High-Energy Operator Product Expansion at NLO

Giovanni Antonio Chirilli

JLAB & ODU
LPT-Orsay & CPHT-Polytechnique

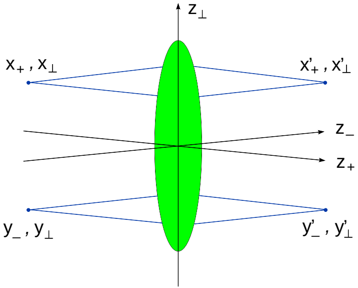
Low-x meeting, Ischia Sept 9-12, 2009

Outline

- Regge limit in the coordinate space.
- Light-cone OPE versus OPE in color dipoles.
- High-energy scattering and Wilson lines.
- Factorization in rapidity: Feynman diagrams in a shock-wave background.
- LO and NLO Impact Factor in QCD in coordinate space.
- NLO BK kernel in $\mathcal{N} = 4$ SYM and in QCD.
- Conclusions.

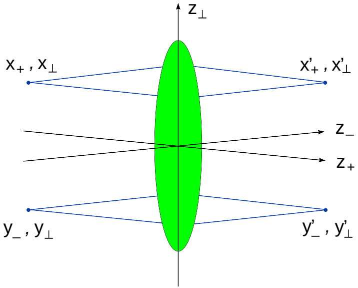
Small-x (Regge) limit in the coordinate space

Regge limit: $x_+ \to \rho x_+,\ x'_+ \to \rho x'_+, y_- \to \rho' y_-,\ y'_- \to \rho' y-_-$ $\quad \rho, \rho' \to \infty$



Regge limit symmetry in a conformal theory: 2-dim conformal Möbius group $SL(2, C)$.

Regge limit: $x_+ \to \rho x_+,\ x'_+ \to \rho x'_+, y_- \to \rho' y_-,\ y'_- \to \rho' y-_- \qquad \rho, \rho' \to \infty$

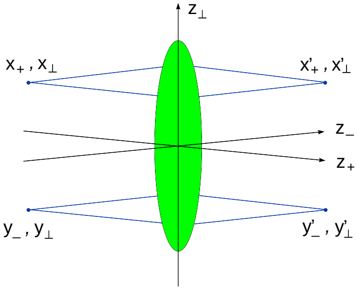


LLA: $\alpha_s \ll 1$, $\alpha_s \ln \rho \sim 1$, $\Rightarrow \sum(\alpha_s \ln \rho)^n \equiv$ BFKL pomeron.
LLA $\Leftrightarrow$ tree diagrams $\Rightarrow$ the BFKL pomeron is Möbius invariant .

NLO LLA: extra α_s: $\sum \alpha_s(\alpha_s \ln \rho)^n \equiv$ NLO BFKL
In a conformal theory ($\mathcal{N} = 4$ SYM) we expect NLO BFKL to be Möbius invariant.

Small-x (Regge) limit in the coordinate space

Regge limit: $x_+ \to \rho x_+,\ x'_+ \to \rho x'_+, y_- \to \rho' y_-,\ y'_- \to \rho' y-_-$ $\quad \rho, \rho' \to \infty$

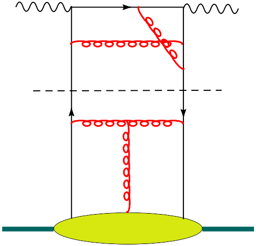


LLA: $\alpha_s \ll 1,\ \alpha_s \ln \rho \sim 1,\ \Rightarrow \sum(\alpha_s \ln \rho)^n \equiv$ BFKL pomeron.
LLA $\Leftrightarrow$ tree diagrams $\Rightarrow$ the BFKL pomeron is Möbius invariant .

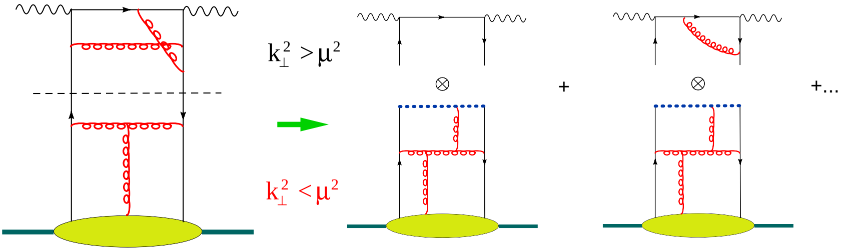
NLO LLA: extra α_s: $\sum \alpha_s(\alpha_s \ln \rho)^n \equiv$ NLO BFKL
In a conformal theory ($\mathcal{N} = 4$ SYM) we expect NLO BFKL to be Möbius invariant.

In QCD, we expect to have running coupling part plus conformal part.

Light-cone expansion and DGLAP evolution in the NLO



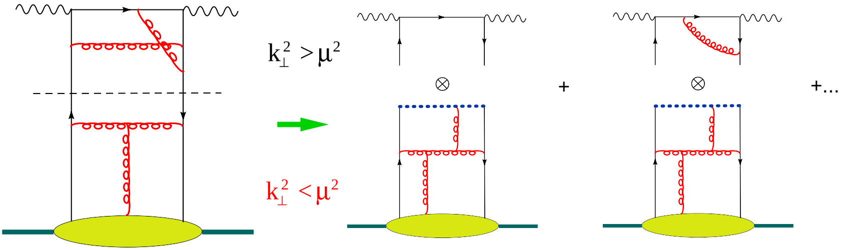
Light-cone expansion and DGLAP evolution in the NLO



μ^2 - factorization scale (normalization point)

$k_\perp^2 \; > \; \mu^2$ - coefficient functions
$k_\perp^2 \; < \; \mu^2$ - matrix elements of light-ray operators (normalized at μ^2)

Light-cone expansion and DGLAP evolution in the NLO



μ^2 - factorization scale (normalization point)

$k_\perp^2 \; > \; \mu^2$ - coefficient functions

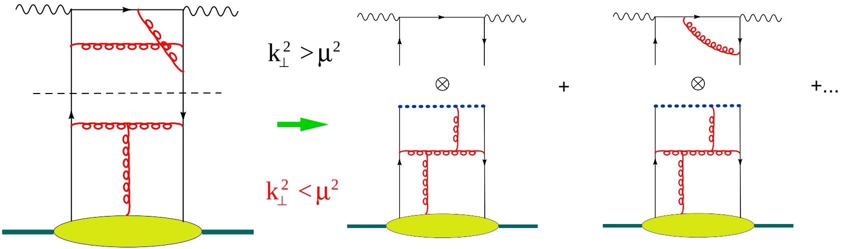
$k_\perp^2 \; < \; \mu^2$ - matrix elements of light-ray operators (normalized at μ^2)

OPE in light-ray operators $\qquad (x-y)^2 \to 0$

$$T\{j_\mu(x)j_\nu(y)\} \; = \; \frac{x_\xi}{2\pi^2 x^4}\Big[1 + \frac{\alpha_s}{\pi}(\ln x^2\mu^2 + C)\Big]\bar{\psi}(x)\gamma_\mu\gamma^\xi\gamma_\nu[x,y]\psi(y) + O(\frac{1}{x^2})$$

$$[x,y] \; \equiv \; Pe^{ig\int_0^1 du \; (x-y)^\mu A_\mu(ux+(1-u)y)}$$ - gauge link

Light-cone expansion and DGLAP evolution in the NLO

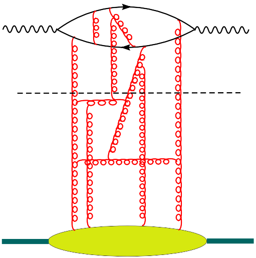


μ^2 - factorization scale (normalization point)

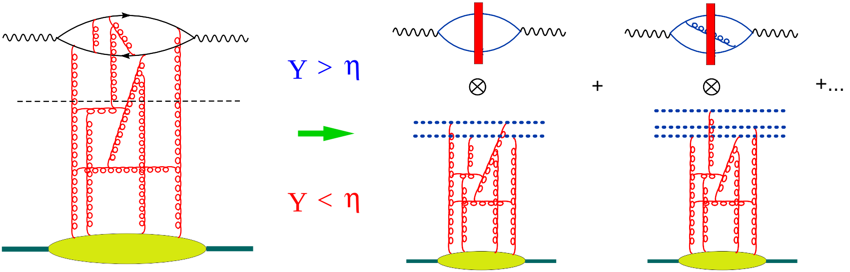
$k_\perp^2 > \mu^2$ - coefficient functions
$k_\perp^2 < \mu^2$ - matrix elements of light-ray operators (normalized at μ^2)
Renorm-group equation for light-ray operators $\Rightarrow$ DGLAP evolution of parton densities $(x-y)^2 = 0$

$$\mu^2 \frac{d}{d\mu^2}\bar{\psi}(x)[x,y]\psi(y) = K_{\rm LO}\bar{\psi}(x)[x,y]\psi(y) + \alpha_s K_{\rm NLO}\bar{\psi}(x)[x,y]\psi(y)$$

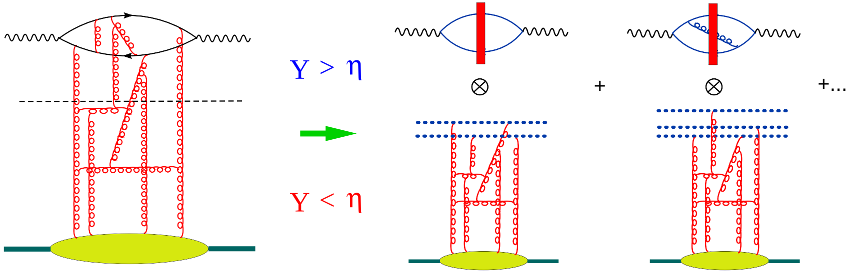
High-energy expansion in color dipoles in the NLO



High-energy expansion in color dipoles in the NLO



High-energy expansion in color dipoles in the NLO



η - rapidity factorization scale

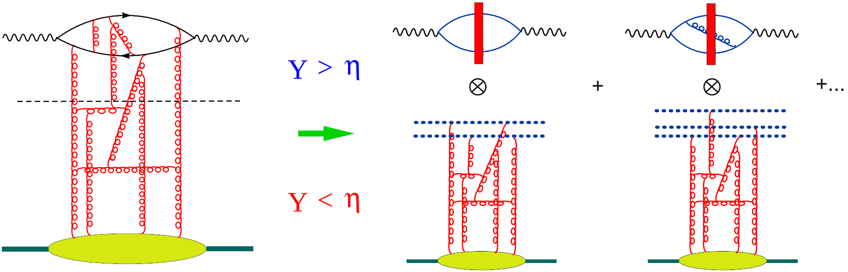
Rapidity Y > η - coefficient function (“impact factor”)

Rapidity Y < η - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

$$U_x^\eta = \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} dx^+ A_+^\eta(x_+, x_\perp)\Big]$$

$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

High-energy expansion in color dipoles in the NLO



The high-energy operator expansion is

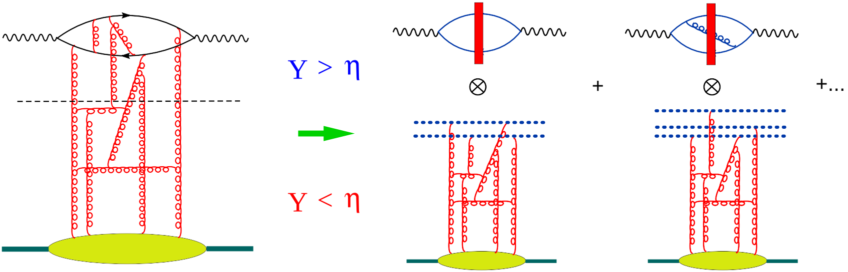
$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 \; I^{\rm LO}_{\mu\nu}(z_1,z_2)\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$+ \int d^2z_1 d^2z_2 d^2z_3 \; I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3)[\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

In the leading order the impact factor is Möbius invariant

In the NLO one should also expect conf. invariance since $I^{\rm NLO}_{\mu\nu}$ is given by tree diagrams

High-energy expansion in color dipoles in the NLO



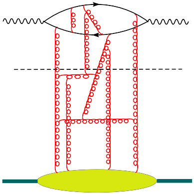
η - rapidity factorization scale

Evolution equation for color dipoles

$$\frac{d}{d\eta}\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\} = \frac{\alpha_s}{2\pi^2}\int d^2z \frac{(x-y)^2}{(x-z)^2(y-z)^2}[\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\}\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\} - N_c\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\}] + \alpha_s K_{\mathrm{NLO}}\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\} + O(\alpha_s^2)$$

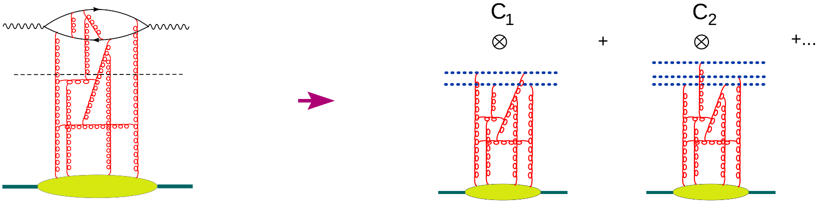
K_{NLO}=? (Linear part of K_{NLO} = $K_{\mathrm{NLO\ BFKL}}$)

- At high energies, particles move along straight lines $\Rightarrow$ the amplitude of $\gamma^* A \rightarrow \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



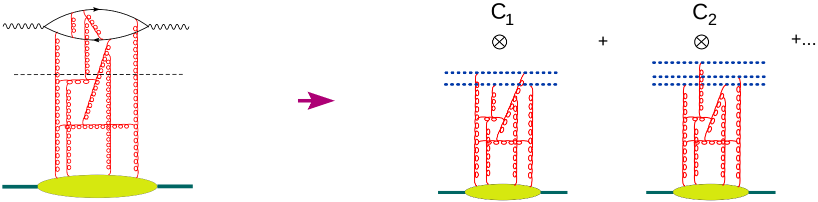
DIS at high energy

- At high energies, particles move along straight lines $\Rightarrow$ the amplitude of $\gamma^* A \to \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



DIS at high energy

- At high energies, particles move along straight lines $\Rightarrow$ the amplitude of $\gamma^* A \to \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



$$A(s) = \int \frac{d^2k_\perp}{4\pi^2} I^A(k_\perp)\langle B|\mathrm{Tr}\{U(k_\perp)U^\dagger(-k_\perp)\}|B\rangle + \dots$$

$$U(x_\perp) = Pe^{ig\int_{-\infty}^{\infty} du\ n^\mu A_\mu(un+x_\perp)} \qquad \text{Wilson line}$$

Propagation in the shock wave: Wilson line (Spectator frame)

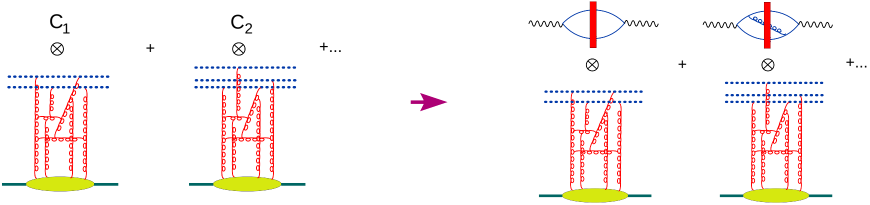
Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.

$$U_z = [\infty p_1 + z_\perp, -\infty p_1 + z_\perp]$$

$$[x, y] = Pe^{ig\int_0^1 du(x-y)^\mu A_\mu(ux+(1-u)y)} \qquad p^\mu = \alpha p_1^\mu + \beta p_2^\mu + p_\perp^\mu$$

Propagation in the shock wave: Wilson line (Spectator frame)

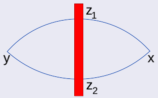
Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.



LO and NLO Impact Factor

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\, I^{\rm LO}_{\mu\nu}(z_1,z_2){\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$
$$+ \int d^2z_1 d^2z_2 d^2z_3\, I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3)[{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

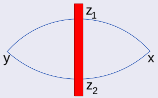
LO Impact Factor diagram: $I^{\rm LO}$



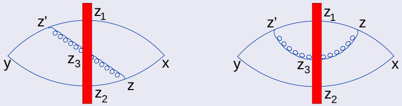
LO and NLO Impact Factor

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}_{\mu\nu}(z_1,z_2){\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$
$$+\int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3)[{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

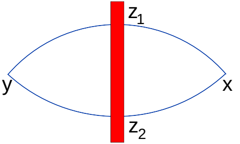
LO Impact Factor diagram: $I^{\rm LO}$



NLO Impact Factor diagrams: $I^{\rm NLO}$



LO Impact Factor



Conformal vectors:

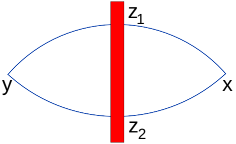
$$\kappa = \frac{\sqrt{s}}{2x_*}\left(\frac{p_1}{s} - x^2 p_2 + x_\perp\right) - \frac{\sqrt{s}}{2y_*}\left(\frac{p_1}{s} - y^2 p_2 + y_\perp\right)$$

$$\kappa' = \frac{\sqrt{s}}{2x'_\bullet}\left(\frac{p_1}{s} - x'^2 p_2 + x'_\perp\right) - \frac{\sqrt{s}}{2y'_\bullet}\left(\frac{p_1}{s} - y'^2 p_2 + y'_\perp\right)$$

$$\zeta_1 = \sqrt{s}\left(\frac{p_1}{s} + z_{1\perp}^2 p_2 + z_{1\perp}\right), \qquad \zeta_2 = \sqrt{s}\left(\frac{p_1}{s} + z_{2\perp}^2 p_2 + z_{2\perp}\right)$$

Here $x^2 = -x_\perp^2, \quad x'^2 = -x'^2_\perp$ (similarly for y)

LO Impact Factor



Conformal vectors:

$$\kappa = \frac{\sqrt{s}}{2x_*}(\frac{p_1}{s} - x^2 p_2 + x_\perp) - \frac{\sqrt{s}}{2y_*}(\frac{p_1}{s} - y^2 p_2 + y_\perp)$$

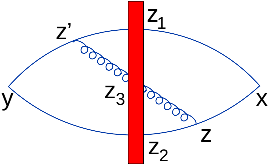
$$\kappa' = \frac{\sqrt{s}}{2x'_\bullet}(\frac{p_1}{s} - x'^2 p_2 + x'_\perp) - \frac{\sqrt{s}}{2y'_\bullet}(\frac{p_1}{s} - y'^2 p_2 + y'_\perp)$$

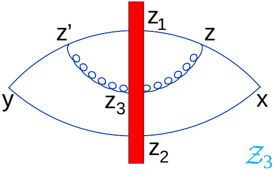
$$\zeta_1 = \sqrt{s}(\frac{p_1}{s} + z_{1\perp}^2 p_2 + z_{1\perp}), \qquad \zeta_2 = \sqrt{s}(\frac{p_1}{s} + z_{2\perp}^2 p_2 + z_{2\perp})$$

Here $x^2 = -x_\perp^2, \quad x'^2 = -x'^2_\perp$ (similarly for y)

$$I^{LO} \propto \frac{2}{\pi^6} \int d^2 z_{1\perp} d^2 z_{2\perp} \mathrm{tr}\{U_{z_{1\perp}} U^\dagger_{z_{2\perp}}\} \frac{z_{12\perp}^2}{x_*^2 y_*^2 (\kappa \cdot \zeta_1)^3 (\kappa \cdot \zeta_2)^3}$$
$$\times \frac{\partial^2}{\partial x^\mu \partial y^\nu} \big[-2(\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2) + \kappa^2 (\zeta_1 \cdot \zeta_2) \big]$$

NLO Impact Factor

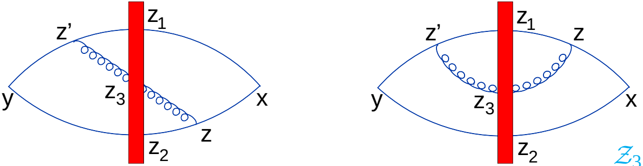




$$\mathcal{Z}_3 \equiv \frac{(x-z_3)^2_\perp}{x^+} - \frac{(y-z_3)^2_\perp}{y^+}$$

$$I^{\rm NLO}_{\mu\nu}(x,y;z_1,z_2,z_3;\eta) \;=\; -\,I^{\rm LO}_{\mu\nu} \times \frac{\alpha_s}{2\pi}\frac{z_{13}^2}{z_{12}^2 z_{23}^2}\ln\frac{\sigma s}{4}\mathcal{Z}_3 \;+\; \text{conf.}$$

The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff $\eta = \ln\sigma$ is not invariant.



$$\mathcal{Z}_3 \equiv \frac{(x-z_3)_\perp^2}{x^+} - \frac{(y-z_3)_\perp^2}{y^+}$$

$$I_{\mu\nu}^{\rm NLO}(x,y;z_1,z_2,z_3;\eta) = -I_{\mu\nu}^{\rm LO} \times \frac{\alpha_s}{2\pi}\frac{z_{13}^2}{z_{12}^2 z_{23}^2}\ln\frac{\sigma s}{4}\mathcal{Z}_3 + \text{conf.}$$

The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff $\eta = \ln\sigma$ is not invariant.

However, if we define a composite operator (a - analog of μ^{-2} for usual OPE)

$$[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf} = \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$
$$+ \frac{\alpha_s}{4\pi}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\frac{1}{N_c}\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{a z_{12}^2}{z_{13}^2 z_{23}^2} + O(\alpha_s^2)$$

the impact factor becomes conformal in the NLO.

Operator expansion in conformal dipoles

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}_{\mu\nu}(z_1,z_2){\rm tr}[\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\rm conf}$$

$$+\int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3)[\frac{1}{N_c}{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

Operator expansion in conformal dipoles

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}_{\mu\nu}(z_1,z_2){\rm tr}[\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\rm conf}$$

$$+\int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3)[\frac{1}{N_c}{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

$$I^{\rm NLO}_{\mu\nu} = -I^{\rm LO}_{\mu\nu}\ \frac{\alpha_s N_c}{4\pi}\int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\ln\frac{z_{12}^2 e^{2\eta} a s^2}{z_{13}^2 z_{23}^2}\mathcal{Z}_3^2 + {\rm conf.}$$

The new NLO impact factor is conformally invariant.

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}_{\mu\nu}(z_1,z_2){\rm tr}[\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\rm conf}$$
$$+\int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3)[\frac{1}{N_c}{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

$$I^{\rm NLO}_{\mu\nu} = -I^{\rm LO}_{\mu\nu}\ \frac{\alpha_s N_c}{4\pi}\int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\ln\frac{z_{12}^2 e^{2\eta} a s^2}{z_{13}^2 z_{23}^2}\mathcal{Z}_3^2 + {\rm conf.}$$

The new NLO impact factor is conformally invariant.

In conformal $\mathcal{N}=4$ SYM theory (where the β-function vanishes) one can construct the composite conformal dipole operator order by order in perturbation theory.

Operator expansion in conformal dipoles

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 \; I^{\rm LO}_{\mu\nu}(z_1,z_2){\rm tr}[\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\rm conf}$$

$$+\int d^2z_1 d^2z_2 d^2z_3 \; I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3)[\frac{1}{N_c}{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

$$I^{\rm NLO}_{\mu\nu} = \; - I^{\rm LO}_{\mu\nu}\;\frac{\alpha_s N_c}{4\pi}\int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\ln\frac{z_{12}^2 e^{2\eta} a s^2}{z_{13}^2 z_{23}^2}\mathcal{Z}_3^2 + {\rm conf.}$$

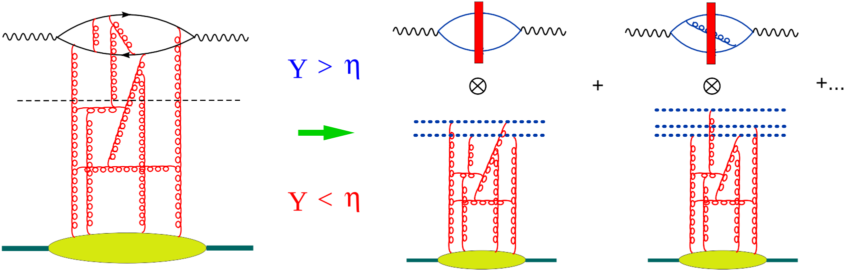
The new NLO impact factor is conformally invariant.

In conformal $\mathcal{N}=4$ SYM theory (where the β-function vanishes) one can construct the composite conformal dipole operator order by order in perturbation theory.

Analogy:

When the UV cutoff does not respect the symmetry of a local operator, the composite local renormalized operator must be corrected by finite counterterms order by order in perturbaton theory.

Expansion of $F_2(x)$ in color dipoles in the next-to-leading order



$$F_2(x_B) \simeq \int d^2z_1 d^2z_2 \; I^{LO}(z_1, z_2)\langle \mathrm{tr}\{U^{\eta}_{z_1} U^{\dagger\,\eta}_{z_2}\}\rangle \qquad \eta = \ln \frac{1}{x_B}$$

$$+ \frac{\alpha_s}{\pi} \int d^2z_1 d^2z_2 d^2z_3 \; I^{NLO}(z_1, z_2, z_3)\langle \mathrm{tr}\{U^{\eta}_{z_1} U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{U_{z_3} U^{\dagger\eta}_{z_2}\}\rangle$$

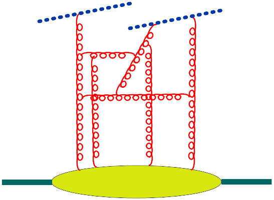
Operator expansion in conformal dipoles in $\mathcal{N}=4$ SYM

$\mathcal{O} \equiv \frac{4\pi^2\sqrt{2}}{\sqrt{N_c^2-1}}\mathrm{Tr}\{Z^2\} \quad (Z = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2))$

$$
\begin{aligned}
(x-y)^4 T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} \;=\; & \frac{(x-y)^4}{\pi^2(N_c^2-1)}\int d^2z_1 d^2z_2 \frac{(x_*y_*)^{-2}}{\mathcal{Z}_1^2\mathcal{Z}_2^2}\,[\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} \\
& - \frac{\alpha_s(x-y)^4}{2\pi^4(N_c^2-1)}\int d^2z_1 d^2z_2 d^2z_3 \frac{z_{12}^2(x_*y_*)^{-2}}{z_{13}^2 z_{23}^2\mathcal{Z}_1^2\mathcal{Z}_2^2} \\
& \times\Big(\ln\frac{x_*y_*z_{12}^2 e^{2\eta}}{16(x-y)_\perp^2 z_{13}^2 z_{23}^2}\Big[\frac{(x-z_3)^2}{x_*} - \frac{(y-z_3)^2}{y_*}\Big]^2 - i\pi + 2C\Big) \\
& \times\,[\mathrm{Tr}\{T^n\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]
\end{aligned}
$$

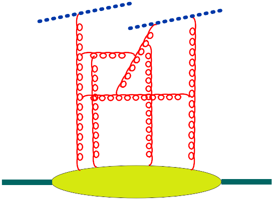
The impact factor is Möbius invariant and does not scale with the energy.

Regularization of the rapidity divergence



For light-like Wilson lines loop integrals are divergent in the longitudinal direction

$$\int_0^{\infty} \frac{d\alpha}{\alpha} = \int_{-\infty}^{\infty} d\eta = \infty$$



For light-like Wilson lines loop integrals are divergent in the longitudinal direction

$$\int_0^\infty \frac{d\alpha}{\alpha} = \int_{-\infty}^{\infty} d\eta = \infty$$

Regularization by: slope

$$U^\eta(x_\perp) = \mathrm{Pexp}\Big\{ig\int_{-\infty}^{\infty} du\, n_\mu\, A^\mu(un + x_\perp)\Big\} \qquad n^\mu = p_1^\mu + e^{-2\eta} p_2^\mu$$

Evolution Equation

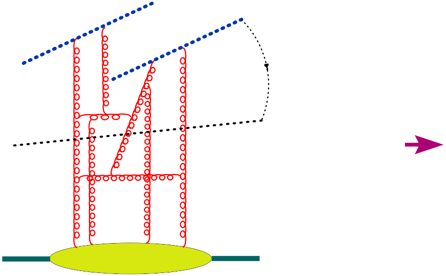
$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} \quad \Rightarrow \quad \frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle$$

Evolution Equation

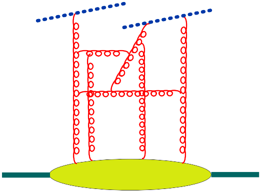
$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} \quad \Rightarrow \quad \frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle$$

To get the evolution equation, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidity $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidity up to η_2).

In the frame $||$ to η_1 the gluons with $\eta < \eta_1$ are seen as pancake.



$$\alpha_s(\eta_1 - \eta_2)K_{\mathrm{evol}} \quad \otimes$$



Particles with different rapidity perceive each other as Wilson lines.

Leading order: BK equation

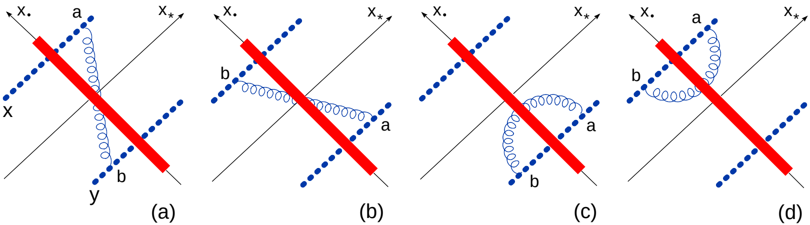
$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}}$$

Leading order: BK equation

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}}$$



$x_\bullet = \sqrt{\frac{s}{2}}x^-$ $\qquad$ $x_* = \sqrt{\frac{s}{2}}x^+$

Non linear evolution equation: BK equation

Non linear evolution equation: BK equation

$$U_z^{ab} = \mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \to (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

Non linear evolution equation: BK equation

$$U_z^{ab} = \mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \to (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

BK equation: **Ian Balitsky (1996), Yu. Kovchegov (1999)**

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) \;=\; \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\;(x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z) + \hat{\mathcal{U}}(z,y) - \hat{\mathcal{U}}(x,y) - \hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1,\; \alpha_s\eta \sim 1$)

LLA for DIS in sQCD $\Rightarrow$ BK eqn (LLA: $\alpha_s \ll 1,\; \alpha_s\eta \sim 1,\; \alpha_s^2 A^{1/3} \sim 1$)

(s for semiclassical)

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] = \mathrm{P} e^{ig\int_{-\infty}^{+\infty} dx^+ A_+(x^+, x_\perp)}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] = \mathrm{P}e^{ig\int_{-\infty}^{+\infty} dx^+ A_+(x^+, x_\perp)}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

Indeed,

$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after inversion $x_\perp \to \frac{x_\perp}{x_\perp^2}$ and $x^+ \to \frac{x^+}{x_\perp^2} \Rightarrow$

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] = \mathrm{P}e^{ig\int_{-\infty}^{+\infty} dx^+ A_+(x^+, x_\perp)}$$

is invariant under inversion (with respect to the point with $x^- = 0$).
Indeed,
$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after inversion $x_\perp \to \frac{x_\perp}{x_\perp^2}$ and $x^+ \to \frac{x^+}{x_\perp^2} \Rightarrow$

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \to \mathrm{P}e^{ig\int_{-\infty}^{+\infty} d\frac{x^+}{x_\perp^2} A_+(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2})} = [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}]$$

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] = \mathrm{P}e^{ig\int_{-\infty}^{+\infty} dx^+ A_+(x^+, x_\perp)}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

Indeed,

$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after inversion $x_\perp \to \frac{x_\perp}{x_\perp^2}$ and $x^+ \to \frac{x^+}{x_\perp^2} \Rightarrow$

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \to \mathrm{P}e^{ig\int_{-\infty}^{+\infty} d\frac{x^+}{x_\perp^2} A_+(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2})} = [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}]$$

$\Rightarrow$ The dipole kernel is invariant under the inversion $V(x_\perp) = U(x_\perp/x_\perp^2)$

$$\frac{d}{d\eta}\mathrm{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int \frac{d^2z}{\not{z}^4}\frac{(x-y)^2\not{z}^4}{(x-z)^2(y-z)^2}[\mathrm{Tr}\{V_x V_z^\dagger\}\mathrm{Tr}\{V_z V_y^\dagger\} - N_c \mathrm{Tr}\{V_x V_y^\dagger\}]$$

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] = \mathrm{P}e^{ig\int_{-\infty}^{+\infty} dx^+ A_+(x^+, x_\perp)}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

Indeed,

$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after inversion $x_\perp \to \frac{x_\perp}{x_\perp^2}$ and $x^+ \to \frac{x^+}{x_\perp^2} \Rightarrow$

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \to \mathrm{P}e^{ig\int_{-\infty}^{+\infty} d\frac{x^+}{x_\perp^2} A_+(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2})} = [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}]$$

$\Rightarrow$ The dipole kernel is invariant under the inversion $V(x_\perp) = U(x_\perp/x_\perp^2)$

$$\frac{d}{d\eta}\mathrm{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int \frac{d^2z}{\not{z}^4}\frac{(x-y)^2\not{z}^4}{(x-z)^2(y-z)^2}[\mathrm{Tr}\{V_x V_z^\dagger\}\mathrm{Tr}\{V_z V_y^\dagger\} - N_c\mathrm{Tr}\{V_x V_y^\dagger\}]$$

In the leading order we have conformal invariance. What about the next-to-leading order?

Non linear evolution equation in the NLO

$$\frac{d}{d\eta}Tr\{U_xU_y^\dagger\} =$$
$$\int \frac{d^2z}{2\pi^2}\left(\alpha_s\frac{(x-y)^2}{(x-z)^2(z-y)^2} + \alpha_s^2 K_{NLO}(x,y,z)\right)[Tr\{U_xU_z^\dagger\}Tr\{U_zU_y^\dagger\} - N_c Tr\{U_xU_y^\dagger\}] +$$
$$\alpha_s^2\int d^2z d^2z'\left(K_4(x,y,z,z')\{U_x, U_{z'}^\dagger, U_z, U_y^\dagger\} + K_6(x,y,z,z')\{U_x, U_{z'}^\dagger, U_{z'}, U_z, U_z^\dagger, U_y^\dagger\}\right)$$

K_{NLO} is the next-to-leading order correction to the dipole kernel and K4 and K6 are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.

Regularization of the rapidity divergence

Regularization by: slope (used in leading order)

$$U^{\eta}(x_\perp) = \mathrm{Pexp}\Big\{ig\int_{-\infty}^{\infty} du\; n_\mu\, A^\mu(un + x_\perp)\Big\} \qquad n^\mu = p_1^\mu + e^{-2\eta} p_2^\mu$$

Regularization of the rapidity divergence

Regularization by: slope (used in leading order)

$$U^{\eta}(x_\perp) = \text{Pexp}\Big\{ig\int_{-\infty}^{\infty} du\, n_\mu\, A^\mu(un + x_\perp)\Big\} \qquad n^\mu = p_1^\mu + e^{-2\eta} p_2^\mu$$

Regularization by: Rigid cut-off (used in NLO)

$$U_x^\eta = \text{Pexp}\Big[ig\int_{-\infty}^{\infty} du\, p_1^\mu A_\mu^\eta(up_1 + x_\perp)\Big]$$

$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

$k^\mu = \alpha_k p_1^\mu + \beta_k p_2^\mu + k_\perp^\mu$

The rigid cut-off leads to (almost) conformal result

Definition of the NLO kernel

The NLO kernel is obtained in the same way as the NLO DGLAP kernel:

1. Write down the general form of the operator equation

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + O(\alpha_s^3)$$

Definition of the NLO kernel

The NLO kernel is obtained in the same way as the NLO DGLAP kernel:

1. Write down the general form of the operator equation

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + O(\alpha_s^3)$$

$$\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} - \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + O(\alpha_s^3)$$

Definition of the NLO kernel

The NLO kernel is obtained in the same way as the NLO DGLAP kernel:

1. Write down the general form of the operator equation

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + O(\alpha_s^3)$$

$$\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = \frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} - \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + O(\alpha_s^3)$$

2. Calculate the “matrix element” of the r.h.s. in the shock-wave background

$$\langle\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle = \frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle - \langle\alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle + O(\alpha_s^3)$$

Definition of the NLO kernel

The NLO kernel is obtained in the same way as the NLO DGLAP kernel:

1. Write down the general form of the operator equation

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + O(\alpha_s^3)$$

$$\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} - \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + O(\alpha_s^3)$$

2. Calculate the “matrix element” of the r.h.s. in the shock-wave background

$$\langle\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle = \frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle - \langle\alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle + O(\alpha_s^3)$$

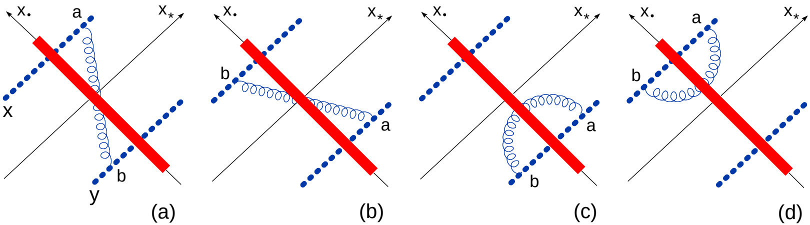
3. Subtract the LO contribution

$\Rightarrow$ $\left[\frac{1}{v}\right]_+$ prescription in the integrals over Feynman parameter v

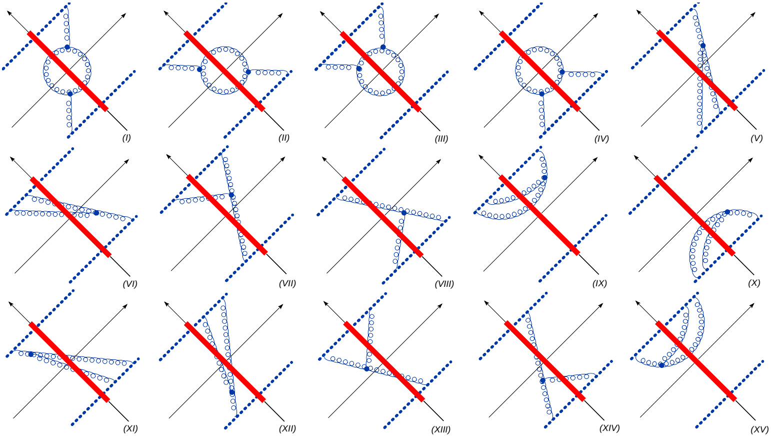
Typical integral

$$\int_0^1 dv\, \frac{1}{(k-p)_\perp^2 v + p_\perp^2(1-v)}\left[\frac{1}{v}\right]_+ = \frac{1}{p_\perp^2}\ln\frac{(k-p)_\perp^2}{p_\perp^2}$$

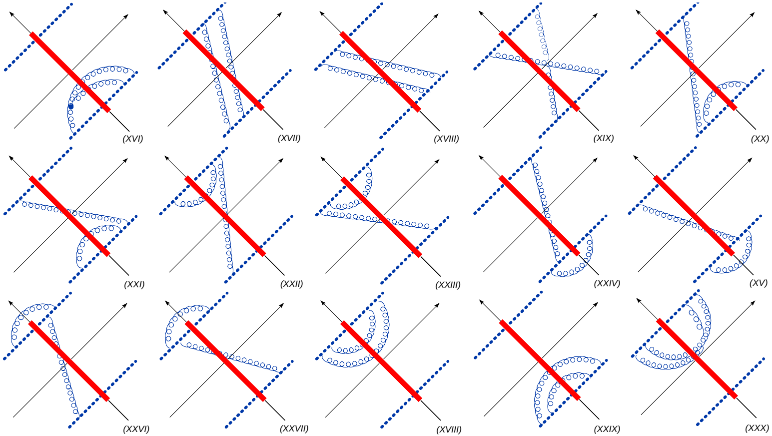
Add an extra gluon to the leading order diagrams



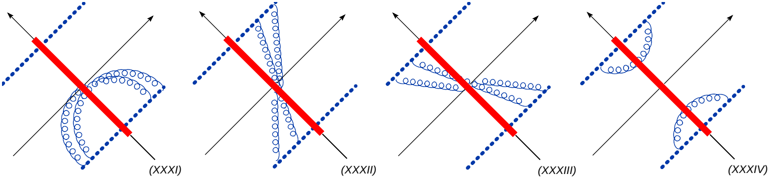
Diagrams with 2 gluons interaction



Diagrams with 2 gluons interaction

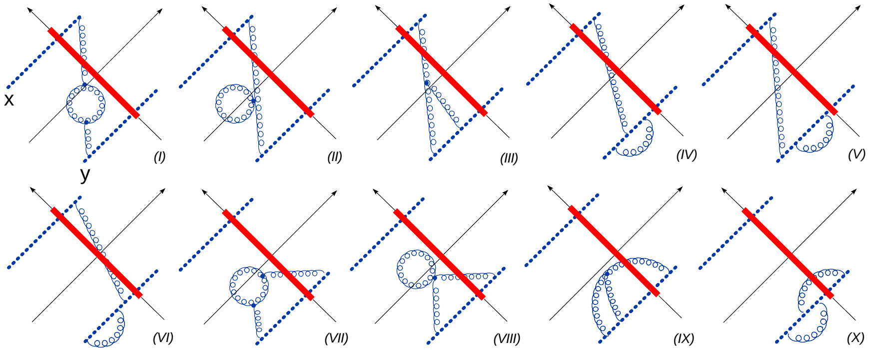


Diagrams with 2 gluons interaction

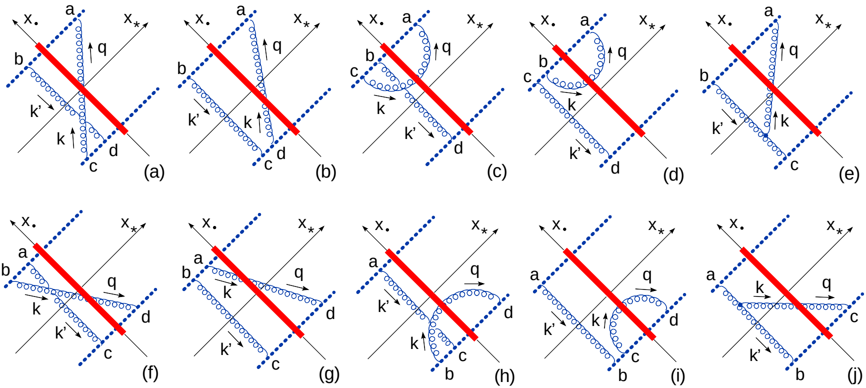


Diagrams of the NLO gluon contribution

"Running coupling" diagrams



$1 \rightarrow 2$ dipole transition diagrams



Gluon contribution to the NLO kernel (I. Balitsky and G.A.C, 2007)

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_x U_y^\dagger\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c\mathrm{Tr}\{U_x U_y^\dagger\}] \\
& \times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_s N_c}{4\pi}\big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\big)\Big] \\
& -\frac{11}{3}\frac{\alpha_s N_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} \;-\; \frac{\alpha_s N_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\} \\
& +\frac{\alpha_s^2}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_z U_{z'}^\dagger\} \\
& -(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big] \\
& +[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\mathrm{Tr}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\} - (z'\to z)] \\
& \times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big)
\end{aligned}
$$

Our result Agrees with NLO BFKL

(Comparing the eigenvalue of the forward kernel)

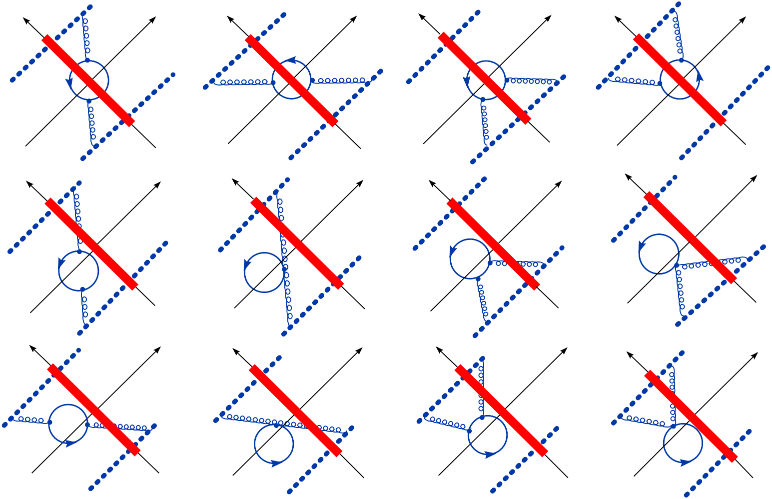
It respects unitarity

Gluon contribution to the NLO kernel (I. Balitsky and G.A.C, 2007)

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_x U_y^\dagger\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c\mathrm{Tr}\{U_x U_y^\dagger\}] \\
& \times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_s N_c}{4\pi}\big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\big)\Big] \\
& -\frac{11}{3}\frac{\alpha_s N_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\} \\
& +\frac{\alpha_s^2}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'} U_y^\dagger\} - \mathrm{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \\
& -(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big] \\
& +[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'} U_y^\dagger\} - \mathrm{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z'\to z)] \\
& \times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big)
\end{aligned}
$$

NLO kernel = Running coupling terms + Non-conformal term + Conformal term

$\mathcal{N}=4$ SYM diagrams (scalar and gluino loops)



Evolution equation for color dipoles in $\mathcal{N}=4$

(I. Balitsky and G.A.C. 2009)

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$
$$= \frac{\alpha_s}{\pi^2}\int d^2z_3\, \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big\{1-\frac{\alpha_s N_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}$$
$$\times\,[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]$$
$$-\frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\,\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}$$
$$\times\,\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4}-\hat U^\eta_{z_3})^{bb'}$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha<\sigma=e^{2\eta}$ in the rapidity of Wilson lines.

Evolution equation for color dipoles in $\mathcal{N}=4$

(I. Balitsky and G.A.C. 2009)

$$
\begin{aligned}
&\frac{d}{d\eta}\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\\
&= \frac{\alpha_s}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_sN_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}\\
&\times\,[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\\
&-\frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\,\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\\
&\times\,\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4}-\hat U^\eta_{z_3})^{bb'}
\end{aligned}
$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha<\sigma=e^{2\eta}$ in the rapidity of Wilson lines.

For the conformal composite dipole the result is Möbius invariant

Evolution equation for composite conformal dipoles in $\mathcal{N}=4$ SYM

$$[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf} = \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$
$$+\frac{\alpha_s}{4\pi}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\frac{1}{N_c}\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2}\;+\;O(\alpha_s^2)$$

$$\frac{d}{d\eta}\big[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\big]^{\rm conf}$$
$$=\frac{\alpha_s}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1-\frac{\alpha_sN_c}{4\pi}\frac{\pi^2}{3}\Big]\big[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\big]^{\rm conf}$$
$$-\frac{\alpha_s^2}{4\pi^4}\int d^2z_3d^2z_4\frac{z_{12}^2}{z_{13}^2z_{24}^2z_{34}^2}\Big\{2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big\}$$
$$\times\,\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}[(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4})^{bb'}-(z_4\to z_3)]$$

Now Möbius invariant!

NLO BFKL equation in $\mathcal{N}=4$ SYM

To find $A(x,y;x',y')$ we need the linearized (NLO BFKL) equation. With two-gluon accuracy

$$\hat{\mathcal{U}}^{\eta}(x,y) \;=\; 1-\frac{1}{N_c^2-1}\mathrm{Tr}\{\hat{U}_x^{\eta}\hat{U}_y^{\dagger\eta}\}$$

Conformal dipole operator in the BFKL approximation

$$\hat{\mathcal{U}}^{\eta}_{\rm conf}(z_1,z_2) \;=\; \hat{\mathcal{U}}^{\eta}(z_1,z_2)+\frac{\alpha_s N_c}{4\pi^2}\int d^2z\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2}[\hat{\mathcal{U}}^{\eta}(z_1,z_3)+\hat{\mathcal{U}}^{\eta}(z_2,z_3)-\hat{\mathcal{U}}^{\eta}(z_1,z_2)]$$

To find $A(x,y;x',y')$ we need the linearized (NLO BFKL) equation. With two-gluon accuracy

$$\hat{\mathcal{U}}^{\eta}(x,y) \;=\; 1-\frac{1}{N_c^2-1}\mathrm{Tr}\{\hat{U}^{\eta}_x\hat{U}^{\dagger\eta}_y\}$$

Conformal dipole operator in the BFKL approximation

$$\hat{\mathcal{U}}^{\eta}_{\mathrm{conf}}(z_1,z_2) \;=\; \hat{\mathcal{U}}^{\eta}(z_1,z_2)+\frac{\alpha_s N_c}{4\pi^2}\int d^2z\frac{z_{12}^2}{z_{13}^2z_{23}^2}\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2}[\hat{\mathcal{U}}^{\eta}(z_1,z_3)+\hat{\mathcal{U}}^{\eta}(z_2,z_3)-\hat{\mathcal{U}}^{\eta}(z_1,z_2)]$$

NLO BFKL

$$\begin{aligned}
&\frac{d}{d\eta}\hat{\mathcal{U}}^{\eta}_{\mathrm{conf}}(z_1,z_2)\\
&= \frac{\alpha_s N_c}{2\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1-\frac{\alpha_s N_c}{4\pi}\frac{\pi^2}{3}\Big][\hat{\mathcal{U}}^{\eta}_{\mathrm{conf}}(z_1,z_3)+\hat{\mathcal{U}}^{\eta}_{\mathrm{conf}}(z_2,z_3)-\hat{\mathcal{U}}^{\eta}_{\mathrm{conf}}(z_1,z_2)]\\
&+\frac{\alpha_s^2N_c^2}{8\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\,\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big\{2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big\}\hat{\mathcal{U}}^{\eta}_{\mathrm{conf}}(z_3,z_4)\\
&\qquad+\frac{3\alpha_s^2N_c^2}{2\pi^3}\zeta(3)\hat{\mathcal{U}}^{\eta}_{\mathrm{conf}}(z_1,z_2)
\end{aligned}$$

Eigenvalues agree with Kotikov and Lipatov (2000)

NLO evolution of composite "conformal" dipoles in QCD

$$\frac{d}{d\eta}[\mathrm{tr}\{\hat U_{z_1} U^\dagger_{z_2}\}]^{\mathrm{conf}} = \frac{\alpha_s}{2\pi^2}\int d^2z_3 \Big([\mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^\dagger_{z_2}\} - N_c \mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\}]^{\mathrm{conf}}$$

$$\times \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[1 + \frac{\alpha_s N_c}{4\pi}\Big(b\ln z_{12}^2\mu^2 + b\frac{z_{13}^2 - z_{23}^2}{z_{13}^2 z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2} + \frac{67}{9} - \frac{\pi^2}{3}\Big)\Big]$$

$$+ \frac{\alpha_s}{4\pi^2}\int \frac{d^2z_4}{z_{34}^4}\Big\{\Big[-2 + \frac{z_{14}{}^2 z_{23}^2 + z_{24}{}^2 z_{13}^2 - 4z_{12}^2 z_{34}^2}{2(z_{14}{}^2 z_{23}^2 - z_{24}{}^2 z_{13}^2)}\ln\frac{z_{14}{}^2 z_{23}^2}{z_{24}{}^2 z_{13}^2}\Big]$$

$$\times [\mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^\dagger_{z_4}\}\{\hat U_{z_4}\hat U^\dagger_{z_2}\} - \mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\hat U_{z_4}\hat U^\dagger_{z_2}\hat U_{z_3}\hat U^\dagger_{z_4}\} - (z_4 \to z_3)]$$

$$+ \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}{}^2}\Big[2\ln\frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2} + \Big(1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2}\Big)\ln\frac{z_{13}^2 z_{24}{}^2}{z_{14}{}^2 z_{23}^2}\Big]$$

$$\times [\mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^\dagger_{z_4}\}\mathrm{tr}\{\hat U_{z_4}\hat U^\dagger_{z_2}\} - \mathrm{tr}\{\hat U_{z_1}\hat U^\dagger_{z_4}\hat U_{z_3}\hat U^\dagger_{z_2}\hat U_{z_4}\hat U^\dagger_{z_3}\} - (z_4 \to z_3)]\Big\}$$

$$b = \tfrac{11}{3}N_c - \tfrac{2}{3}n_f$$

$K_{\mathrm{NLO\ BK}}$ = Running coupling part + Conformal "non-analytic" (in j) part + Conformal analytic ($\mathcal{N} = 4$) part

Linearized $K_{\mathrm{NLO\ BK}}$ reproduces the known result for the forward NLO BFKL kernel.

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.
- The NLO BK kernel in QCD and $\mathcal{N}=4$ SYM agrees with NLO BFKL eigenvalues.
- The NLO BK kernel for the evolution of conformal composite dipoles in $\mathcal{N}=4$ SYM is Möbius invariant in the transverse plane.
- The NLO BK kernel in QCD is a sum of the running-coupling part and conformal part.