Möbius representation of small-x evolution kernels at NLO

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Contents

Problem: NLO BFKL kernel in the coordinate space.

Collaborators: R.Fiore, A.V. Grabovsky, A. Papa.

Motivation:

Investigation of conformal properties

Understanding of relation between the BFKL kernel and the kernel of the colour dipole model

Hope for simplification.

Introduction

The NLO BFKL kernel was calculated long ago.

For the forward scattering (i.e. for $t = 0$ and color singlet in the t-channel) it is known more than 10 years

V.S. F., L.N. Lipatov, 1998,

M. Ciafaloni, G. Camicci, 1998.

Almost five years ago the kernel was found also for any fixed (not growing with energy) momentum transfer t and any possible color state in the t-channel

V.S. F., R. Fiore, 2005.

All these results were obtained in the momentum space.

Introduction

The BFKL approach is based on the remarkable property of QCD – gluon reggeization. The high-energy QCD can be reformulated in terms of the gauge-invariant effective field theory for reggeized gluon interactions,

L.N. Lipatov, 1995,

so that the primary reggeon in QCD is not the Pomeron, but the reggeized gluon.

The most interesting for phenomenological applications is the Pomeron (colour singlet) exchange. If scattering particles are colourless, it is described by the colour dipole model

N.N. Nikolaev and B.G. Zakharov, 1994,

A. H. Mueller, 1994,

which is formulated in the impact parameter space.

Introduction

Just for scattering of colourless particles the LO BFKL kernel has the remarkable property: it can be written in the Möbius representation, which is invariant in regard to the conformal transformations of the transverse coordinates
L.N. Lipatov, 1989.

The conformal invariance has great consequences. It permits to find all eigenfunction of the kernel and to the write explicitly the Green function of the BFKL equation in the coordinate space.

It was extremely interesting to find the Möbius representation of the BFKL kernel in the NLO in order to check conformal properties.

Evidently, the conformal invariance is violated by renormalization. One may wonder, however, whether the renormalization is the only source of the violation. If so, one could expect the conformal invariance of the NLO BFKL kernel in N=4 supersymmetric Yang-Mills theory (N=4 SUSY).

The direct way of finding of the Möbius representation of the BFKL kernel is to transform it from momentum to coordinate space.

Such transformation permits also to perform explicit comparison of the Möbius representation of the BFKL kernel and the kernel of the colour dipole model.

Introduction

In the LO such transformation makes evident the conformal invariance of the Möbius representation of the BFKL kernel and coincidence of this representation with the kernel of the colour dipole approach

V.F., R. Fiore, A. Papa, 2006.

Starting from the papers

Yu.V. Kovchegov, H. Weigert, 2006

I. Balitsky, 2006

the comparison became possible also in the NLO for the quark contribution to the kernel.

In the NLO one could also expect coincidence of the Möbius representation of the BFKL kernel and the kernel of the colour dipole approach.

However, the situation is not so simple.

The NLO kernels are not unambiguously defined.

The ambiguity of the NLO kernels is analogous to the ambiguity of the NLO anomalous dimensions. It is caused by the possibility to redistribute radiative corrections between the kernels and the impact factors.

The Möbius form of the NLO BFKL kernel was interesting also from the point of view of searches of a simple representation of the NLO BFKL kernel.

In the momentum representation the kernel is rather complicated. The colour singlet kernel for $t \neq 0$ is found in the NLO in the form of the intricate two-dimensional integral. Its simplification was extremely desirable.

Möbius representation of the BFKL kernel

For colourless objects the impact factors in the representation

$$\delta(\vec{q}_A - \vec{q}_B) disc_s \mathcal{A}_{AB}^{A'B'} = \frac{i}{4(2\pi)^{D-2}} \langle A'\bar{A}| e^{Y\widehat{\mathcal{K}}} \frac{1}{\hat{\vec{q}}_1^{\,2} \hat{\vec{q}}_2^{\,2}} |\bar{B}'B\rangle$$

are "gauge invariant":

$$\langle A'\bar{A}|\vec{q}, 0\rangle = \langle A'\bar{A}|0, \vec{q}\rangle = 0 \; .$$

Therefore $\langle A'\bar{A}|\Psi\rangle = 0$ if $\langle \vec{r}_1, \vec{r}_2|\Psi\rangle$ does not depend either on $\vec{r}_1$ or on $\vec{r}_2$.
$\langle A'\bar{A}|\widehat{\mathcal{K}}$ is "gauge invariant" as well, because $\langle \vec{q}_1, \vec{q}_2|\widehat{\mathcal{K}}_r|\vec{q}_1^{\,'}, \vec{q}_2^{\,'}\rangle$ vanishes at $\vec{q}_1^{\,'} = 0$ or $\vec{q}_2^{\,'} = 0$.
It means that we can change $|In\rangle \equiv \left(\hat{\vec{q}}_1^{\,2} \hat{\vec{q}}_2^{\,2}\right)^{-1} |\bar{B}'B\rangle$ for $|In_d\rangle$, where $|In_d\rangle$ has the "dipole " property $\langle \vec{r}, \vec{r}|In_d\rangle = 0$.
After this one can omit the terms in the kernel proportional to $\delta(\vec{r}_{1'2'})$, as well as change the terms independent either of $\vec{r}_1$ or of $\vec{r}_2$ in such a way that the resulting kernel becomes conserving the "dipole " property.

Möbius representation of the BFKL kernel

The kernel obtained in this way is called Möbius form of the BFKL kernel. It can be written as

$$\langle\vec{r}_1\vec{r}_2|\hat{\mathcal{K}}_M|\vec{r}_1{}'\vec{r}_2{}'\rangle = \delta(\vec{r}_{11'})\delta(\vec{r}_{22'})\int d\vec{r}_0\, g_0(\vec{r}_1,\vec{r}_2;\vec{r}_0)$$

$$+\delta(\vec{r}_{11'})g_1(\vec{r}_1,\vec{r}_2;\vec{r}_2{}') + \delta(\vec{r}_{22'})g_1(\vec{r}_2,\vec{r}_1;\vec{r}_1{}') + \frac{1}{\pi}g_2(\vec{r}_1,\vec{r}_2;\vec{r}_1{}',\vec{r}_2{}')$$

with the functions $g_{1,2}$ turning into zero when their first two arguments coincide. The first three terms contain ultraviolet singularities which cancel in their sum, as well as in the LO, with account of the "dipole" property of the "target" impact factors. The coefficient of $\delta(\vec{r}_{11'})\delta(\vec{r}_{22'})$ is written in the integral form in order to make the cancellation evident.

The term $g(\vec{r}_1,\vec{r}_2;\vec{r}_1{}',\vec{r}_2{}')$ is absent in the LO because the LO kernel in the momentum space does not contain terms depending on all three independent momenta simultaneously.

Möbius representation of the BFKL kernel

In the LO $g_1(\vec{r}_1, \vec{r}_2; \vec{r}_0) = -\,g_0(\vec{r}_1, \vec{r}_2; \vec{r}_0) = \frac{\alpha_s N_c}{2\pi^2} \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{10}^{\,2} \vec{r}_{20}^{\,2}},$ so that the $\langle \vec{r}_1 \vec{r}_2 | \hat{\mathcal{K}}_M | \vec{r}_1{}' \vec{r}_2{}' \rangle$ coincides with the colour dipole kernel and is explicitly conformal invariant.

In QCD the NLO kernel contains quark and gluon contributions. In ones turn, the quark contribution is divided into two pieces: non-Abelian" (leading in N_c) and Abelian" (suppressed by N_c^{-2}). Their Möbius forms

V.S. F., R. Fiore, A. Papa, 2006, 2007

agrees, with account of the ambiguity of the kernel, with the results

V.V. Kovchegov, H. Weigert, 2006,

I. Balitsky, 2006,

obtained by direct calculation in the dipole picture. The Abelian part is greatly simplified in comparison with the momentum representation. Moreover, this part is conformal invariant. It could be important for the QED Pomeron.

Möbius representation of the BFKL kernel

The most important contribution to the BFKL kernel is the gluon one. In the momentum representation in the NLO for arbitrary momentum transfer it is very complicated
V.S. F, R. Fiore 2005.
The Möbius form of this contribution
V.S. F, R. Fiore, A.V. Grabovsky, A. Papa, 2007
turned out strikingly simple.

However, the conformal invariance is broken not only by the terms related to the renormalization.

Moreover, it occurred afterwards that the NLO gluon contribution to the kernel of the colour dipole approach
I. Balitsky, G.A. Chirilli, 2008
does not agree with the Möbius form of the same contribution to the BFKL kernel.

Möbius representation of the BFKL kernel

Supersymmetric Yang-Mills theories contain gluons and Maiorana fermions in the adjoint representation of the colour group. The gluon contribution is the same as in QCD. The fermion one can be obtained by change of the group coefficients: $n_f \to n_M N_c$ for the "non-Abelian" part, and $n_f \to -n_M N_c^3$ for the "Abelian" part; n_M is the number of flavours of Maiorana quarks. For N–extended SUSY $n_M = N$.

At $N > 1$ besides quarks there are n_S scalar particles; $n_S = 2$ at $N = 2$ and $n_S = 6$ at $N = 4$. At $N = 4$ $\beta_0 = \frac{11}{3} - \frac{2}{3}n_M - \frac{1}{6}n_S = 0$ and α_s is not running. Nevertheless, the Möbius form of the NLO kernel
V.S. F, R. Fiore, 2007
is not conformal invariant.

Möbius representation of the BFKL kernel

In the theory with n_M Maiorana fermions and n_S scalars in the adjoint representation we have

$$g_1(\vec{r}_1,\vec{r}_2;\vec{r}_2') = \frac{\alpha_s(\frac{4e^{-2C}}{\vec{r}^{\,2}})N_c}{2\pi^2}\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{22'}^{\,2}\vec{r}_{12'}^{\,2}}\left[1+\frac{\alpha_s N_c}{2\pi}\left(\frac{67}{18}-\zeta(2)-\frac{5n_M}{9}-\frac{2n_S}{9}\right.\right.$$

$$\left.\left.+\frac{\beta_0}{2N_c}\frac{\vec{r}_{12'}^{\,2}-\vec{r}_{22'}^{\,2}}{\vec{r}_{12}^{\,2}}\ln\left(\frac{\vec{r}_{22'}^{\,2}}{\vec{r}_{12'}^{\,2}}\right)-\frac{1}{2}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{22'}^{\,2}}\right)\ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{12'}^{\,2}}\right)+\frac{\vec{r}_{12'}^{\,2}}{2\vec{r}_{12}^{\,2}}\ln\left(\frac{\vec{r}_{12'}^{\,2}}{\vec{r}_{22'}^{\,2}}\right)\ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{12'}^{\,2}}\right)\right)\right].$$

Since only the integral of g^0 is fixed, it can be written in different forms. One of them is

$$g_0(\vec{r}_1,\vec{r}_2;\vec{r}_0) = -g(\vec{r}_1,\vec{r}_2;\rho)+\frac{\alpha_s^2N_c^2}{4\pi^3}\delta(\vec{r}_0)2\pi\zeta(3)\,.$$

The function $g_1(\vec{r}_1,\vec{r}_2;\vec{\rho})$ vanish at $\vec{r}_1=\vec{r}_2$. Then, these functions turn into zero for $\vec{\rho}^{\,2}\to\infty$ faster than $(\vec{\rho}^{\,2})^{-1}$ to provide the infrared safety. The ultraviolet singularities of this function at $\vec{\rho}=\vec{r}_2$ and $\vec{\rho}=\vec{r}_1$ cancel with the singularities of $g^0(\vec{r}_1,\vec{r}_2;\vec{\rho})$ on account of the “dipole” property of the “target” impact factors.

Möbius representation of the BFKL kernel

$$g_2(\vec r_1,\vec r_2;\vec r_1\,',\vec r_2\,') = \frac{\alpha_s^2 N_c^2}{4\pi^3}\Bigg[\frac{1}{2\vec r^{\,4}_{1'2'}}\left(\frac{\vec r^{\,2}_{12'}\,\vec r^{\,2}_{21'}}{d}\ln\left(\frac{\vec r^{\,2}_{12'}\,\vec r^{\,2}_{21'}}{\vec r^{\,2}_{11'}\vec r^{\,2}_{22'}}\right)-1\right)\left(1-n_M+\frac{n_S}{2}\right)$$

$$-\left(\frac{(4-n_M)}{4\vec r^{\,4}_{1'2'}}\frac{\vec r^{\,2}_{12}\,\vec r^{\,2}_{1'2'}}{d}-\frac{1}{4\vec r^{\,2}_{11'}\,\vec r^{\,2}_{22'}}\left(\frac{\vec r^{\,4}_{12}}{d}-\frac{\vec r^{\,2}_{12}}{\vec r^{\,2}_{1'2'}}\right)\right)\ln\left(\frac{\vec r^{\,2}_{12'}\,\vec r^{\,2}_{21'}}{\vec r^{\,2}_{11'}\vec r^{\,2}_{22'}}\right)$$

$$+\frac{\ln\left(\frac{\vec r^{\,2}_{12}}{\vec r^{\,2}_{1'2'}}\right)}{4\vec r^{\,2}_{11'}\vec r^{\,2}_{22'}}+\frac{\ln\left(\frac{\vec r^{\,2}_{12}\,\vec r^{\,2}_{1'2'}}{\vec r^{\,2}_{11'}\vec r^{\,2}_{22'}}\right)}{2\vec r^{\,2}_{12'}\vec r^{\,2}_{21'}}\left(\frac{\vec r^{\,2}_{12}}{2\vec r^{\,2}_{1'2'}}+\frac{1}{2}-\frac{\vec r^{\,2}_{22'}}{\vec r^{\,2}_{1'2'}}\right)+\frac{\vec r^{\,2}_{12}\ln\left(\frac{\vec r^{\,2}_{12}\vec r^{\,2}_{1'2'}}{\vec r^{\,2}_{12'}\,\vec r^{\,2}_{21'}}\right)}{4\vec r^{\,2}_{11'}\vec r^{\,2}_{22'}\vec r^{\,2}_{1'2'}}$$

$$+\frac{\ln\left(\frac{\vec r^{\,2}_{22'}}{\vec r^{\,2}_{12}}\right)}{2\vec r^{\,2}_{11'}\,\vec r^{\,2}_{12'}}+\frac{\ln\left(\frac{\vec r^{\,2}_{12}\vec r^{\,2}_{1'2'}}{\vec r^{\,2}_{12'}\vec r^{\,2}_{22'}}\right)}{2\vec r^{\,2}_{11'}\,\vec r^{\,2}_{1'2'}}+\frac{\ln\left(\frac{\vec r^{\,2}_{12}\vec r^{\,2}_{11'}}{\vec r^{\,2}_{22'}\,\vec r^{\,2}_{1'2'}}\right)}{2\vec r^{\,2}_{12'}\vec r^{\,2}_{1'2'}}$$

$$+\frac{\vec r^{\,2}_{12}\ln\left(\frac{\vec r^{\,2}_{11'}}{\vec r^{\,2}_{1'2'}}\right)}{2\vec r^{\,2}_{11'}\vec r^{\,2}_{12'}\,\vec r^{\,2}_{22'}}+\frac{\vec r^{\,2}_{21'}\ln\left(\frac{\vec r^{\,2}_{21'}\vec r^{\,2}_{1'2'}}{\vec r^{\,2}_{12}\,\vec r^{\,2}_{11'}}\right)}{2\vec r^{\,2}_{11'}\vec r^{\,2}_{22'}\vec r^{\,2}_{1'2'}}+(1\leftrightarrow 2)\Bigg],\quad d=\vec r^{\,2}_{12'}\vec r^{\,2}_{21'}-\vec r^{\,2}_{11'}\vec r^{\,2}_{22'}.$$

Ambiguities of the kernel

However, the hope for conformal invariance still remained.

The reason is the ambiguity of the NLO kernel.

The BFKL kernel has an evident ambiguity connected with impact factors.

The discontinuity

$$\langle A'\bar{A}|e^{Y\hat{\mathcal{K}}}\frac{1}{\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}}|\bar{B}'B\rangle$$

remains intact under the transformation

$$\hat{\mathcal{K}}\to\hat{\mathcal{O}}^{-1}\hat{\mathcal{K}}\hat{\mathcal{O}}\;,\;\;\langle A'\bar{A}|\to\langle A'\bar{A}|\hat{\mathcal{O}}\;,\;\;\frac{1}{\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}}|\bar{B}'B\rangle\to\hat{\mathcal{O}}^{-1}\frac{1}{\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}}|\bar{B}'B\rangle.$$

If the LO kernel is fixed, one can take $\hat{\mathcal{O}}=1-\alpha_s\hat{U}$, and get

$$\hat{\mathcal{K}}\to\hat{\mathcal{K}}-\alpha_s[\hat{\mathcal{K}}^{(B)},\hat{U}].$$

Secondly, there is a freedom in the energy scale s_0. At first sight, it can lead to an additional ambiguity of the NLO kernel. However, it is not so.

Ambiguities of the kernel

It was shown

V.F., 1986

that any change of the energy scale can be compensated by the corresponding redefinition of the impact factors. Alternatively, we can leave one of the impact factored unchanged, changing the kernel. In this case the change will have the form $\widehat{\mathcal{K}} \to \widehat{\mathcal{K}} - \alpha_s[\widehat{\mathcal{K}}^B \hat{U}]$ with a specific form of the operator $\hat{U}$

V. S. F., R. Fiore, A. V. Grabovsky, 2009.

Therefore, the freedom in a choice of the energy scale does not give anything new. In the NLO dependence on s_0 of the energy factor is cancelled by the dependence of the impact factors, so that s_0 can be taken as a free parameter. This freedom can be used for optimization of perturbative results

D.Yu. Ivanov, A. Papa, 2006.

Elimination of the discrepancy

To get a hint on possible form of the transformation which can eliminate the discrepancy with BC-2008 the forward scattering was considered V. S. F., R. Fiore, A. V. Grabovsky, 2009.

It was shown that in this case the discrepancy can be removed by the transformation

$$\widehat{\mathcal{K}} \to \widehat{\mathcal{K}} + \frac{1}{2}[\widehat{\mathcal{K}}^B, \ln \hat{\vec{q}}^{\,2} \widehat{\mathcal{K}}^B] \ ,$$

up to the term with $\zeta(3)$ and to the difference in the renormalization scales. In the BFKL approach the term with $\zeta(3)$ passed through a great number of verifications. In particular, it is necessary for fulfillment of the bootstrap conditions for the gluon reggeization. We have no doubt that this term is correct.

Fortunately, it was recognized

I. Balitsky, G.A. Chirilli, 2009

that there was an error in their calculation of this term.

Elimination of the discrepancy

In principle, one can easily write a formal expression for the operator $\hat{U}$ eliminating the discrepancy. Indeed, let us denote $\hat{\mathcal{K}}_M - \hat{\mathcal{K}}_{BC} = \hat{\Delta}$, the Born kernel $\hat{\mathcal{K}}^B$ eigenstates $|\mu\rangle$, and corresponding eigenvalues ω_μ^B. Then, if $\hat{\Delta} = \alpha_s \left[\hat{\mathcal{K}}^B, \hat{U}\right]$, one has

$$\left(\omega_{\mu'}^B - \omega_\mu^B\right) \langle\mu'|\alpha_s \hat{U}|\mu\rangle = \langle\mu'|\hat{\Delta}|\mu\rangle.$$

It is seen from here that the operator $\hat{U}$ exists only if the operator $\hat{\Delta}$ has zero matrix elements between states of equal energies. If so, supposing that the states $|\mu\rangle$ form a complete set, one has

$$\langle\mu'|\alpha_s \hat{U}|\mu\rangle = \sum_{\mu,\mu'} \frac{|\mu'\rangle\langle\mu'|\hat{\Delta}|\mu\rangle\langle\mu|}{\omega_{\mu'}^B - \omega_{\mu'}^B}.$$

Elimination of the discrepancy

Finally, it was found quite recently

V.S. F., R.Fiore, A.V. Grabovsky, 2009

that with account of the error in the $\zeta(3)$ term and the difference of the renormalization scheme used by Balitsky and Chirilli from the $\overline{MS}$ one, their result agree with the Möbius form of the BFKL kernel. The agreement is reached by the transformation

$$\widehat{\mathcal{K}} \to \widehat{\mathcal{K}} - \alpha_s [\widehat{\mathcal{K}}^B \hat{U}]$$

with

$$\langle \vec{q}_1, \vec{q}_2 | \alpha_s \hat{U} | \vec{q}_1{}', \vec{q}_2{}' \rangle = -\delta(\vec{q}_{11'} + \vec{q}_{22'}) \frac{\mathcal{K}_r^B(\vec{q}_1, \vec{q}_1{}'; \vec{q})}{2\vec{q}_1{}^2 \vec{q}_2{}^2} \ln \vec{q}_{11'}^{\,2}$$

$$+ \frac{\alpha_s N_c}{4\pi^2} \delta(\vec{q}_{22'}) \delta(\vec{q}_{11'}) \int d^{2+2\epsilon} k \ln \vec{k}^{\,2} \left(\frac{2}{\vec{k}^{\,2}} - \frac{\vec{k}(\vec{k} - \vec{q}_1)}{\vec{k}^{\,2}(\vec{k} - \vec{q}_1)^2} - \frac{\vec{k}(\vec{k} - \vec{q}_2)}{\vec{k}^{\,2}(\vec{k} - \vec{q}_2)^2} \right).$$

Restoration of conformal invariance in N=4 SUSY

Moreover, an additional transformation (analogous to one used in BC-2009)

$$\widehat{\mathcal{K}} \to \widehat{\mathcal{K}} - [\widehat{\mathcal{K}}^B \hat{U}_1]$$

with

$$\langle \vec{r}_1 \vec{r}_2 | \hat{U}_1 | \vec{r}_1{}' \vec{r}_2{}' \rangle = \frac{\alpha_s N_c}{4\pi^2} \int d\vec{r}_0 \frac{\vec{r}_{12}{}^2}{\vec{r}_{10}^{\,2} \vec{r}_{20}^{\,2}} \ln\left(\frac{\vec{r}_{12}{}^2}{\vec{r}_{10}^{\,2} \vec{r}_{20}^{\,2}} \right)$$

$$\times \Big[\delta(\vec{r}_{11'})\delta(\vec{r}_{2'0}) + \delta(\vec{r}_{1'0})\delta(\vec{r}_{22'}) - \delta(\vec{r}_{11'})\delta(r_{22'}) \Big]$$

removes the pieces on the Möbius form of the BFKL kernel, which are not related to the renormalization, but nevertheless are non-conformal. Therefore, in N=4 SUSY it makes the kernel conformal invariant.

Summary

- In the case of scattering of colourless objects the BFKL kernel can be written in the Möbius form.
- The Möbius form is greatly simplified in comparison with the BFKL kernel in the momentum representation.
- There was an evident discrepancy between the Möbius form of the BFKL kernel and the BC kernel.
- It was recognized that the discrepancy can be removed by the ambiguities of the NLO kernels.
- The ambiguity is caused by the possibility to redistribute radiative corrections between the kernels and the impact factors.

Summary

- Now it is proved that this ambiguity permits to match the Möbius form of BFKL kernel and the BC kernel and to construct the Möbius invariant NLO BFKL kernel in $N = 4$ SUSY.