Calculations at $1/m_b^{n>3}$ and $1/(m_b^3\, m_c^m)$ Order

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in collaboration with

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Theoretische Physik 1
Universität Siegen

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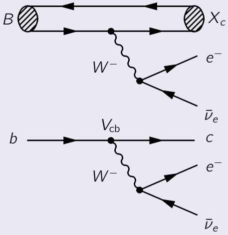
Outline

Inclusive $B \to X_c \ell \bar{\nu}_\ell$-Decays

Topic and Goal

- Consider inclusive semi-leptonic decay $B \to X_c \ell \bar{\nu}_\ell$
- Clean mode to extract CKM matrix element V_{cb}
- Use Heavy Quark Expansion (HQE) to classify non-perturbative effects
- Systematics of higher order corrections in HQE



State of the Art

$1/m_b^n$, rows: α_s^n

n	0	2	3	4
0	●	●	●	●[a]
1	●	○[b]	—	—
2	●[c]	—	—	—
3	○[d]	—	—	—

[a] Recent result: [JHEP **0703** (2007) 087]

[b] Only combined corrections to μ_π^2 calculated arXiv:0708.0855

[c] Recent results: arXiv:0803.0951, arXiv:0803.0960.

[d] Only for special kinematic point and BLM

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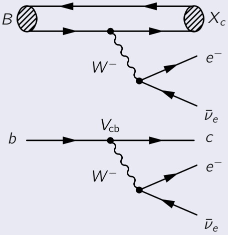

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Standard Setup of the $1/m_b$ Expansion

Differential Rate

$$d\Gamma = 16\pi\, G_F^2\, |V_{cb}|^2\, W_{\mu\nu}\, L^{\mu\nu}\, d\phi$$

- $L^{\mu\nu}$: Leptonic tensor, $W_{\mu\nu}$: Hadronic tensor, $d\phi$: Phasespace

- Starting point:
  Correlator of two hadronic currents

$$2M_B\, T_{\mu\nu} = \langle B(p)|\bar{b}_v \gamma_\nu P_L S_{\mathrm{BGF}} \gamma_\mu P_L b_v |B(p)\rangle$$

- Expand Background field propagator (geometric series)

$$iS_{\mathrm{BGF}} = \frac{i}{\not{p} + i\not{D} - m_c} = \sum_{n=0} (-1)^n \frac{1}{\not{p} - m_c} \left(i\not{D} \frac{1}{\not{p} - m_c} \right)^n, \; p_\mu := m_b v_\mu - q_\mu$$

- Optical theorem relates $W_{\mu\nu}$ to $T_{\mu\nu}$: $\mathrm{Im}\, T_{\mu\nu} \propto W_{\mu\nu}$

$$\mathrm{Im} \frac{1}{(p^2 - m_c^2 + i\epsilon)^{n+1}} \propto \delta^{(n)}(p^2 - m_c^2)$$

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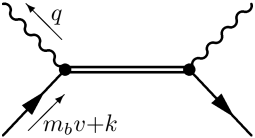

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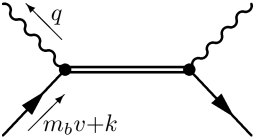

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Setup of the Differential Rate

Double Differential Rate

- Consider differential rate in $v{\cdot}p$ and p^2, where $p = m_b v - q$

$$\frac{\mathrm{d}^2\Gamma}{\mathrm{d}v{\cdot}p\,\mathrm{d}p^2} = \frac{G_F^2|V_{cb}|^2}{6\pi^3}\sqrt{(v{\cdot}p)^2 - p^2}\; W^{\mu\nu}$$
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- Hadronic tensor
- From leptonic tensor

General Structure

$$\frac{\mathrm{d}^2\Gamma}{\mathrm{d}v{\cdot}p\,\mathrm{d}p^2} = \frac{G_F^2|V_{cb}|^2}{6\pi^3}\sqrt{(v{\cdot}p)^2 - p^2}\sum_{n=0}^{\infty} P_n(v{\cdot}p, p^2, m_c)\,\delta^{(n)}(p^2 - m_c^2)$$

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Phase-Space Integration to Total Width

- $\sqrt{p^2} \leq v{\cdot}p \leq \frac{m_b^2+p^2}{2m_b}$ and $0 \leq p^2 \leq m_b^2$
- Phase-space factor $\sqrt{(v{\cdot}p)^2 - p^2} = |\vec{p}|$ leads to elliptic integral

$$\int \mathrm{d}v{\cdot}p \sqrt{(v{\cdot}p)^2 - p^2}\ v{\cdot}p^{2n+1} = \text{regular}$$

$$\int \mathrm{d}v{\cdot}p \sqrt{(v{\cdot}p)^2 - p^2}\ v{\cdot}p^{2n} = C_n\ (p^2)^{n+1} \log p^2 + \text{regular}$$

How Infrared Sensitive Terms emerge

- Due to integration by parts the logarithm will be differentiated

$$(p^2)^l \log \frac{p^2}{m_b^2} \delta^{(l)}(p^2 - m_c^2) = (-1)^l\, l!\, \log \frac{m_c^2}{m_b^2} \delta(p^2 - m_c^2) + \text{regular}$$

$$(p^2)^l \log \frac{p^2}{m_b^2} \delta^{(n)}(p^2 - m_c^2) = (-1)^{n-l-1}\, l!\, (n-l-1)! \left(\frac{1}{m_c^2}\right)^{n-l} \qquad n > l$$

$\Rightarrow$ At some order $1/m_c^2$ terms will appear!

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The Infrared Sensitive Terms

General Remarks

- Need even power in $v \cdot p$ for infrared sensitive terms
- Left-handed current: $m_c \leftrightarrow -m_c$ symmetry $\Rightarrow$ Only even powers of m_c can appear at tree level
- Sort leptonic tensor according to powers of p

$$L_{\mu\nu}^{\text{lead}} = m_b^2 \Big(v_\mu v_\nu - g_{\mu\nu} \Big) \quad , \qquad L_{\mu\nu}^{\text{sub}} = -2 m_b \Big(\frac{v_\mu p_\nu + v_\nu p_\mu}{2} - g_{\mu\nu} v \cdot p \Big)$$

$$L_{\mu\nu}^{\text{subsub}} = p_\mu p_\nu - g_{\mu\nu} p^2$$

Orders Contributing to Infrared Sensitive Terms

- Considering expansion up to dimension $n+3$
  - Odd power in $n \Rightarrow$ even power in $\not{Q} - m_c$, odd in iD and $L_{\mu\nu}^{\text{lead,subsub}}$
  - Even power in $n \Rightarrow$ odd power in $\not{Q} - m_c$, even in iD and $L_{\mu\nu}^{\text{sub}}$

$\Rightarrow$ At least 1 gluon matrixelement for leading IR sensitive terms

First Appearance of $1/m_c$ Terms

Origin

$$\int \mathrm{d}v{\cdot}p\sqrt{(v{\cdot}p)^2 - p^2}\ v{\cdot}p^{2n}(p^2)^k = C_n\ (p^2)^{n+k+1}\log p^2 + \text{regular}$$

- Project out most singular contribution
  - Take leading order leptonic tensor and odd number of derivatives
  - To collect all terms: Start with only m_c from propagators, then replace m_c^2 by p^2 or $v{\cdot}p^2$ stepwise

Determine Leading Order

- We have in the order $1/m_b^i$ (for simplicity $n = k = 0$)

$$\Gamma \sim \int \mathrm{d}p^2\, m_c^{i+1} p^2 \log p^2\, \delta^{(i)}(p^2 - m_c^2)$$
$$\sim (m_c^2)^{\frac{3-i}{2}}$$

$\Rightarrow$ For $i = 5$ the first $1/(m_b^3 m_c^2)$ terms appear

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Explicit Expressions

Partonic Rate and $1/m_b^2$ Terms

$$P_0^{\text{lead}} \quad \Rightarrow \quad \text{No IR terms}$$

$$P_0^{\text{sub}} \quad \Rightarrow \quad \frac{m_c^4}{m_b^4} \log \frac{m_c^2}{m_b^2}$$

$$P_1^{\text{subsub}} \quad \Rightarrow \quad \frac{\mu_\pi^2 - \mu_G^2}{m_b^2} \frac{m_c^4}{m_b^4} \log \frac{m_c^2}{m_b^2}$$

Darwin Term at $1/m_b^3$

$$P_3^{\text{Darwin, lead}} \quad \Rightarrow \quad (p^2)^3 \ln \frac{p^2}{m_b^2} \,, \quad m_c^2 (p^2)^2 \ln \frac{p^2}{m_b^2} \,, \quad m_c^4 (p^2) \ln \frac{p^2}{m_b^2}$$

$$\Rightarrow \quad 8 \frac{\rho_D^3}{m_b^3} \log \frac{m_c^2}{m_b^2}$$

A First Summary

- Integrate out both bottom- and charm-quark together
- IR sensitive terms stem from lower integration limit of $v \cdot p$ due to presence of non analytic factor $\sqrt{(v \cdot p)^2 - p^2} = |\vec{p}|$
  - Effects emerge for fully integrated width or higher moments of the distribution
  - The strength and order it first appears is determinded by the kinematic observable

$\Rightarrow$ A $1/m_c$ expansion emerges besides $1/m_b$ starting at $1/m_b^3$

  - Leading order: $\frac{1}{m_b^3}\left(\log\rho + \frac{\Lambda_{QCD}^2}{m_c^2} + \frac{\Lambda_{QCD}^4}{m_c^4} + \ldots\right)$
  - Subleading order: $\frac{1}{m_b^4}\left(\log\rho + \frac{\Lambda_{QCD}^2}{m_c^2} + \frac{\Lambda_{QCD}^4}{m_c^4} + \ldots\right)$

The Second Road

Scales

- Two distinct scales in problem : m_b and m_c
- Limit $m_c \to 0$: Infrared sensitivity

Charm-Quark Operators

- First integrate out heavy DOF with momenta μ: $m_b > \mu > m_c$

$\Rightarrow$ Now include explicit four-quark operators with dynamical charm

- Charm can both be soft and hard

$$\langle B|\bar{b}\,\bar{\Gamma}_\nu\, c\,(x)\,\bar{c}\,\Gamma_\mu\, b(0)|B\rangle$$
$$= \langle B|\bar{b}\,(x)\,\bar{\Gamma}_\nu\,\langle c\,(x)\,\bar{c}(0)\rangle\,\Gamma_\mu\, b(0)|B\rangle\Big|_{>\mu} + \langle B|\bar{b}\,\bar{\Gamma}_\nu\, c\,(x)\,\bar{c}\,\Gamma_\mu\, b(0)|B\rangle\Big|_{<\mu}$$

- μ: Scale to distinguish physics of the two terms
- First term: Calculated in the usual way (kinematics change!)
- Second term: Now has to be added, nonperturbative dynamics

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Kinematic Restriction

Assumptions

- Introduce hard separation scale μ: $0 \leq q^2 < (m_b - \mu)^2$

$\Rightarrow$ Non-analytic terms change to $\log \mu^2/m_b^2$

$\Rightarrow$ functions are analytic in m_c

Cutoff Mass μ

- $(\mu, \Lambda_{QCD}) \ll m_c \hat{=} \mu/m_c$ corrections; $\mu \to 0$: Second term vanishes
- Raising μ: Shifts corrections to width from first to second term
- $\mu \gg m_c \hat{=}$ usual calculation: Λ_{QCD}/m_c expansion

Remark on Loop vs Straight Calculation

- Analytically the same terms if m_c large and charm is integrated out
- Only commutators appear, no residual momentum of b-quark!

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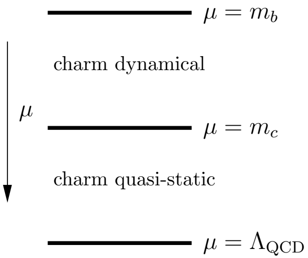
- First step: Match at high scale $\mu \sim m_b$

$\Rightarrow$ Charm still dynamical and "intrinsic-charm" operators appear in the OPE

- Next step: Use RGE to scale down to semi-hard scale $\mu_{\text{sh}} \sim m_c$

$\Rightarrow$ Integrate out charm-quark and match "intrinsic- charm" operators onto local operators built from light fields as before



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- Expand in powers of the external field
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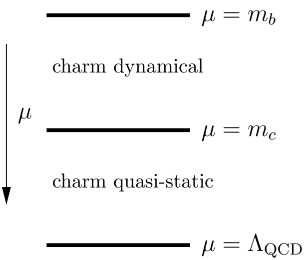

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Reminder: Darwin Term using RGE   arXiv:0805.0971 [hep-ph]

Operator Basis

$$\mathcal{O}_{\rho_D} = \bar{b}_v\,(iD_\mu)(ivD)(iD^\mu)\,b_v$$

$$\mathcal{O}_{T_1} = (4\pi)^2\,\mu^{2\epsilon}\,\frac{1}{3}\left(\bar{b}_v\,\gamma_\mu P_L\,c\;\bar{c}\,\gamma^\mu P_L\,b_v - \bar{b}_v\,\not{v} P_L\,c\;\bar{c}\,\not{v} P_L\,b_v\right)$$

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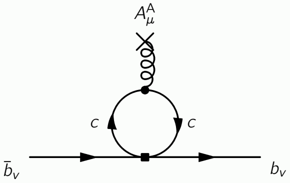
- Quark field has dimension $[c] = 3/2 - \epsilon \Rightarrow$ extract factor of $\mu^{2\epsilon}$
- With this convention, the anomalous-dimension matrix is $\mathcal{O}(\alpha_s^0)$

Leading Contribution: One Loop



- Calculate one-loop matrix elements of $\mathcal{O}_{T_{1,2}}$ for the partonic transition $b \to b$ in the presence of a soft background field $A_\mu(k)$
- Compare result with one-gluon matrix element of Darwin term

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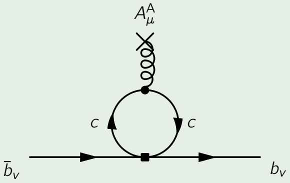

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Identify the Operator Mixing

One-Gluon Matrix Element of Darwin Term

$$\langle b|\mathcal{O}_{\rho_D}|b\rangle_{\text{tree}} = \frac{1}{2}\langle b|\bar{b}_v\,[iD_\mu,[(iv\cdot D),iD^\mu]]\,b_v|b\rangle_{\text{tree}} + \mathcal{O}(1/m_b)$$
$$= \frac{g_s}{2}\left((v\cdot k)(k\cdot A) - k^2\,(v\cdot A)\right)\bar{u}_b\,u_b + \dots$$

- Parton level $\Rightarrow$ Mixing to leading order in α_s^0

Result of Loop Calculation

$$\langle b|\mathcal{O}_{T_1}|b\rangle^{(0)} = +\frac{1}{3}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{m_c^2}\right)\langle b|\mathcal{O}_{\rho_D}|b\rangle_{\text{tree}}$$
$$\langle b|\mathcal{O}_{T_2}|b\rangle^{(0)} = -\frac{2}{3}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{m_c^2}\right)\langle b|\mathcal{O}_{\rho_D}|b\rangle_{\text{tree}}$$

- Result obtained in $D = 4 - 2\epsilon$ dimensions and $\overline{MS}$-scheme
- For $\mu \le m_c$ $T_i = 0$ and logarithmic Darwin term reproduced

Identify the Operator Mixing

One-Gluon Matrix Element of Darwin Term

$$\langle b|\mathcal{O}_{\rho_D}|b\rangle_{\text{tree}} = \frac{1}{2}\langle b|\bar{b}_v\,[iD_\mu,[(iv\cdot D),iD^\mu]]\,b_v|b\rangle_{\text{tree}} + \mathcal{O}(1/m_b)$$
$$= \frac{g_s}{2}\left((v\cdot k)(k\cdot A) - k^2\,(v\cdot A)\right)\bar{u}_b\,u_b + \ldots$$

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Powerlike IR Sensitive Terms with Cutoff μ hep-ph/0511158

Calculation of Charm Loop in External Background Field

- Now collect finite terms for $1/m_c$ expansion

→ Calculate in Fock-Schwinger gauge

- One loop results with two insertions accounting for $1/m_c^2$:

$$\langle \bar{c}_\alpha \gamma_\nu \gamma_5 c_\beta \rangle_A = \frac{1}{48\pi^2 m_c^2} \left(2\left\{\left[D_\kappa\,,\,G^{\kappa\lambda}\right]\,,\,\tilde{G}_{\nu\lambda}\right\} + \left\{\left[D_\kappa\,,\,\tilde{G}_{\nu\lambda}\right]\,,\,G^{\kappa\lambda}\right\}\right)_{\beta\alpha} + \cdots$$

$$\langle \bar{c}_\alpha \gamma_\nu c_\beta \rangle_A = \frac{i}{240\pi^2 m_c^2} \left(13\left[D^\kappa\,,\,\left[G_{\lambda\nu}\,,\,G^{\lambda,\kappa}\right]\right] + 8i\left[D^\kappa\,,\,\left[D^\lambda\,,\,\left[D_\lambda\,,\,G_{\kappa\nu}\right]\right]\right]\right.$$
$$\left. -4i\left[D^\lambda\,,\,\left[D^\kappa\,,\,\left[D_\lambda\,,\,G_{\kappa\nu}\right]\right]\right]\right)_{\beta\alpha} + \cdots$$

Remarks

- At least one commutator $\hat{=}$ at least one gluon matrix element
- Calculation assumes $\mu \gg m_c$, neglects $(m_c/\mu)^k$

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Reordering of Terms in the Expansion

- OPE is an expansion in both m_b and m_c
- $\Lambda_{\mathrm{QCD}}^2/m_c^2$ counts as $\Lambda_{\mathrm{QCD}}/m_b$
- Higher dimensional operators are partially relevant in lower dimension

$\Rightarrow$ Sum all contributions in same color for consistency

$1/m_b^n$, $1/m_c^n$

n	0	2	3	4	5	6
0	●	●	●	●	○	○
2	–	–	○	○	○	○
4	–	–	○	○	○	○
6	–	–	○	○	○	○

Implications for Higher Orders

- Not sufficient to calculate only dimension n for $n > 4$.
- Gets worse for higher orders: Need to sum up diagonal terms (see table)

$\Rightarrow$ More operators than naturally appear in dimension $n + 3$

Appearing Non-Perturbative Parameters

$$2M_B\tilde{f}_1 = \langle B|\bar{b}_v\left[iD_\kappa,\left[iD_\lambda,[iD^\lambda,iG^{\kappa\alpha}]\right]\right]b_v|B\rangle\ v_\alpha$$

$$2M_B\tilde{f}_2 = \langle B|\bar{b}_v\left[iD_\lambda,\left[iD_\kappa,[iD^\lambda,iG^{\kappa\alpha}]\right]\right]b_v|B\rangle\ v_\alpha$$

$$2M_B\tilde{f}_3 = \langle B|\bar{b}_v\left[iD_\kappa,\left[iG_{\lambda\alpha},iG^{\lambda\kappa}\right]\right]b_v|B\rangle\ v^\alpha$$

$$2M_B\tilde{f}_4 = \langle B|\bar{b}_v\left\{\left[iD^\rho,iG_{\rho\lambda}\right],iG_{\delta\gamma}\right\}(-i\sigma_{\alpha\beta})b_v|B\rangle \times\frac{1}{2}\left(g^{\lambda\alpha}g^{\delta\beta}v^\gamma-g^{\lambda\alpha}g^{\gamma\beta}v^\delta+g^{\delta\alpha}g^{\gamma\beta}v^\lambda\right)$$

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IC Part of Total Rate

$$\frac{m_b^3m_c^2}{\Gamma_0}\Gamma\Big|_{\frac{1}{m_c^2}} = -\frac{3}{2}\frac{2}{15}\left(-8\tilde{f}_1+4\tilde{f}_2-13\tilde{f}_3\right)+\frac{1}{2}\frac{2}{3}(-2\tilde{f}_4-\tilde{f}_5)\,.$$

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Numerical Impact

Using ground state saturation estimate

Non-Perturbative Parameters

$$\tilde{f}_1 \approx 0.31\ \text{GeV}^5$$
$$\tilde{f}_2 \approx 0.25\ \text{GeV}^5$$
$$\tilde{f}_3 \approx 0.14\ \text{GeV}^5$$
$$\tilde{f}_4 \approx 0.34\ \text{GeV}^5$$
$$\tilde{f}_5 \approx -0.40\ \text{GeV}^5$$

Contribution to Total Rate

$$\delta\Gamma\Big|_{\frac{1}{m_b^3 m_c^2}} \approx (0.45\%) \times \Gamma_{\text{Parton}} \qquad \delta\Gamma\Big|_{\frac{1}{m_b^4}} \approx (0.29\%) \times \Gamma_{\text{Parton}}$$

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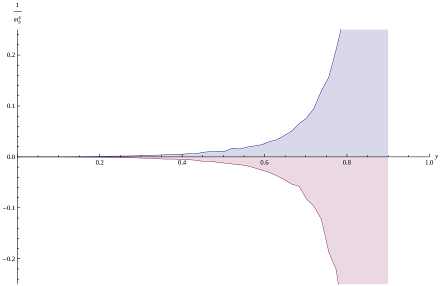
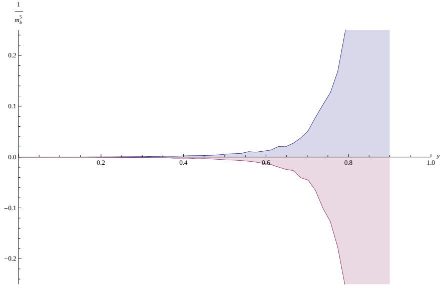
Estimates Using Variation **Preliminary!!**

- Vary all non-perturbative parameters in $1/m_b^n$ between $\pm(0.7\text{ GeV})^n$

$\Rightarrow$ $\delta\Gamma = (\Gamma|_{1/m_b^n} - \Gamma|_{1/m_b^{n-1}})/\Gamma|_{\text{parton}}$: $-3.5\% < \delta\Gamma|_{1/m_b^4;1/m_b^5} < 3.5\%$

- $\delta\frac{d\Gamma}{dE_e}|_{1/m_b^n} = (\frac{d\Gamma}{dE_e}|_{1/m_b^n} - \frac{d\Gamma}{dE_e}|_{1/m_b^{n-1}})/\frac{d\Gamma}{dE_e}|_{1/m_b^0}$
- Full $1/m_b^5$ result: $1/m_b^3 m_c^2$ have to be projected out!
- Overestimating, since same parameters should have a definite sign!





Summary

What has been shown

- Conventional OPE for $B \to X_c \ell \bar{\nu}_\ell$ contains terms with infrared sensitive parts to the charm quark mass
  - Leading order: $\frac{1}{m_b^3}\left(\log\rho + \frac{\Lambda_{QCD}^2}{m_c^2} + \frac{\Lambda_{QCD}^4}{m_c^4} + \dots\right)$
  - Subleading order: $\frac{1}{m_b^4}\left(\log\rho + \frac{\Lambda_{QCD}^2}{m_c^2} + \frac{\Lambda_{QCD}^4}{m_c^4} + \dots\right)$

  $\Rightarrow$ Need to include IR terms for consistency: "Intrinsic charm"
- For large m_c both ways of calculation are equivalent

$\Rightarrow$ Way to treat light final state quark in higher orders

What has been calculated

- $1/(m_b^3 m_c^2)$ terms are calculated to complete $1/m_b^4$

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- To investigate: Implications for higher orders in $B \to X_u \ell \bar{\nu}_\ell$

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Backup Slides

Explicit Expressions

Partonic Rate and $1/m_b^2$ Terms

$$P_0^{\text{lead}} = \frac{3}{2} mb^2 (v \cdot p) \quad \Rightarrow \quad \text{No IR terms}$$

$$P_0^{\text{sub}} = -2 m_b (v \cdot p)^2 + p^2 m_b \quad \Rightarrow \quad \frac{m_c^4}{m_b^4} \log \frac{m_c^2}{m_b^2}$$

$$P_1^{\text{subsub}} = 5p^4 + 7m_c^2 p^2 - 20 (v \cdot p)^2 p^2 - 10 m_c^2 (v \cdot p)^2 \quad \Rightarrow \quad \frac{\mu_\pi^2 - \mu_G^2}{m_b^2} \frac{m_c^4}{m_b^4} \log \frac{m_c^2}{m_b^2}$$

Darwin Term at $1/m_b^3$

$$P_3^{\text{Darwin, lead}} = -\frac{\rho_D^3}{12} m_b^2 \left(3m_c^4 - 6p^2 m_c^2 + 3p^4 + 8(v \cdot p)^4 - 8p^2 (v \cdot p)^2\right)$$

$$\Rightarrow \quad (p^2)^3 \ln \frac{p^2}{m_b^2}\,, \quad m_c^2 (p^2)^2 \ln \frac{p^2}{m_b^2}\,, \quad m_c^4 (p^2) \ln \frac{p^2}{m_b^2}$$

$$\Rightarrow \quad 8 \frac{\rho_D^3}{m_b^3} \log \frac{m_c^2}{m_b^2}$$

Ground State Factorization Estimate

Assumptions

- Factorization: A full set of states can be inserted
- Approximation: Assume that matrix elements are mostly satured by "ground state" $|\Omega_0\rangle$
- $\bar{\epsilon}$ corresponds to an average excitation energy

$$iD_\mu^\perp \equiv (v_\mu v_\nu - g_{\mu\nu}) iD^\nu \to \Pi_k \qquad \sigma^{\mu\nu} \to \sigma^{kl} = \epsilon^{klm}\sigma_m$$

$$\langle\Omega_0|\Pi_k\Pi_l|\Omega_0\rangle = \frac{\mu_\pi^2}{3} - \frac{\mu_G^2}{6}\sigma_{kl} \qquad \langle\Omega_0|\Pi_k\Pi_0\Pi_l|\Omega_0\rangle = -\frac{\rho_D^3}{3} - \frac{\rho_{LS}^3}{6}\sigma_{kl}$$

$$\langle\Omega_0|\Pi_k\Pi_0^3\Pi_l|\Omega_0\rangle = -\bar{\epsilon}^3\left(\frac{\mu_\pi^2}{3} - \frac{\mu_G^2}{6}\sigma_{kl}\right) \qquad \langle\Omega_0|\Pi_k\Pi_l\Pi_0|\Omega_0\rangle = 0$$

Modell for Weak-Annihilation **Preliminary!!**

- Assumption: Leading IR terms stem from same source
- "Resum" into logarithmic Darwin term

$\Rightarrow$ $\rho_D^3 \log m_c^2/m_b^2 \quad \to \rho_D^3 \log(m_c^2 + M_*^2)/m_b^2$

- Compare with known $1/(m_b^3 m_c^2)$ terms

$$M_*^2 = \frac{\tilde{f}_1}{5\rho_D^3} - \frac{\tilde{f}_2}{10\rho_D^3} + \frac{13\tilde{f}_3}{40\rho_D^3} - \frac{\tilde{f}_4}{12\rho_D^3} - \frac{\tilde{f}_5}{24\rho_D^3} \simeq (0.65\text{ GeV})^2$$

$\Rightarrow$ Using factorization estimates, WA takes the form

$$\frac{\delta\,\Gamma_{\rm sl}}{\Gamma_{\rm sl}} \simeq -\frac{8\rho_D^3}{m_b^3}\left(\ln\frac{m_b^2}{m_c^2+M_*^2} - \frac{77}{48}\right)$$