The Top quark mass at the LHC

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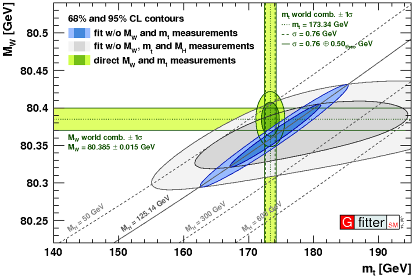
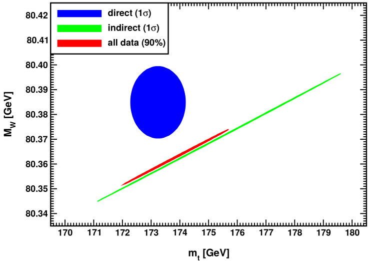
Outline

- Top, precision physics, vacuum stability
- Current measurements
- Theoretical work
- Theoretical issues on the top mass measurements:
- The mass renormalon
- Measurements from the reconstructed mass
- A study with Monte Carlo of increasing accuracy

Top and precision physics



From PDG:

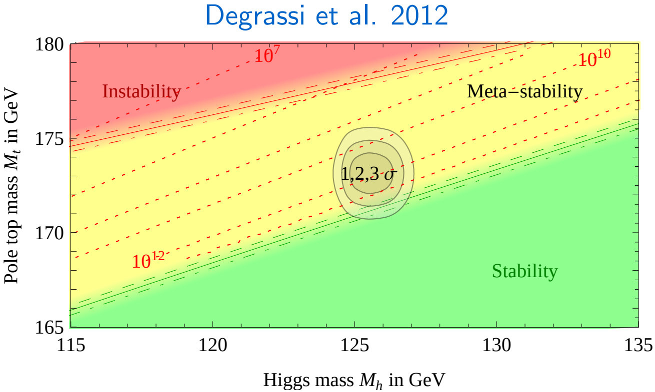


$$\Delta G_\mu / G_\mu = 5 \cdot 10^{-7}; \quad \Delta M_Z / M_Z = 2 \cdot 10^{-5};$$

$$\Delta\alpha(M_Z)/\alpha(M_Z) = \begin{cases} 1 \cdot 10^{-4} (\text{Davier et al.; PDG}) \\ 3.3 \cdot 10^{-4} (\text{Burkhardt, Pietrzyk}) \end{cases}$$

M_W can be predicted from the above with high precision, provided M_H and M_T (entering radiative corrections) are also known (and depending on how aggressive is the error on $\alpha(M_Z)$).

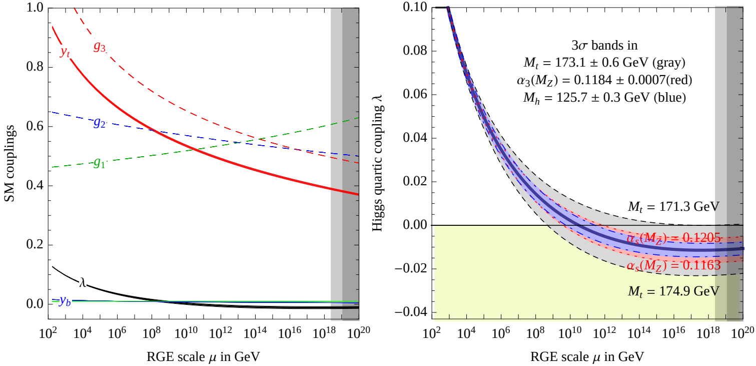
Top and vacuum stability



With current value of M_t and M_H the vacuum is metastable.
No indication of new physics up to the Plank scale from this.

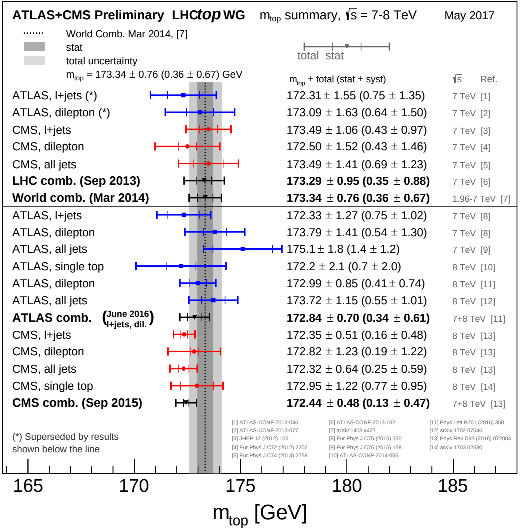
Top and vacuum stability

Degrassi et al. 2012



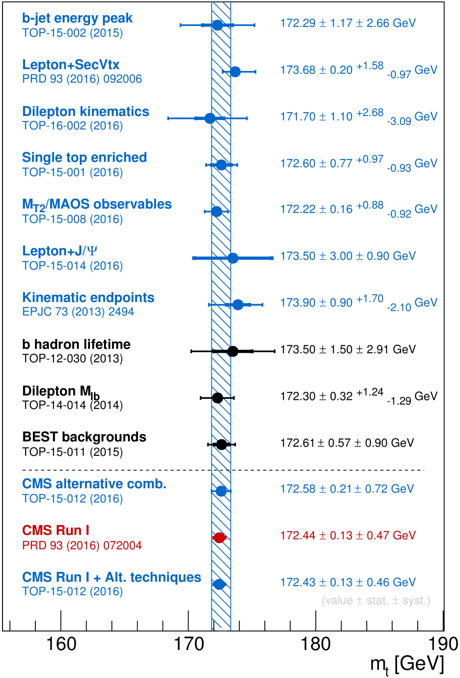
The quartic coupling λ_H becomes tiny at very high field values, and may turn negative, leading to vacuum instability.
M_t as low as 171 GeV leads to $\lambda_H \to 0$ at the Plank scale.

Top Mass Measurements



From kinematic reconstruction (In essence, from the mass of the system of decay products).

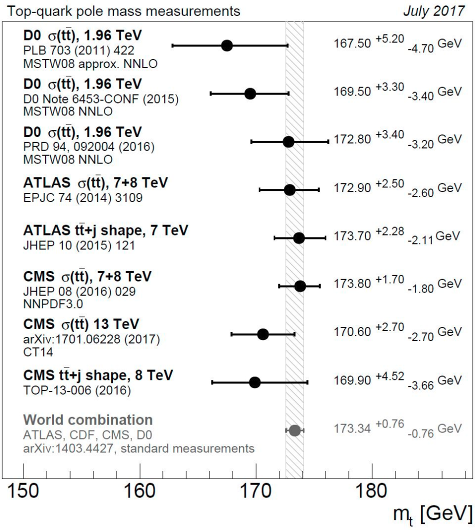
The most precise method as of now.



Several methods explored by CMS (see PAS TOP-15-012).

Notice: they do not increase precision with respect to PRD 93 (2016) 072004, but show amazing consistency.

From total cross section and $t\bar{t}j$ kinematics



Sometimes quoted as "pole mass measurement"
(but also the others are ...)

Theoretical work

Proposal for alternative mass-sensitive observables:

- Butenschoen,Dehnadi,Hoang,Mateu,Preisser,Stewart,2016 Use boosted top jet mass + SCET.
- Agashe,Franceschini,Kim,Schulze,2016: peak of b-jet energy insensitive to production dynamics.
- Kawabata,Shimizu,Sumino,Yokoya,2014: shape of lepton spectrum. Insensitive to production dynamics and reduced sensitivity to strong interaction effects.
- Frixione, Mitov: Use only lepton observables.
- Alioli, Fernandez, Fuster, Irles, Moch, Uwer, Vos ,2013; Bayu etal: M_t from $t\bar{t}j$ kinematics.

NLO and NNLO, PS+NLO results relevant to the top mass

- Narrow width $t\bar{t}$ production and decay at NLO, Bernreuther,Brandenbourg,Si,Uwer 2004, Melnikov,Schulze 2009.
- $l\nu l\nu b\bar{b}$ final states with massive b, Frederix, 2013, Cascioli,Kallweit,Maierhöfer,Pozzorini, 2013.
- NNLO differential top decay, Brucherseifer,Caola,Melnikof 2013.
- NNLO production, Czakon,Heymes,Mitov,2015.
- $l\nu l\nu b\bar{b} + \mathrm{jet}$ Bevilacqua,Hartanto,Kraus,Worek 2016.
- Approx. NNLO in production and exact NNLO in decay for $t\bar{t}$. Gao,Papanastasiou 2017.
- Resonance aware formalism for NLO+PS: Ježo,P.N. 2015;
- Off shell + interference effects+PS, Single top, Frederix,Frixione,Papanastasiou,Prestel,Torielli, 2016
- Off shell + interference effects+PS, $l\nu l\nu b\bar{b}$, Jeo,Lindert,Oleari,Pozzorini,P.N., 2016.

Issues about the top mass

Heavily debated theory issues about the top mass measurement. The problem is:

- There is no particle level definition of the system of decay products of the top.
- We use theoretical tools to compute mass sensitive distributions and extract the top mass parameter by fitting the computed distributions to measured data.

It has been argued that the mass used in the Monte Carlo does not bear a clear relation to a well defined field theoretical parameter (see "pole mass measurements" vs. "direct measurements").

- Assuming that it does: pole mass renormalon problem.

Reminder

The relation of the pole mass m_p to the $\overline{\mathrm{MS}}$ mass m is (Marquard,A.V.Smirnov,V.A.Smirnov,Steinhauser, 2015)

$m_p = m(1 + 0.4244\alpha_s + 0.8345\alpha_s^2 + 2.375\alpha_s^3 + (8.49 \pm 0.25)\alpha_s^4)$
(typical size: $m_p = m + 7.557 + 1.617 + 0.501 + 0.195$ GeV).

It was pointed out (P.N., TOP2015) that high order terms match well the asymptotic formula (Beneke,Braun,1994; Beneke 1994)

$$r_n \to N\, m_t (2b_0)^n \Gamma(n+1+b) \left(1 + \sum_{k=1}^{\infty} \frac{s_k}{n^k}\right), \qquad b = \frac{b_1}{b_0^2},$$

so that N can be estimated, and a more refined formulation of the mass relation can be carried out.
(Notice the factorial growth due to the IR renormalon.)

Mass renormalon

The asymptotic nature of the relation between the $\overline{\mathrm{MS}}$ and the pole mass leads to an irreducible ambiguity the order of typical hadronic scales.

Some authors have quoted an ambiguity of 1 GeV (Hoang, 2014).

Recent calculations give much smaller results:

- Beneke, Marquard, Steinhauser, P.N. 2016:
  - V1: 70 MeV;
  - V2 (9 Jun 2017): 110 MeV.

  V2: bottom and charm mass effects accounted for.
- Hoang, Lepenik, Preisser, 26 Jun 2017: 250 MeV.

Below currently quoted errors.

Mass Renormalon: size of the ambiguity

The Pole Mass m_P is given in terms of the $\overline{\mathrm{MS}}$ mass m by an expansion of the form

$$m_P = m + N\alpha_s \sum_{n=0}^{\infty} c_n(\mu, m)\alpha_s^n. \tag{1}$$

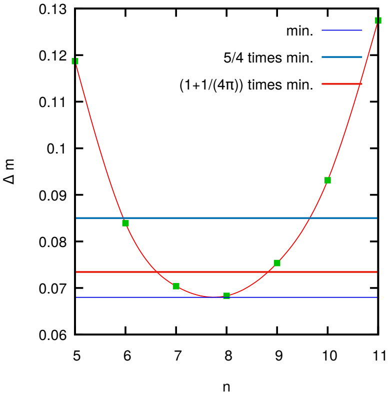
The coefficients grow as the factorial of n. Can also be written as

$$m_P = m + N \int_0^{\infty} \mathrm{d}r\, e^{-\frac{r}{\alpha_s}} \sum_{n=0}^{\infty} \frac{c_n(\mu, m)}{n!} r^n. \tag{2}$$

$c_n \propto n! \longrightarrow c_n/n! \propto \mathrm{const.}$, i.e.: geometric divergence for some r.

Prescription used by Beneke etal: take the principal value of the integral as its central value, and (the absolute value of) its imaginary part divided by Pi as the estimate of the error.

Mass Renormalon: size of the ambiguity



Hoang etal prescription:

Take as error half of the sum of all terms that do not exceed the smallest term by more than a factor f.

f is defined to be "a number larger but close to unity" and $f = 5/4 = 1 + 0.25$ is chosen.

It is not difficult to make contact among the two procedures, and show that the Beneke etal method roughly corresponds to the above with $f = 1 + 1/(4\pi)$ (which explains a good part of the difference).

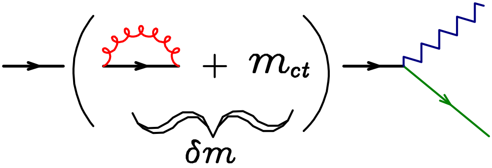
Mass Renormalon: size of the ambiguity

It is clear that the choice of f in Hoang etal, as well as the choice of the factor in front of Im/Pi in Beneke etal, are rather arbitrary (and slightly reminiscent of scale variation issues).

The motivation for the Im/Pi choice in Beneke etal is that it works well in context where the renormalon effect can be related to some physical observable (Beneke 1999).

In all cases, the message is: the renormalon problem cannot be used as an excuse to abandon pole mass measurements.

Reconstructed mass measurements



If we DO NOT use the pole mass, the term in the round bracket differs from zero near the mass peak. This leads to an NLO correction of the form

$$1 + \delta m \frac{\partial}{\partial m} \tag{3}$$

to be applied to the amplitude, i.e. a shift in mass.

Thus, even when using LO Monte Carlo, we better think of it as using the pole mass, as far as measurements of the mass of the decay products are concerned.

Reconstructed mass measurements

Still, one may wonder about corrections that the Monte Carlo includes, and that are only at leading order or leading logs, as are multiple radiations that can affect the *b* jet shape.

However:

- There are ways to estimate these perturbative errors within a Monte Carlo, and include them in the error estimate.
- There are Monte Carlo with increasing accuracy, where at least the hardest radiation is NLO accurate (i.e. MC@NLO and POWHEG), using the pole mass parameter.

Generators for $t\bar{t}$

- `MC@NLO` Frixione,Webber,P.N. and `POWHEG-hvq` Frixione,Ridolfi,P.N.. Include NLO radiation in production.
  `hvq`: `User-Processes-V2/hvq`
- The above with Shower Monte Carlo that do MEC corrections to top decay (Pythia8, Herwig7).
- `tt̄_dec` Campbell,Ellis,Re,P.N.. Includes exact spin correlations and NLO corrections in decay in NWA.
  `User-Processes-V2/ttb_NLO_dec`
- `bb̄4l` Ježo,Lindert,Nason,Oleari,Pozzorini,P.N. 2016 Includes exact NLO matrix element for $pp \to l\bar{\nu}_l\bar{\ell}\nu_\ell b\bar{b}$, thus finite width effects and interference between radiation in production and decay is included.
  `User-Processes-RES/b_bbar_4l`

A study with generators of increasing accuracy

(Ferrario-Ravasio,Ježo,Oleari,P.N.)

- We focus upon the $pp \to l\bar{\nu}_l\bar{\ell}\nu_\ell b\bar{b}$ process. Can be studied with the `hvq`, `tt̄_dec`, and `bb̄4l` generators.
- We make the simplifying assumption that the W can be fully reconstructed.
- We consider the top mass determination from mass distribution of the system comprising the W and a (charge matched) b jet, the b-jet energy spetrum, and the leptonic observables proposed by Frixione and Mitov.
- We study the effect of scale variation, PDF and α_s sensitivity, and the differences between the Pythia8 and Herwig7 shower interface.

m_{W-bj}

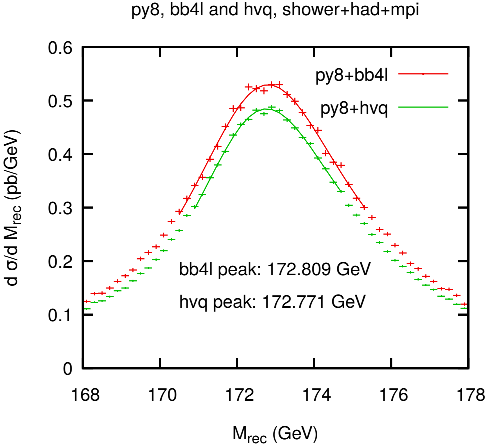
$W - bj$ is defined in the following way:

- Jets are defined using the anti-k_T algorithm with $R = 0.5$. The $b/\bar{b}$ jet is defined as the jet containing the hardest $b/\bar{b}$.
- $W^{\pm}$ is defined as the hardest $l^{\pm}$ paired with the hardest matching neutrino.
- The $W - bj$ system is obtained by matching a $W^{+/-}$ with a $b/\bar{b}$ jet (i.e. we assume we know the sign of the b).

A difference δm_{rec} in the reconstructed mass peak between two generator with the same m_t parameter will lead to a $\delta m_t = -\delta m_{rec}$ in the mass extracted by fitting a given data set.

Pythia8, POWHEG-hvq - POWHEG-$b\bar{b}4l$ comparison

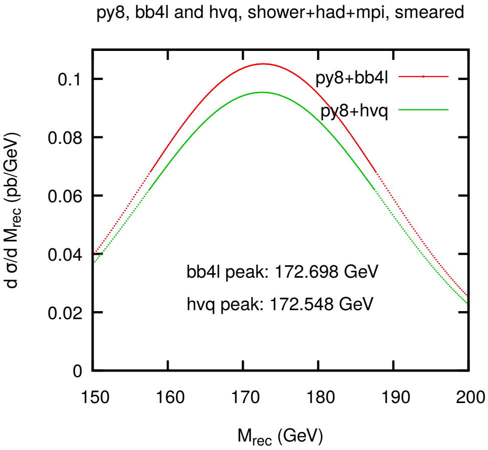
We compare the new $b\bar{b}4l$ NLO+PS generator with the old hvq, using Pythia8 for the shower.



$b\bar{b}4l$ − hvq: 38 MeV

Pythia8, POWHEG-hvq - POWHEG-$b\bar{b}4l$ comparison

Same, accounting for experimental errors by smearing the peak with a gaussian distribution with a width of 15 GeV.



$$f_{sm}(x) \propto \int \mathrm{d}y\, f(y) \times \exp\left[-\frac{(y-x)^2}{2\sigma^2}\right],$$

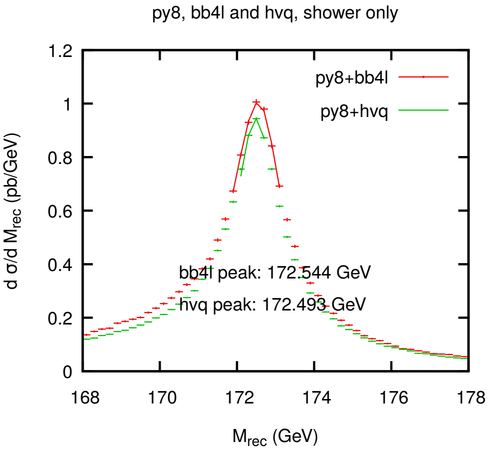
$\sigma = 15$ GeV,

Peak from a fit with a 4th degree polynomial.

$b\bar{b}4l$ – hvq: 150 MeV

Pythia8, POWHEG-hvq - POWHEG-$b\bar{b}4l$ comparison

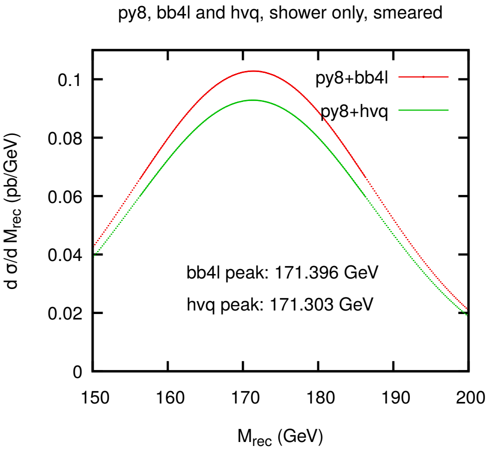
Same stuff, no hadronization and mpi;



$\mathtt{b\bar{b}4l} - \mathtt{hvq}$: 51 MeV

Pythia8, POWHEG-hvq - POWHEG-$b\bar{b}4l$ comparison

No hadronization and mpi, with smearing;



Smearing, $\sigma = 15$ GeV,

$\mathtt{b\bar{b}4l} - \mathtt{hvq}$: 93 MeV

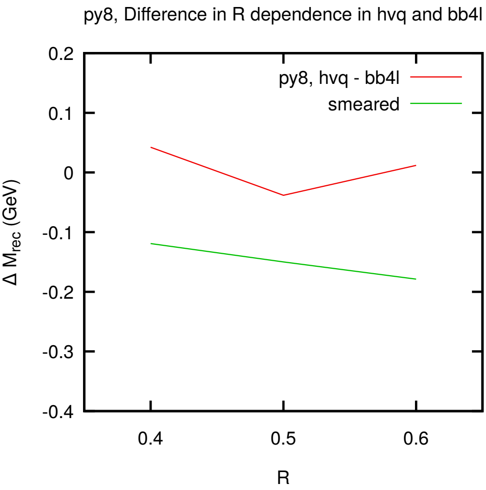
Pythia8, POWHEG-hvq - POWHEG-$b\bar{b}4l$ comparison

Summary of POWHEG-hvq hw7 - POWHEG-$b\bar{b}4l$ (with Pythia8) comparison:

$M_{\rm rec}$ (GeV)						
	Full			Shower only		
	$b\bar{b}4l$	hvq	Δ	$b\bar{b}4l$	hvq	Δ
$\sigma = 0$	172.809	172.771	0.038	172.544	172.493	0.051
$\sigma = 15$	172.698	172.548	0.150	171.396	171.303	0.093

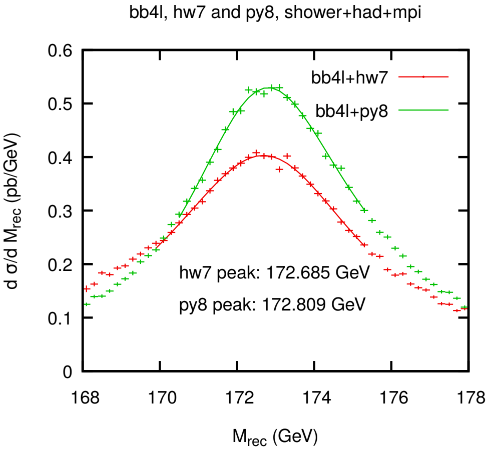
Very modest difference! Is it stable under change of the R parameter?

Pythia8, POWHEG-hvq - POWHEG-$b\bar{b}4l$ comparison



Fairly stable! From this, we are tempted to conclude that switching to the new generator makes no difference ...

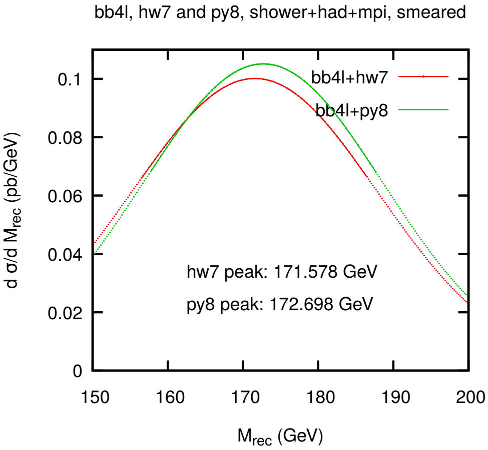
POWHEG-$b\bar{b}4l$, Herwig7 - Pythia8 comparison



hw7 − py8: −124 MeV

POWHEG-$b\bar{b}4l$, Herwig7 - Pythia8 comparison

Same, accounting for experimental errors by smearing the peak with a gaussian distribution with a width of 15 GeV.

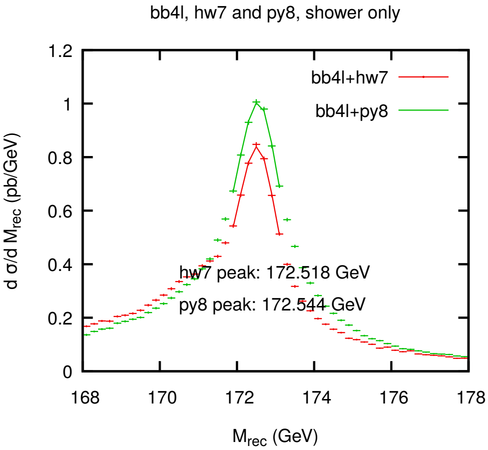


Smearing, $\sigma = 15$ GeV,

hw7 − py8: −1.12 GeV

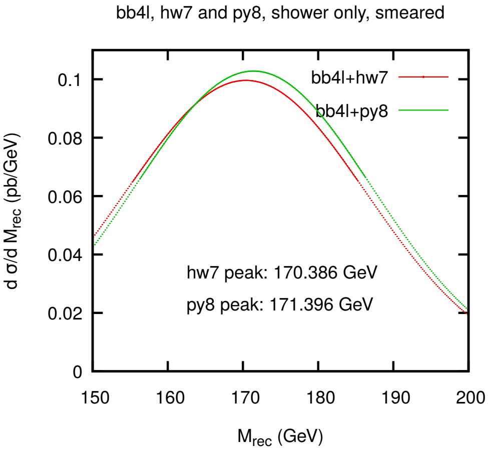
POWHEG-$b\bar{b}4l$, Herwig7 - Pythia8 comparison

Same stuff, no hadronization and mpi;



hw7 − py8: −26 MeV

No hadronization and mpi, with smearing;



Smearing, $\sigma = 15$ GeV,

hw7 − py8: -1.01 GeV

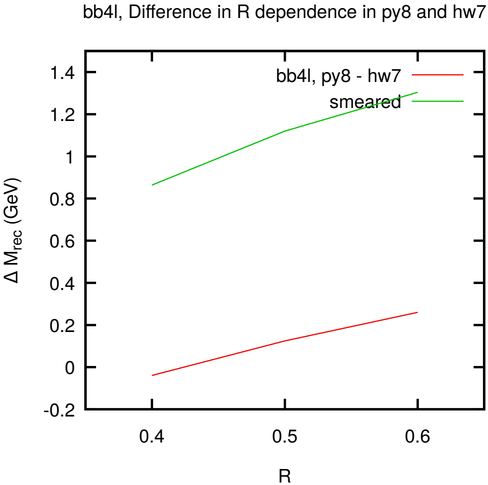
POWHEG-$\mathtt{b\bar{b}4l}$, Herwig7 - Pythia8 comparison

Summary of POWHEG-$\mathtt{b\bar{b}4l}$ hw7 - py8 comparison:

M_{rec} (GeV)						
	Full			Shower only		
	hw7	py8	Δ	hw7	py8	Δ
$\sigma = 0$	172.685	172.809	0.124	172.518	172.544	0.026
$\sigma = 15$	171.578	172.698	1.12	170.386	171.396	1.01

Modest differences in the unsmeared case; but with smearing, we see very large differences.

POWHEG-$b\bar{b}4l$, Herwig7 - Pythia8 comparison



Different slope in their R dependence: at least one of them needs tuning to fit it. This may reduce the differences in extracted mass.

Differences mainly caused by Shower/Matching effects.

Summary of the comparison

- No important differences between `bb̄4l` and ttbdec. Thus, finite width and interference effects do not seem crucial.
- With Pythia8: no relevant differences between `hvq` and `bb̄4l`. (not unespected, Pythia8 implements MEC in top decay).
- With Herwig7: large differences between `hvq` and `bb̄4l`. (Puzzling: Herwig7 should be as accurate as Pythia8.)
- Large differences between Pythia8 and Herwig7 with the `bb̄4l` generator, mostly due to the shower; and disturbing differences when `hvq` is used, mostly due to hadronization effects.
- Important Pyhtia8/Herwig7 differences also seen in leptonic observables (would have hoped to be less sensitive to shower and hadronization effects).

Checks and attempts to solve the Herwig7/Pythia8 issue

- B radiation in POWHEG: new impementation of B radiation Buonocore, Tramontano, P.N., from Buonocore master thesis.
- 3 alternative (and orthogonal) implementations of NLO+PS shower matching in Herwig7 (with help from the authors). 2.5 alternative implementations of the interface with Pythia8.
- Herwig7 is angular ordered. There are issues related to the need of truncated-vetoed shower in the POWHEG interface. Herwig7 provides a variant of the shower initial conditions equivalent to the inclusion of truncated shower.

Disclaimer:

The message is NOT that the shower/matching error is 1 GeV!
It seems clear, even only from the r dependence, that it is unlikely that both MC's can fit the same data.

However, it is important to explore this further. We may either

- Find a problem in Hw7 and/or in our matching procedures.
- Find that by tuning observables related to the b-jet shape the difference goes away.
- Find that, after tuning the tuneable, a difference persists.
  Only at this point this difference becomes an uncertainty!

Herwig7, with the angular ordered shower is drastically different from the other Monte Carlo: this exercise should be carried out.

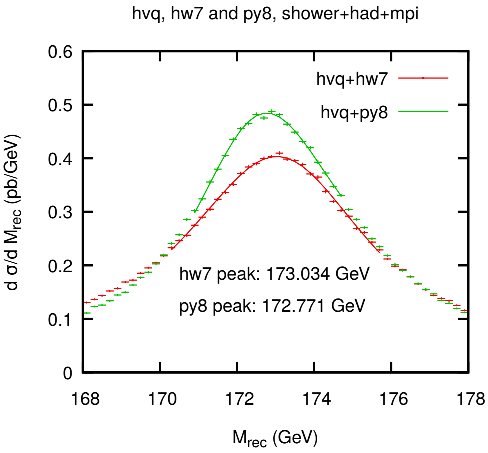
Conclusions

- A precise determination of m_t at the LHC seems possible, although not easy.
- The pole mass renormalon poblem does not seem to be an urgent one.
- Best to think of the measurements as fitting a parameter in the calculation/generator with a given accuracy. Must assess the errors by judicious variation of parameters, and by comparison of different generators.
- Studies with generators of increasing accuracy seem to indicate that uncertainties related to the shower model/matching are still the most dangerous ones.

BACKUP MATERIAL

Example: POWHEG-hvq, Pythia8 - Herwig7 comparison

The POWHEG-hvq generator interfaced to Pythia8 is widely used now by the experimental collaborations. We consider the differences we get when switching to Herwig7.

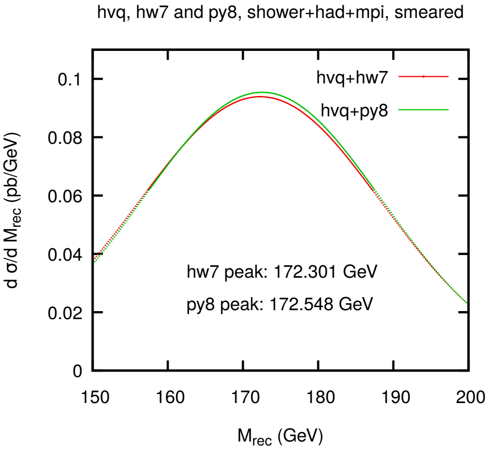


Peak position obtained by fitting peak distribution with a skewed Lorentian:

hw7 – py8: 263 MeV

Example: POWHEG-hvq, Pythia8 - Herwig7 comparison

Same, accounting for experimental errors by smearing the peak with a gaussian distribution with a width of 15 GeV.

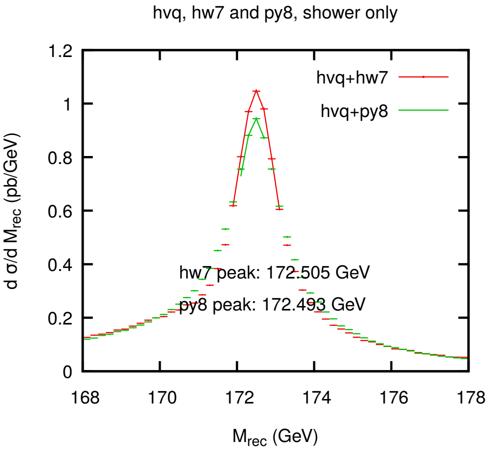


smeared,

hw7 − py8: −247 MeV

Example: POWHEG-hvq, Pythia8 - Herwig7 comparison

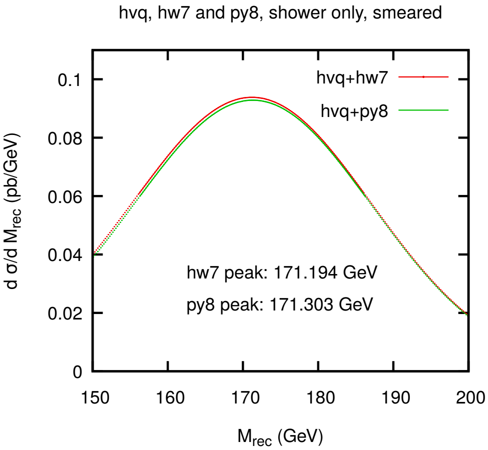
Same stuff, no hadronization and mpi;



hw7 − py8: 12 MeV

Example: POWHEG-hvq, Pythia8 - Herwig7 comparison

No hadronization and mpi, with smearing;



Smearing, $\sigma = 15$ GeV,

hw7 − py8: −109 MeV

Example: POWHEG-hvq, Pythia8 - Herwig7 comparison

Summary of POWHEG-hvq py8 - hw7 comparison:

M_{rec} (GeV)						
	Full			Shower only		
	Herwig7	Pythia8	Δ	Herwig7	Pythia8	Δ
$\sigma = 0$	173.034	172.771	0.263	172.505	172.493	0.012
$\sigma = 15$	172.301	172.548	-0.247	171.194	171.303	-0.109

Sizable difference, but well below the current $\pm 0.5 GeV$ experimental results.
The different shape around the peak region is worrisome.
Hadronization seems to be responsible for the discrepancy.

General approach

Assuming we have an observable O sensitive to the top mass, we will have in general

$$O = O_{\rm c} + B(m_t - m_{t,{\rm c}}) + \mathcal{O}((m_t - m_{t,{\rm c}})^2)$$

where $m_{t,{\rm c}} = 172.5$ GeV is our central value for the top mass. $O_{\rm c}$ and B differ for different generator setup. Given an experimental result for O, the extracted mass value is

$$m_t = m_{t,{\rm c}} + (O_{\rm exp} - O_{\rm c})/B$$

By changing the generator setup $O_{\rm c}, B \to O'_{\rm c}, B'$:

$$m_t - m'_t = -\frac{O_{\rm c} - O'_{\rm c}}{B} - (O_{\rm exp} - O'_{\rm c})(B - B')/(BB') \approx -\frac{O_{\rm c} - O'_{\rm c}}{B}.$$

where we neglect differences in B among different generators.

Jet energy peak

Agashe,Franceschini,Kim,Schulze,2016

	$E_{b-\text{jet peak}}$ (GeV)	
	`bb̄4l`	`hvq`
hw7	68.88 ± 0.40	69.67 ± 0.26
py8	71.24 ± 0.40	70.77 ± 0.27
hw7, no had.	68.09 ± 0.45	68.30 ± 0.28
py8, no had	69.64 ± 0.44	69.04 ± 0.27

Here $B = 0.45$, so:

- `bb̄4l`, hw7 - py8: $\Delta m_t = 5$ GeV, (only shower: 3.4 GeV)
- `hvq`, hw7 - py8: $\Delta m_t = 2.4$ GeV (only shower: 0.74 GeV)

Lepton Observables

Frixione, Mitov, 2014

Deviations in top mass values:

comparison of `bb̄4l` and `tt̄_dec`, both with Pythia8

	$\Delta M_{\rm top}$ (GeV)		
	Mom 1	Mom 2	Mom 3
$\langle[p_t(l^+)]^k\rangle$	-0.8 $\pm$ 0.4	-0.7 $\pm$ 0.3	-0.6 $\pm$ 0.5
$\langle[p_t(l^+l^-)]^k\rangle$	1.1 $\pm$ 0.3	1.6 $\pm$ 0.2	2.6 $\pm$ 0.3
$\langle[m(l^+,l^-)]^k\rangle$	-0.8 $\pm$ 0.6	-0.6 $\pm$ 0.4	-0.1 $\pm$ 0.7
$\langle[E(l^+l^-)]^k\rangle$	-0.3 $\pm$ 0.5	-0.4 $\pm$ 0.4	-0.3 $\pm$ 0.5
$\langle[p_t(l^+)+p_t(l^-)]^k\rangle$	-0.4 $\pm$ 0.4	-0.5 $\pm$ 0.3	-0.9 $\pm$ 0.4

Generally good agreement between the two;
the only (marginal) exception of $p_t(l^+l^-)$.

Lepton Observables

Deviations in top mass values:
comparison of `bb̄4l` and `hvq`, both with Pythia8

	$\Delta M_{\rm top}$ (GeV)		
k	1	2	3
$\langle [p_t(l^+)]^k \rangle$	-0.1 $\pm$ 0.4	0.2 $\pm$ 0.3	0.6 $\pm$ 0.5
$\langle [p_t(l^+ l^-)]^k \rangle$	2.4 $\pm$ 0.3	2.8 $\pm$ 0.2	3.8 $\pm$ 0.3
$\langle [m(l^+, l^-)]^k \rangle$	-1.8 $\pm$ 0.6	-1.2 $\pm$ 0.4	-0.4 $\pm$ 0.6
$\langle [E(l^+ l^-)]^k \rangle$	0.2 $\pm$ 0.5	0.4 $\pm$ 0.4	0.9 $\pm$ 0.5
$\langle [p_t(l^+) + p_t(l^-)]^k \rangle$	-0.1 $\pm$ 0.4	-0.1 $\pm$ 0.3	-0.2 $\pm$ 0.4

Good agreement for 1st, 4th and 5th observable. These are the observables that were argued to be less sensitive to shower and spin correlation effects by Frixione and Mitov.

Lepton Observables

Deviations in top mass values:
comparison of Pythia8 and Herwig7, both with `bb̄4l`

	ΔM_{top} (GeV)		
k	1	2	3
$\langle [p_t(l^+)]^k \rangle$	3.4 ± 0.4	4.0 ± 0.2	4.9 ± 0.4
$\langle [p_t(l^+ l^-)]^k \rangle$	4.6 ± 0.3	5.3 ± 0.2	6.5 ± 0.2
$\langle [m(l^+, l^-)]^k \rangle$	0.7 ± 0.5	1.2 ± 0.3	1.8 ± 0.5
$\langle [E(l^+ l^-)]^k \rangle$	2.8 ± 0.4	3.0 ± 0.3	3.3 ± 0.4
$\langle [p_t(l^+) + p_t(l^-)]^k \rangle$	3.2 ± 0.4	3.7 ± 0.2	4.2 ± 0.3

Bad agreement in general, also for 1st, 4th and 5th observable.

Lepton Observables

Deviations in top mass values:
comparison of Pythia8 and Herwig7, both with `hvq`

	ΔM_{top} (GeV)		
k	1	2	3
$\langle [p_t(l^+)]^k \rangle$	2.0 ± 0.4	2.6 ± 0.3	3.5 ± 0.5
$\langle [p_t(l^+l^-)]^k \rangle$	2.7 ± 0.3	3.3 ± 0.2	4.2 ± 0.3
$\langle [m(l^+, l^-)]^k \rangle$	0.6 ± 0.6	1.2 ± 0.4	2.0 ± 0.7
$\langle [E(l^+l^-)]^k \rangle$	1.4 ± 0.5	1.6 ± 0.4	1.8 ± 0.5
$\langle [p_t(l^+) + p_t(l^-)]^k \rangle$	2.0 ± 0.4	2.5 ± 0.3	3.2 ± 0.4

Still bad, although better than $\mathtt{b\bar{b}4l}$.

Lepton Observables

Deviations in top mass values:
comparison of `bb̄4l` and `hvq`, both with hw7

	ΔM_{top} (GeV)		
k	1	2	3
$\langle [p_t(l^+)]^k \rangle$	-1.5 $\pm$ 0.4	-1.2 $\pm$ 0.3	-0.8 $\pm$ 0.4
$\langle [p_t(l^+l^-)]^k \rangle$	0.5 $\pm$ 0.3	0.8 $\pm$ 0.2	1.4 $\pm$ 0.2
$\langle [m(l^+, l^-)]^k \rangle$	-1.9 $\pm$ 0.5	-1.2 $\pm$ 0.4	-0.2 $\pm$ 0.5
$\langle [E(l^+l^-)]^k \rangle$	-1.2 $\pm$ 0.4	-1.1 $\pm$ 0.3	-0.7 $\pm$ 0.4
$\langle [p_t(l^+) + p_t(l^-)]^k \rangle$	-1.3 $\pm$ 0.4	-1.3 $\pm$ 0.2	-1.2 $\pm$ 0.3

Still bad.

Renormalon Issue: Simplified (1-loop α_s) illustration

$$m_P = m + N\alpha_s \sum_{n=0}^{\infty} c_n(\mu, m)\alpha_s^n,$$

where m_p is the pole mass, m is the $\overline{\mathrm{MS}}$ mass, and $\alpha_s = \alpha_s(\mu)$.
The asymptotic behaviour of the expansion is (in 1-loop α_s)

$$\begin{aligned} \alpha_s^n c_n &\stackrel{n\to\infty}{\longrightarrow} \mu t_a^{(n)}, \\ t_a^{(n)} &\equiv (2b_0\alpha_s)^n n! \approx \sqrt{2\pi} e^{(n+1/2)\log n - n + n\log(2b_0\alpha_s)}, \end{aligned} \quad (4)$$

Minimum at $n_{\mathrm{m}} \approx 1/(2b_0\alpha_s)$. Using $\alpha_s = 1/(b_0 \log[\mu^2/\Lambda^2])$:

$$t_a^{(n_{\mathrm{m}})} = \sqrt{2\pi n_{\mathrm{m}}}\, e^{-n_{\mathrm{m}}} = \sqrt{2\pi n_{\mathrm{m}}}\, \frac{\Lambda}{\mu}$$

The ambiguity of the asymptotic formula should be μ independent.
But the minimal term goes like

$$N\mu\alpha_s t_a^{(n_{\mathrm{m}})} = N\,\alpha_s\sqrt{2\pi n_{\mathrm{m}}}\,\Lambda \quad (5)$$

Needs an extra factor of $\sqrt{n_{\mathrm{m}}}$ to be μ independent.

Around the minimum

$$t_a^{(n)} \approx t_a^{(n_{\mathrm{m}})} \left(1 + \frac{1}{2n}\left(n - n_{\mathrm{m}}\right)^2\right) \tag{6}$$

We can supplement the minimal term by a factor quantifying how many terms are close to the minimum

$$\frac{1}{2n}\left(n - n_{\mathrm{m}}\right)^2 < p \Longrightarrow \Delta_n = \sqrt{2pn_{\mathrm{m}}}$$

Δ_n times the minimal term is in fact μ independent, and equal to

$$N\frac{\sqrt{4\pi p}}{2b_0}\Lambda$$

Borel sum approach

We transform the series in the inverse Borel transform of a convergent series. Order by order in α_s we have the identity

$$N\alpha_s \sum_{n=0}^{a} c_n(\mu, m)\alpha_s^n = N \int_0^\infty \mathrm{d}r\, e^{-\frac{r}{\alpha_s}} \sum_{n=0}^{a} c_n(\mu, m)\, \frac{r^n}{n!}.$$

Plugging in the asymptotic value for the coefficients:

$$N\mu \int_0^\infty \mathrm{d}r\, e^{-\frac{r}{\alpha_s}} \sum_{n=0}^{a} (2b_0 r)^n = N\mu \int_0^\infty \mathrm{d}r\, \frac{e^{-\frac{r}{\alpha_s}}}{1 - 2b_0 r}$$

The singularity in $r = 1/(2b_0)$ is due to the renormalon. One can define the sum as the principal value for the integral, and the ambiguity as the imaginary part of the integral divided by π (Beneke, 1999)

$$N\mu \frac{1}{2b_0} e^{-\frac{1}{2b_0\alpha_s}} = \frac{N}{2b_0}\Lambda$$

Comparison of the two methods

The minimal term method and the Borel method agree in the estimate of the error provided that

- We extrapolate n to non integer values.
- $p = 1/(4\pi) = 0.08$.

Beneke etal vs. Hoang etal

- Beneke, Marquard, Steinhauser, P.N. 2016, v2, 9 Jun 2017
  Uses the Im/Pi prescription
- Hoang, Lepenik, Preisser, 26 Jun 2017:
  - Take half the sum of all terms that are less than the minimal one multiplied by a factor f, where "f is a number larger but close to unity". Choose $f = 5/4$,
  - Do not extrapolate to non integer n.
  - Do further scale variation on the terms they sum

From the previous argument: the "range" factor in H is larger than the one in B by

$$\sqrt{\frac{f-1}{\frac{1}{4\pi}}} = \sqrt{\frac{0.25}{0.080}} = 1.77\,.$$

Further enhancement in H arises from a scale variation procedure.