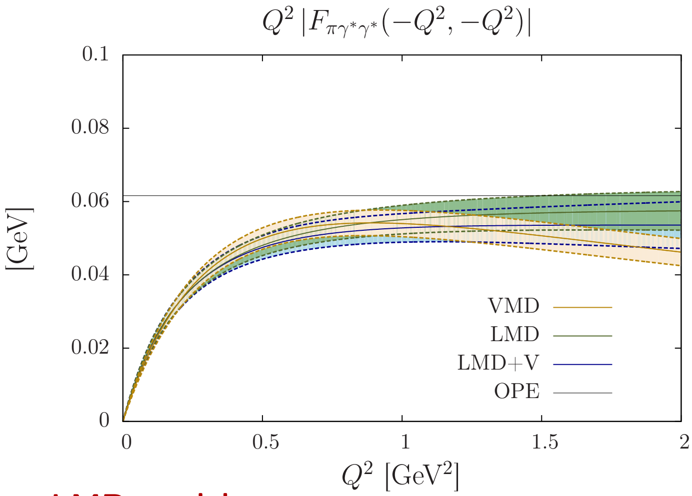
$$\alpha^{\text{LMD+V}} = 0.273(24)\ \text{GeV}^{-1} \quad , \quad \widetilde{h}_2 = 0.345(167)\ \text{GeV}^3 \quad , \quad \widetilde{h}_5 = -0.195(70)\ \text{GeV}\,.$$

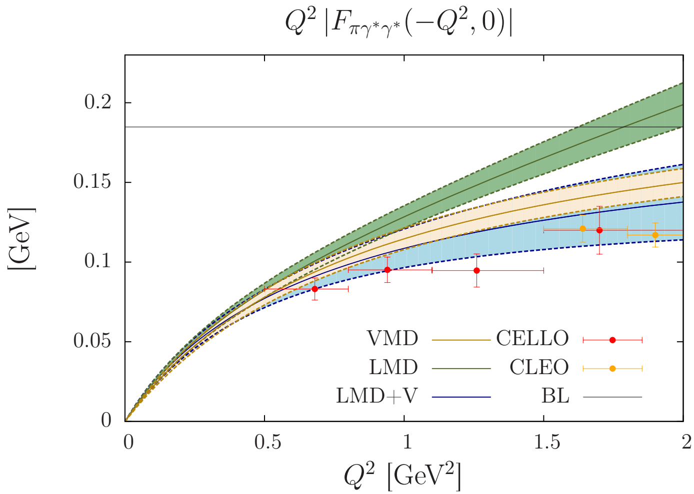
- The data are well describe by this model (same χ^2/d.o.f. as for the LMD model)
- $\alpha^{\text{LMD+V}}$ compatible with the theoretical prediction $\alpha^{\text{th}} = 0.274\ \text{GeV}^{-1}$ (statistical accuracy 9%)
- Fit to CLEO data (single-virtual form factor) : $\widetilde{h}_5 = -0.166(6)\ \text{GeV}$
- $\widetilde{h}_2$ can be fixed by comparing with the subleading term in the OPE [Nesterenko et al '83]

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2,-Q^2) \xrightarrow[Q^2\to\infty]{} \frac{2F_\pi}{3}\left[\frac{1}{Q^2} - \frac{8}{9}\frac{\delta^2}{Q^4} + \mathcal{O}\left(\frac{1}{Q^6}\right)\right]$$

- QCD sum rules : $\delta^2 = 0.20(2)\ \text{GeV}^2$ [Novikov et al '84] $\rightarrow \widetilde{h}_2 = 0.327\ \text{GeV}^3$

Final results





- **LMD model :**

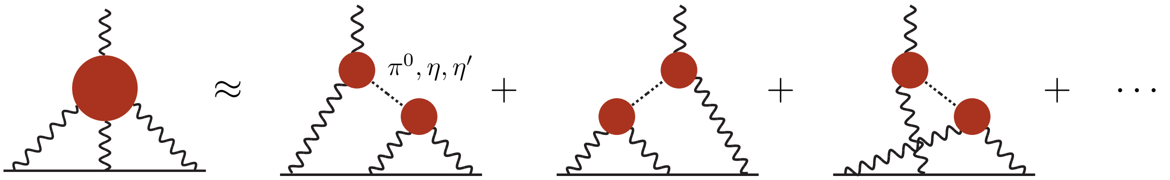
$$\alpha^{\mathrm{LMD}} = 0.275(18)(3)\ \mathrm{GeV}^{-1}\,, \quad \beta = -0.028(4)(1)\ \mathrm{GeV}\,, \quad M_V^{\mathrm{LMD}} = 0.705(24)(21)\ \mathrm{GeV}$$

- **LMD+V model :**

$$\alpha^{\mathrm{LMD+V}} = 0.273(24)(7)\ \mathrm{GeV}^{-1}\,, \quad \widetilde{h}_2 = 0.345(167)(83)\ \mathrm{GeV}^3\,, \quad \widetilde{h}_5 = -0.195(70)(34)\ \mathrm{GeV}$$

with
$$\begin{cases} \widetilde{h}_0 = -F_\pi/3 = -0.0308\ \mathrm{GeV} \\ M_{V_1} = 0.775\ \mathrm{GeV} \\ M_{V_2} = 1.465\ \mathrm{GeV} \end{cases}$$
fixed at the physical point

Back to phenomenology : the pion-pole contribution



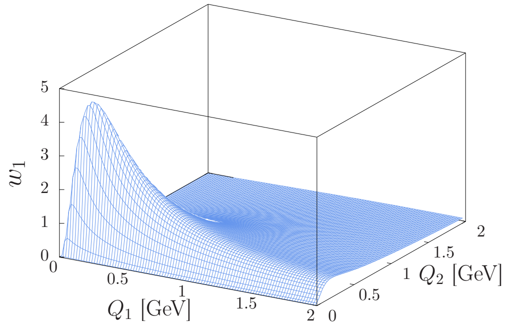
Back to phenomenology : the pion-pole contribution

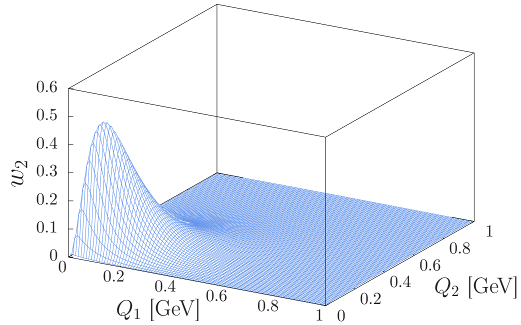
[Jegerlehner & Nyffeler '09] $a_\mu^{\mathrm{HLbL};\pi^0} = \left(\frac{\alpha_e}{\pi}\right)^3 \left(a_\mu^{\mathrm{HLbL};\pi^0(1)} + a_\mu^{\mathrm{HLbL};\pi^0(2)}\right)$

$$a_\mu^{\mathrm{HLbL};\pi^0(1)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_1(Q_1,Q_2,\tau) \; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2) \; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0)$$

$$a_\mu^{\mathrm{HLbL};\pi^0(2)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_2(Q_1,Q_2,\tau) \; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) \; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$

→ $w_{1,2}(Q_1,Q_2,\tau)$ are some model-independent weight functions (concentrated at small momenta below 1 GeV)





Back to phenomenology : the pion-pole contribution

[Jegerlehner & Nyffeler '09]

$$a_\mu^{\mathrm{HLbL};\pi^0} = \left(\frac{\alpha_e}{\pi}\right)^3 \left(a_\mu^{\mathrm{HLbL};\pi^0(1)} + a_\mu^{\mathrm{HLbL};\pi^0(2)}\right)$$

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$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{HLbL};\pi^0} = (65.0 \pm 8.3) \times 10^{-11}$$

→ most model calculations yield results in the range

$$a_\mu^{\mathrm{HLbL};\pi^0} = (50 - 80) \times 10^{-11}$$

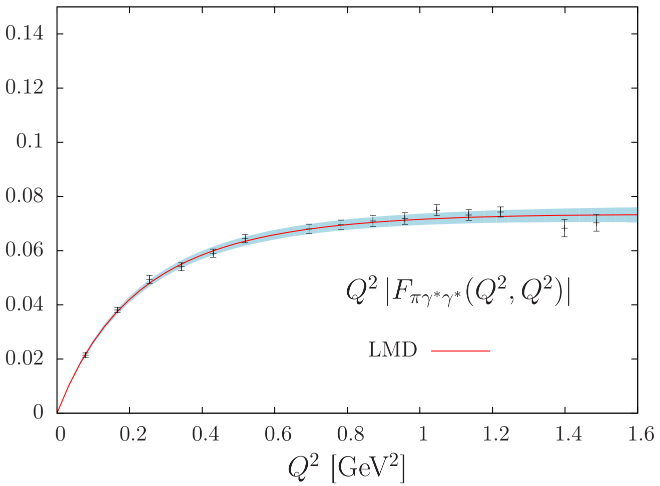
Model	$a_\mu^{\mathrm{HLbL};\pi^0} \times 10^{11}$
LMD (this work)	68.2(7.4)
LMD+V (this work)	65.0(8.3)
VMD (theory)	57.0
LMD (theory)	73.7
LMD+V (theory + phenomenology)	62.9

Λ [GeV]	LMD		LMD+V	
0.25	14.6	(21.4%)	14.4	(22.1%)
0.5	37.9	(55.5%)	37.2	(57.2%)
0.75	50.7	(74.4%)	49.5	(76.1%)
1.0	57.3	(84.0%)	55.5	(85.4%)
1.5	62.9	(92.3%)	60.6	(93.1%)
2.0	65.1	(95.5%)	62.5	(96.1%)
5.0	67.7	(99.2%)	64.6	(99.4%)
20.0	68.2	(100%)	65.0	(100%)

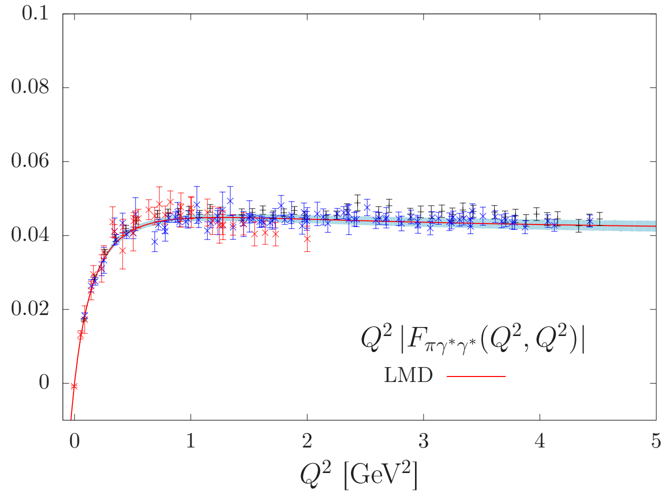
Perspectives : $N_f = 2+1$ simulations

- Use $N_f = 2+1$ CLS ensembles
- Include full $\mathcal{O}(a)$-improvement (to reduce discretisation effects)
  $\rightarrow$ require the calculation of the improvement coefficient c_V
  $\rightarrow$ both for local and conserved vector currents
- Consider the case $\vec{p} \neq \vec{0}$ in addition to $\vec{p} = \vec{0}$

$N_f = 2$ (G8)



$N_f = 2+1$ (D200)



- We can also probe larger virtualities in the single-virtual case

Perspectives : $N_f = 2+1$ simulations

- Use $N_f = 2+1$ CLS ensembles
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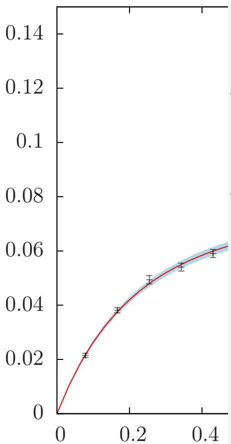
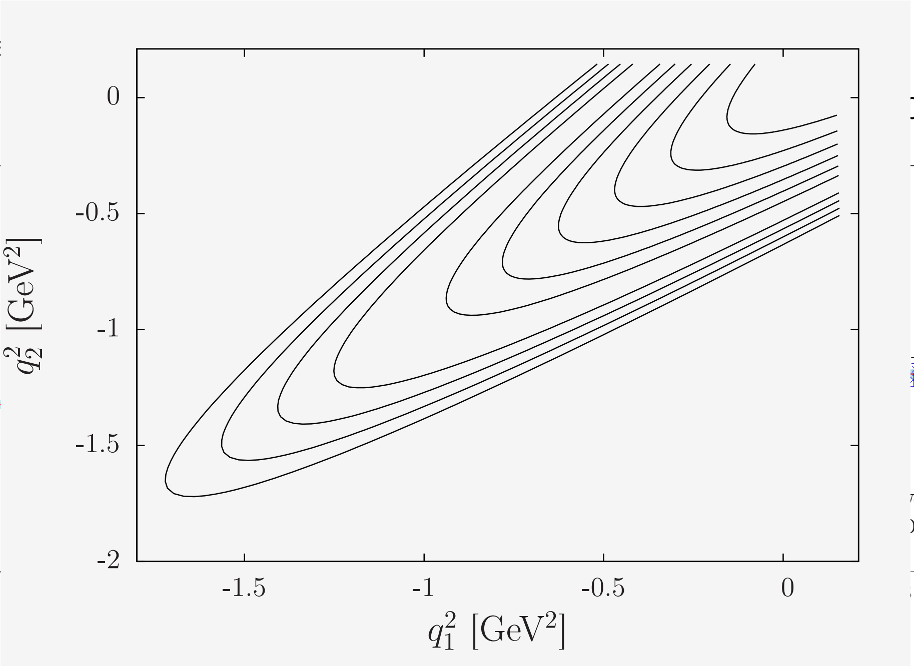
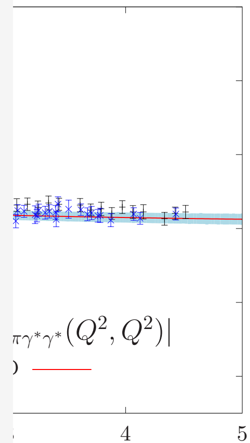
  - $\rightarrow$ both for local and conserved vector currents
- Consider the case





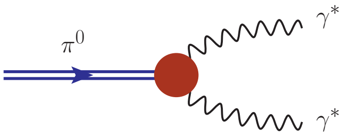


- We can also probe larger virtualities in the single-virtual case

$(g-2)_\mu$: Mainz effort

- **Pseudoscalar transition Form Factor (this talk)**

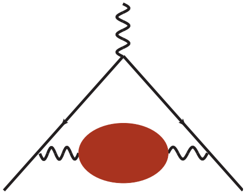
  ↪ Gives the dominant contribution to HLbL from first principles



- **Hadronic Vacuum Polarisation (LO)**

  ↪ Recent publication with $N_f = 2$ flavours [arXiv :1705.1775]

  ↪ Now generating data with $N_f = 2+1$ flavours



- **Hadronic Light-by-Light**

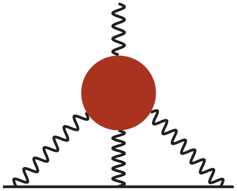
  ↪ Direct lattice calculation : exact QED kernel in position space :

$$a_\mu^{\mathrm{HLbL}} = \frac{me^3}{3} \int \mathrm{d}^4 y \int \mathrm{d}^4 x \; \overline{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) \; i\Pi_{\rho,\mu\nu\lambda\sigma}(x,y)$$

$$i\Pi_{\rho,\mu\nu\lambda\sigma}(x,y) = -\int \mathrm{d}^4 z \, z_\rho \, \langle J_\mu(x) J_\nu(y) J_\sigma(z) J_\lambda(0) \rangle$$

  ↪ Four-point correlation function computed on the lattice

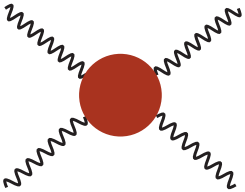
  ↪ Only one collaboration has published results so far [RBC/UKQCD]



- **Light-by-light forward scattering amplitudes**

  ↪ Related to two-photon fusion cross sections via the optical theorem

  ↪ Provides informations about resonances contributions (through TFFs)



Conclusion

- We have performed a first lattice calculation of the pion transition form factor in the momentum region relevant for the $(g-2)_\mu$.
- Our results are compatible with the anomaly contraint ($\alpha^{\mathrm{th}} = 0.274~\mathrm{GeV}^{-1}$)

$$\alpha^{\mathrm{LMD}} = 0.275(18)(3)~\mathrm{GeV}^{-1} \quad , \quad \alpha^{\mathrm{LMD+V}} = 0.273(24)(7)~\mathrm{GeV}^{-1}$$

  $\rightarrow 7-9\%$ accuracy

- Disconnected contributions have been computed on one lattice ensemble.
- Provides a first lattice estimate of the pion-pole contribution to the hadronic light-by-light scattering in the $g-2$ of the muon

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{HLbL};\pi^0} = (65.0 \pm 8.3) \times 10^{-11}$$

- We are currently improving the calculation using $N_f = 2+1$ simulations

  $\rightarrow$ Full $\mathcal{O}(a)$-improvement to reduce discretization effects

  $\rightarrow$ Include a new kinematical configuration where the pion has one unit of momentum

  $\rightarrow$ This allows to probe a larger kinematical range

  $\rightarrow$ Model independent extraction of the form factor