Hadronic contributions to the muon anomalous magnetic moment from lattice QCD

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Searching for Physics Beyond the Standard Models Using Charged Leptons
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Collaborators

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and the RBC/UKQCD collaborations

Outline I

Muon g-2 experimental measurement [Bennett et al., 2006]

E821 at BNL measured relative precession of muon spin to it's momentum

$\omega_a = \frac{g-2}{2}\frac{eB}{m} = a_\mu \frac{eB}{m}$, the muon anomaly

The rate of detected electrons oscillates with ω_a, fit to

$N(t) = Be^{-\lambda t}(1 + A\cos\omega_a t + \phi)$

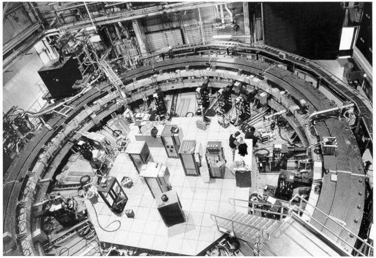


Figure 12. The storage-ring magnet. The cryostats for the inner-radius coils are clearly visible. The kickers have not yet been installed. The racks in the center are the quadrupole pulsers, and a few of the detector stations are installed, especially the quadrant of the ring closest to the person. The magnet power supply is in the upper left, above the plane of the ring. (Courtesy of Brookhaven National Laboratory)

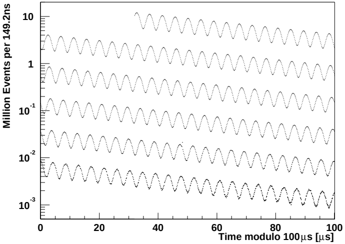
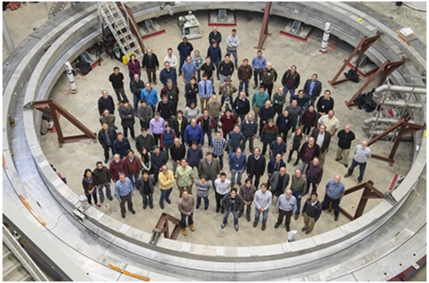


Figure 26. Histogram of the total number of electrons above 1.8 GeV versus time (modulo 100 μ s) from the 2001 μ^- data set. The bin size is the cyclotron period, $\approx$ 149.2 ns, and the total number of electrons is 3.6 billion.

$a_\mu(\mathrm{Expt}) = 11\,659\,208.0(5.4)(3.3) \times 10^{-10}$ 0.54 ppm!

New muon g-2 experiments

Storage ring moved to FNAL for E989, running since 2017!

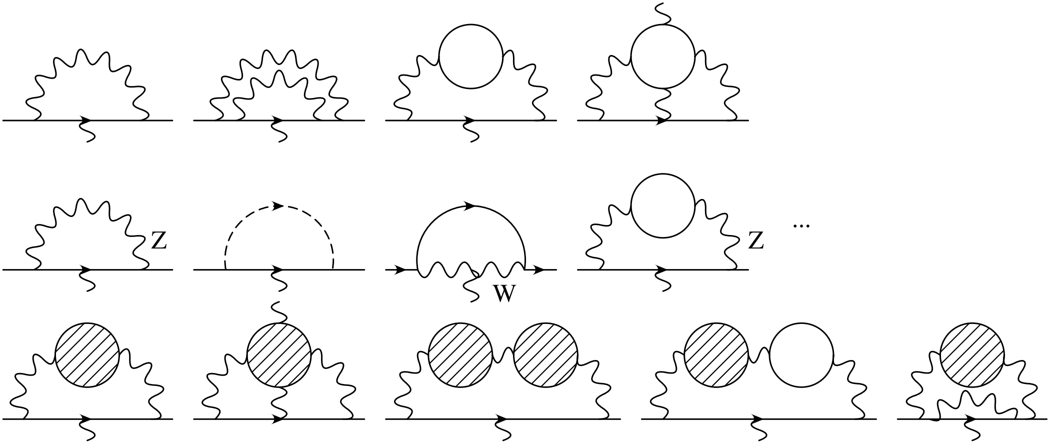


which is aiming for 0.14 ppm, 4$\times$ improvement!

In Japan at J-PARC, the E34 experiment will measure a_μ using ultra-cold muons, different systematics ($\sim$ 2020).

Standard Model Theory: QED+EW+QCD

$$
\begin{aligned}
\langle \mu(\vec{p}')|J_\nu(0)|\mu(\vec{p})\rangle &= -e\bar{u}(\vec{p}')\left(F_1(q^2)\gamma_\nu + i\frac{F_2(q^2)}{4m}[\gamma_\nu,\gamma_\rho]q_\rho\right)u(\vec{p}) \\
a_\mu &= F_2(0)
\end{aligned}
$$



Experiment - Theory

SM Contribution	Value ± Error ($\times 10^{11}$)	Ref	Update
QED (5 loops)	116584718.951 ± 0.080	[Aoyama et al., 2012]	
HVP LO	6923 ± 42	[Davier et al., 2011]	6926 (33) (Davier 16)
	6949 ± 43	[Hagiwara et al., 2011]	6922 (25) (KNT17 preliminary)
	6925 ± 27	[Blum et al., 2018]	combined lattice+R-ratio
HVP NLO	-98.4 ± 0.7	[Hagiwara et al., 2011]	
		[Kurz et al., 2014]	
HVP NNLO	12.4 ± 0.1	[Kurz et al., 2014]	
HLbL	105 ± 26	[Prades et al., 2009]	
HLbL (NLO)	3 ± 2	[Colangelo et al., 2014a]	
Weak (2 loops)	153.6 ± 1.0	[Gnendiger et al., 2013]	
SM Tot (0.42 ppm)	116591802 ± 49	[Davier et al., 2011]	
(0.43 ppm)	116591828 ± 50	[Hagiwara et al., 2011]	
(0.51 ppm)	116591840 ± 59	[Aoyama et al., 2012]	
Exp (0.54 ppm)	116592080 ± 63	[Bennett et al., 2006]	
Diff (Exp − SM)	287 ± 80	[Davier et al., 2011]	$\rightarrow 3.6\sigma$
	261 ± 78	[Hagiwara et al., 2011]	$\rightarrow 3.9\sigma$
	249 ± 87	[Aoyama et al., 2012]	

New experiments+new theory=new physics?

- Fermilab E989 running since 2017, aims for 0.14 ppm

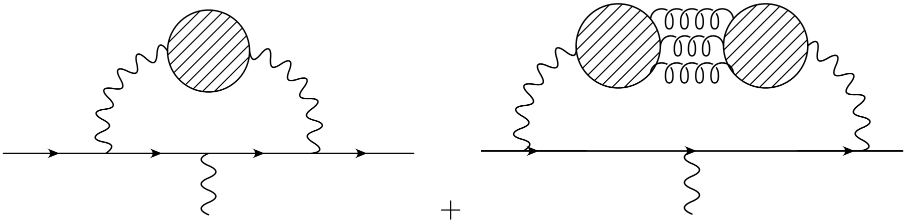
  J-PARC E34 $\sim$2020, aims for 0.3-0.4 ppm

  Today $a_\mu(\mathrm{Expt})$-$a_\mu(\mathrm{SM}) \approx 2.9 - 3.6\sigma$ (possibly more)
- If both central values stay the same,

  E989 ($\sim 4\times$ smaller error) $\rightarrow \ \sim 5\sigma$
  E989+new HLBL theory (models+lattice, 10%) $\rightarrow \ \sim 6\sigma$
  E989+new HLBL +new HVP (50% reduction) $\rightarrow \ \sim 8\sigma$
- Big discrepancy: new Physics $\sim 2\times$ Electroweak
- Lattice calculations important to trust theory errors, may become (part of) the central values (Muon g-2 Theory Initiative)
- Much progress, see many talks at Lattice 2017 (Granada) for latest results, 2nd plenary meeting of Mg-2TI (Mainz June 2018, WP by end of year)

Outline I

HVP contribution to muon g-2 [Blum, 2003, Lautrup et al., 1971]



Using lattice QCD and continuum, ∞-volume pQED

$$a_\mu(\mathrm{HVP}) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dq^2\, f(q^2)\, \hat{\Pi}(q^2)$$

$f(q^2)$ is known, $\hat{\Pi}(q^2)$ is subtracted HVP, $\hat{\Pi}(q^2) = \Pi(q^2) - \Pi(0)$, computed directly on Euclidean space-time lattice

$$\begin{aligned} \Pi^{\mu\nu}(q) &= \int d^4x\, e^{iqx} \langle j^\mu(x) j^\nu(0) \rangle \qquad j^\mu(x) = \sum_i Q_i \bar{\psi}(x)\gamma^\mu\psi(x) \\ &= \Pi(q^2)(q^\mu q^\nu - q^2\delta^{\mu\nu}) \qquad \text{(by Ward Identity)} \end{aligned}$$

Time-momentum representation (double subtraction) Bernecker-Meyer 2011

Useful in lattice (Euclidean time) setup
Switch order of FT and integral over momentum

$$\Pi(q^2) - \Pi(0) = \sum_t \left(\frac{\cos qt - 1}{q^2} + \frac{1}{2}t^2 \right) C(t)$$

$$C(t) = \frac{1}{3} \sum_{x,i} \langle j_i(x) j_i(0) \rangle$$

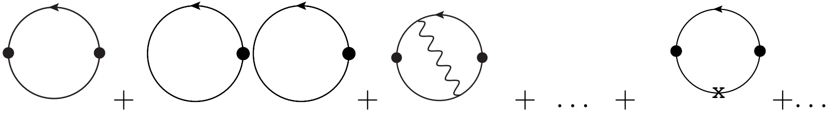
$$w(t) = 2 \int_0^\infty \frac{d\omega}{\omega} f(\omega^2) \left[\frac{\cos \omega t - 1}{(2 \sin \omega t/2)^2} + \frac{t^2}{2} \right]$$

$$a_\mu^{\mathrm{HVP}} = \sum_t w(t) C(t)$$

HVP contributions computed on lattice (u,d,s,c quarks) (RBC/UKQCD)[Blum et al., 2018]

Expand around isospin limit

$$C(t) = C^{(0)}(t) + \alpha C^{(1)}_{\mathrm{QED}} + \sum_f \Delta m_f C^{(1)}_{\Delta m_f}(t) + O(\alpha^2, \alpha \Delta m, \Delta m^2)$$



In units of 10^{-10}: $O(700)$, -11.2(4), $O(10)$

Lattice setup

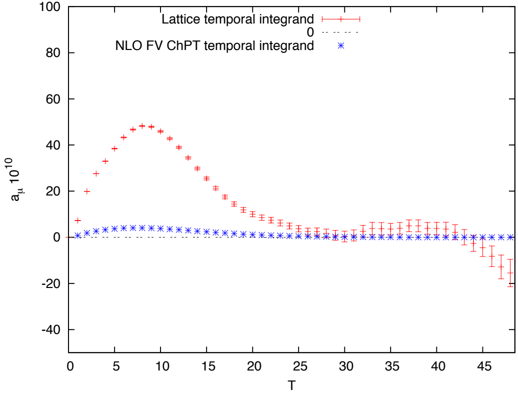
- Photons: Feynman gauge, QED_L [Hayakawa and Uno, 2008] (omit all modes with $\vec{q} = 0$)
- Gluons: Iwasaki gauge action (RG improved, plaquette+rectangle)
- muons: $L_s = \infty$ free domain-wall fermions (DWF)
- quarks: Möbius-DWF

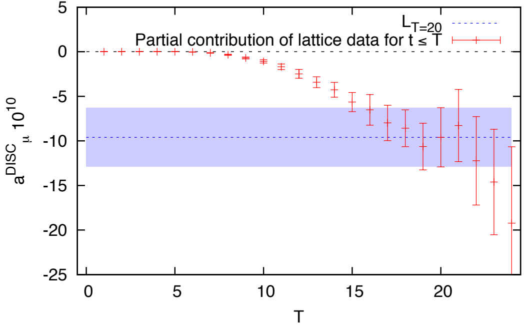
2+1f Möbius-DWF physical point QCD ensembles (RBC/UKQCD) [Blum et al., 2014]

	$48^3 \times 96$	$64^3 \times 128$
a^{-1} (GeV)	1.73	2.36
a (fm)	0.114	0.084
L (fm)	5.47	5.38
L_s	24	12
m_π (MeV)	139	135
m_μ (MeV)	106	106

cHVP and dHVP– results at physical pion mass

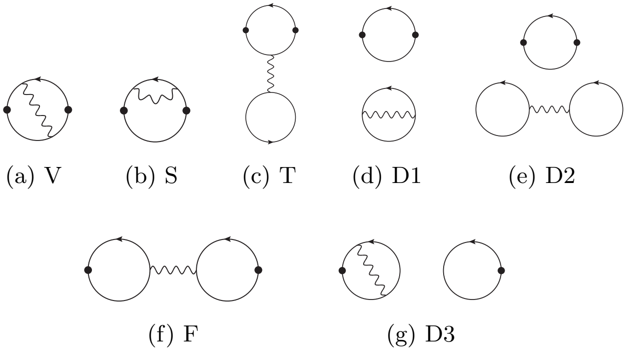
RBC/UKQCD





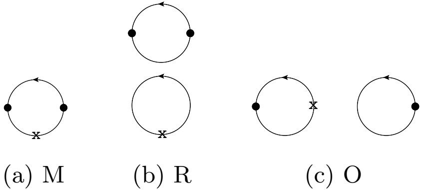
- small statistical errors from improved measurement techniques (full volume, 2000 low-mode average and all mode average AMA)
- statistical errors grow at long distance

QED and strong isospin breaking corrections



(a) V (b) S (c) T (d) D1 (e) D2 (f) F (g) D3

QED



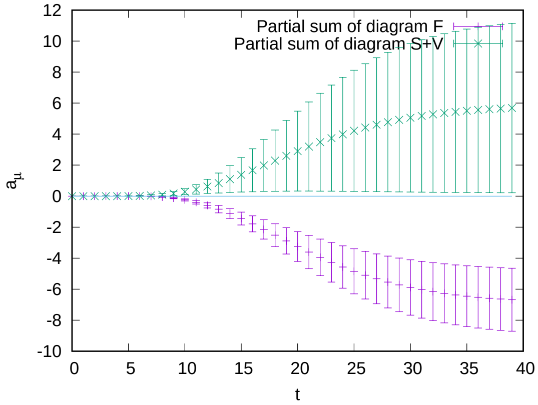
(a) M (b) R (c) O

Strong Isospin

- Focus on V, S, F, and M so far, rest are $1/N_c$ suppressed

QED corrections

RBC/UKQCD, [Blum et al., 2018]



- Importance sampling a'la HLbL calc (now improving statistics with HLbL data)
- physical pion mass, $a^{-1} = 1.73$ GeV, 48^3 lattice

Connection to the R-ratio Bernecker-Meyer 2011

The HVP can also be obtained from a dispersion relation and the experimental cross section for $e^+e^- \to$ hadrons

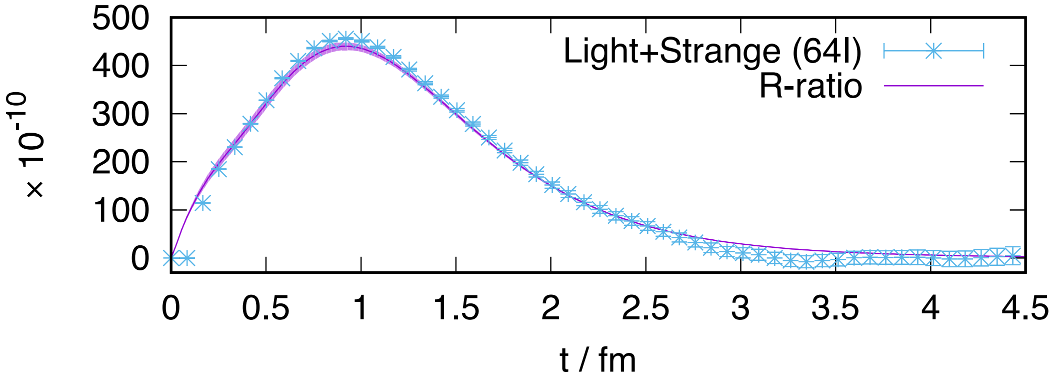
$$\Pi(-Q^2) = \frac{1}{\pi}\int_0^\infty ds \frac{s}{s+Q^2}\sigma(s, e^+e^- \to \text{ had})$$

$$R(s) = \frac{\sigma(s, e^+e^- \to \text{ had})}{\sigma(s, e^+e^- \to \ \mu^+\mu^-, \text{ tree})} = \frac{3s}{4\pi\alpha^2}\sigma(s, e^+e^- \to \text{ had})$$

Fourier transform gives

$$C(t) = \frac{1}{12\pi^2}\int_0^\infty d(\sqrt{s})R(s)se^{-\sqrt{s}t}$$

a_μ from Lattice vs. R-ratio Jegerlehner 2016 comparison Bernecker-Meyer 2011



RBC/UKQCD [Blum et al., 2018]

Lattice more accurate at intermediate distances

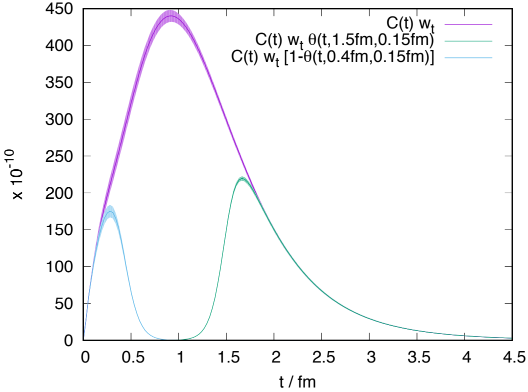
Window method: combine lattice and R-ratio

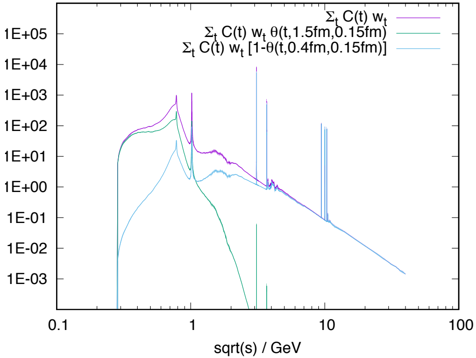
define smooth kernel $\Theta(t, t', \Delta) = [1 + \tanh[(t - t')/\Delta]]/2$, then

$$
\begin{aligned}
a_\mu &= \sum_t w_t C(t) = a_\mu^{\mathrm{SD}} + a_\mu^{\mathrm{W}} + a_\mu^{\mathrm{LD}} \\
a_\mu^{\mathrm{SD}} &= \sum_t w_t C(t)[1 - \Theta(t, t_0, \Delta)] \\
a_\mu^{\mathrm{W}} &= \sum_t w_t C(t)[\Theta(t, t_0, \Delta) - \Theta(t, t_1, \Delta)] \\
a_\mu^{\mathrm{LD}} &= \sum_t w_t C(t)\Theta(t, t_1, \Delta)
\end{aligned}
$$

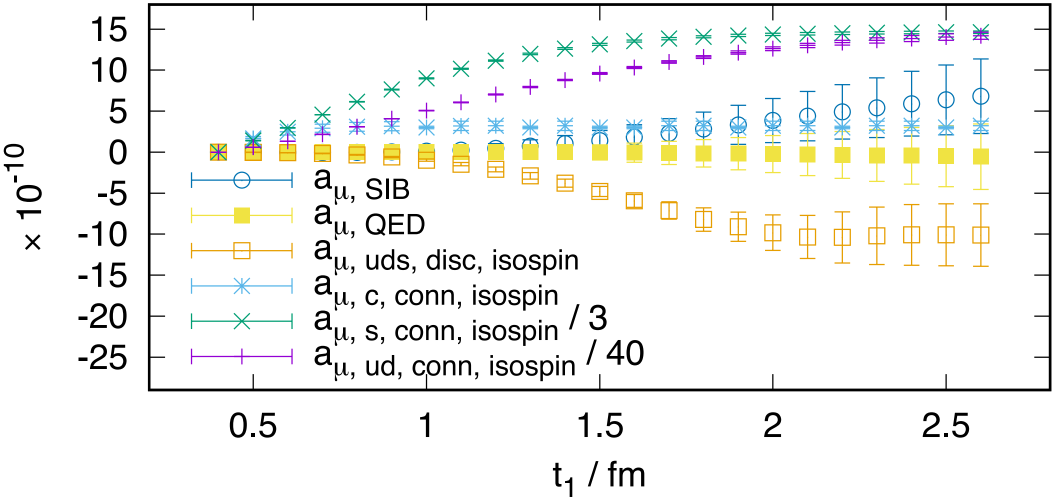
Take a_μ^W from lattice, rest from R-ratio

Select window in t (or, $\equiv \sqrt{s}$)



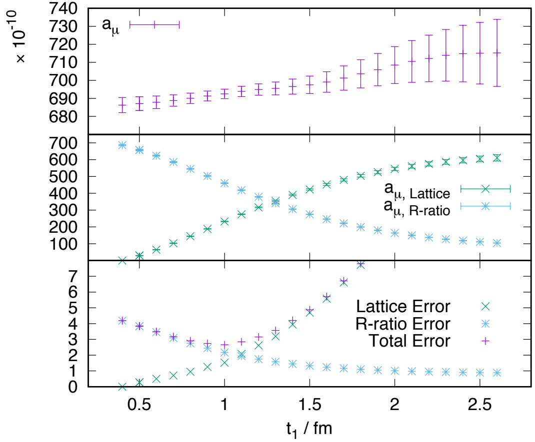


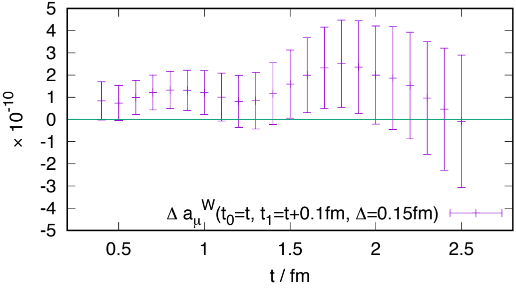
Window method, components ($t_0 = 0.4$ fm) RBC/UKQCD [Blum et al., 2018]



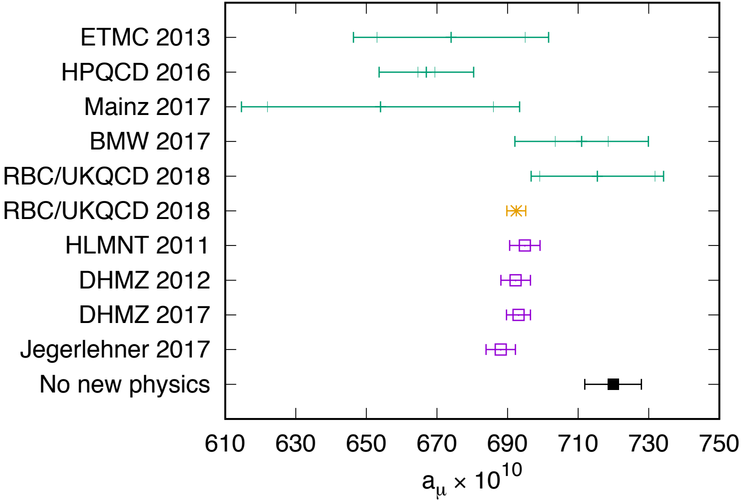
Combined lattice (u,d,s,c) and R-ratio result for a_μ ($t_0 = 0.4\text{fm}$)

RBC/UKQCD [Blum et al., 2018]



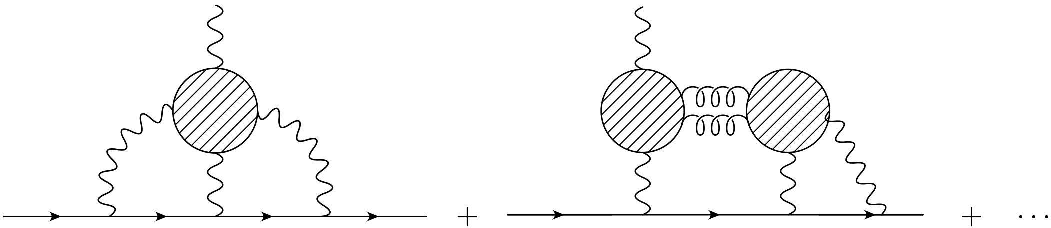


Summary of HVP theory results



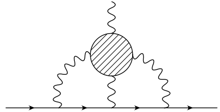
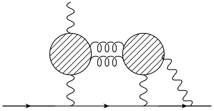
Latest RBC/UKQCD results [Blum et al., 2018]. WM most accurate to date (pub)

Outline I



- Models: $(105 \pm 26) \times 10^{-11}$ [Prades et al., 2009, Benayoun et al., 2014]
  $(116 \pm 40) \times 10^{-11}$ [Jegerlehner and Nyffeler, 2009]
- Model errors difficult to quantify error now compatible with HVP error. see talk by A. Keshavarzi, muon g-2 theory initiative HVP working group workshop, Feb 2018, KEK
- First lattice results promise reliable errors [Blum et al., 2015, Blum et al., 2016, Blum et al., 2017a] see also [Green et al., 2015, Asmussen et al., 2016]
- Dispersive/data approach also systematic [Colangelo et al., 2014b, Pauk and Vanderhaeghen, 2014, Colangelo et al., 2015, Colangelo et al., 2017]

Point source method in QCD+pQED (L. Jin) [Blum et al., 2016]

The desired amplitude  $+$  $+\quad \cdots$ is obtained from a Euclidean space lattice calculation

$$\mathcal{M}_\nu(\vec{q}) \quad = \quad \lim_{\substack{t_{\rm src}\to-\infty \\ t_{\rm snk}\to\infty}} e^{E_{q/2}(t_{\rm snk}-t_{\rm src})} \sum_{\vec{x}_{\rm snk},\vec{x}_{\rm src}} e^{-i\frac{\vec{q}}{2}\cdot(\vec{x}_{\rm snk}+\vec{x}_{\rm src})} e^{i\vec{q}\cdot\vec{x}_{\rm op}} \mathcal{M}_\nu(x_{\rm snk}, x_{\rm op}, x_{\rm src}),$$

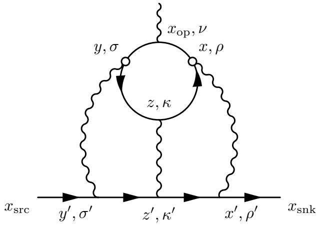
where

$$\begin{aligned} -e\mathcal{M}_\nu(x_{\rm src}, x_{\rm op}, x_{\rm snk}) &= \left\langle \mu(x_{\rm snk}) J_\nu(x_{\rm op}) \overline{\mu}(x_{\rm src}) \right\rangle \\ &= -e \sum_{x,y,z} \sum_{x',y',z'} \mathcal{F}_\nu(x,y,z,x',y',z',x_{\rm op},x_{\rm snk},x_{\rm src}). \end{aligned}$$

and

$$\left[\left(\frac{-i\not{q}^{+} + m_\mu}{2E_{q/2}}\right)\left(F_1(q^2)\gamma_\nu + i\frac{F_2(q^2)}{4m}[\gamma_\nu,\gamma_\rho]q_\rho\right)\left(\frac{-i\not{q}^{-} + m_\mu}{2E_{q/2}}\right)\right]_{\alpha\beta} = \Big(\mathcal{M}_\nu(\vec{q})\Big)_{\alpha\beta},$$

Point source method in QCD+pQED (L. Jin) [Blum et al., 2016]



$$\mathcal{F}^C_\nu\left(\vec{q}; x, y, z, x_{\mathrm{op}}\right) = (-ie)^6 \mathcal{G}_{\rho,\sigma,\kappa}(\vec{q}; x, y, z)\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x, y, z, x_{\mathrm{op}})$$

$$\begin{aligned} & i^4 \mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x, y, z, x_{\mathrm{op}}) \\ = \quad & \sum_{q=u,d,s} \frac{(e_q/e)^4}{6} \left\langle \mathrm{tr}\left[-i \quad \gamma_\rho S_q(x,z) i\gamma_\kappa S_q(z,y) i\gamma_\sigma S_q\left(y, x_{\mathrm{op}}\right) i\gamma_\nu S_q\left(x_{\mathrm{op}}, x\right)\right]\right\rangle_{\mathrm{QCD}} + 5 \text{ permutations} \end{aligned}$$

$$\begin{aligned} & i^3 \mathcal{G}_{\rho,\sigma,\kappa}(\vec{q}; x, y, z) \\ = \quad & e^{\sqrt{m^2+\vec{q}^2/4}(t_{\mathrm{snk}}-t_{\mathrm{src}})} \sum_{x',y',z'} G_{\rho,\rho'}(x,x') G_{\sigma,\sigma'}(y,y') G_{\kappa,\kappa'}(z,z') \\ & \times \sum_{\vec{x}_{\mathrm{snk}},\vec{x}_{\mathrm{src}}} e^{-i\vec{q}/2\cdot(\vec{x}_{\mathrm{snk}}+\vec{x}_{\mathrm{src}})} S\left(x_{\mathrm{snk}}, x'\right) i\gamma_{\rho'} S(x', z') i\gamma_{\kappa'} S(z', y') i\gamma_{\sigma'} S\left(y', x_{\mathrm{src}}\right) + 5 \text{ permutations} \end{aligned}$$

Point source method in QCD+pQED (L. Jin) [Blum et al., 2016]

- Do all sums in the QED part exactly (using FFT's),
- QCD part done stochastically
- Key idea: contribution exponentially suppressed with $r = |x - y|$, so importance sample, concentrate on $r \lesssim \lambda_\pi^{\mathrm{compton}}$
- space-time translational invariance allows coordinates relative to the hadronic loop

$$\mathcal{M}_\nu(\vec{q}) = \sum_r \left\{ \sum_{z, x_{\mathrm{op}}} \mathcal{F}_\nu \left(\vec{q}, \frac{r}{2}, \frac{-r}{2}, z, x_{\mathrm{op}} \right) e^{i \vec{q} \cdot \vec{x}_{\mathrm{op}}} \right\}$$

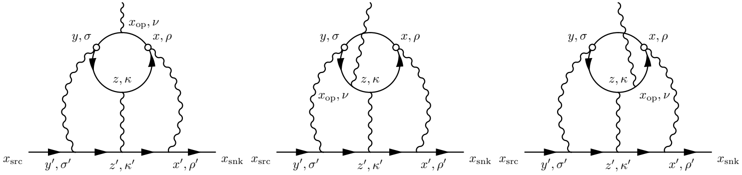
where $r = x - y$, $z \to z - w$, $x_{\mathrm{op}} \to x_{\mathrm{op}} - w$ and $w = (x + y)/2$

- We sum all the internal points over the entire space-time except we fix $x + y = 0$.
- (x, y) pairs stochastically sampled, z and x_{op} sums exact

Point source method in QCD+pQED (L. Jin) [Blum et al., 2016]

$$\langle \mu(\vec{p}')|J_\nu(0)|\mu(\vec{p})\rangle = -e\bar{u}(\vec{p}')\left(F_1(q^2)\gamma_\nu + i\frac{F_2(q^2)}{4m}[\gamma_\nu,\gamma_\rho]q_\rho\right)u(\vec{p})$$

- implies $F_2(0)$ only accessible by extrapolation $q \to 0$.
- Form is due to Ward Identity, or charge conservation
- need WI to be exact on each config, or error blows up as $\vec{q} \to 0$
- To enforce WI compute average of diagrams with all possible insertions of $J_\nu(x_{\rm op})$



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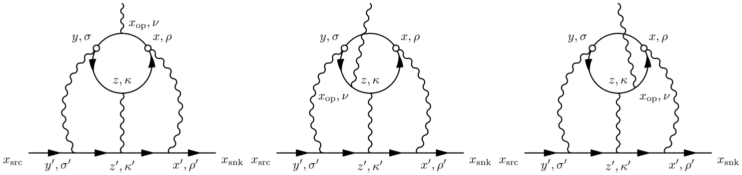

- WI allows a moment method that projects directly to $q = 0$

$$\begin{aligned}
\mathcal{M}_\nu(\vec{q}) &= \sum_{r,z,x_{\rm op}} \mathcal{F}^C_\nu\left(\vec{q}, \frac{r}{2}, -\frac{r}{2}, z, x_{\rm op}\right)\left(e^{i\vec{q}\cdot\vec{x}_{\rm op}} - 1\right) \\
&\approx \sum_{r,z,x_{\rm op}} \mathcal{F}^C_\nu\left(\vec{q}, \frac{r}{2}, -\frac{r}{2}, z, x_{\rm op}\right)(i\vec{q}\cdot\vec{x}_{\rm op}) \\
\frac{\partial}{\partial q_i}\mathcal{M}_\nu(\vec{q})|_{\vec{q}=0} &= i\sum_{r,z,x_{\rm op}} \mathcal{F}^C_\nu\left(\vec{q}=0, r, -r, z, x_{\rm op}\right)(x_{\rm op})_i
\end{aligned}$$

Point source method in QCD+pQED (L. Jin) [Blum et al., 2016]

Sandwich $\mathcal{M}_\nu(\vec{q})$ between positive energy Dirac spinors $u(\vec{0},s)$, $\bar{u}(\vec{0},s)$

$$\overline{u}(\vec{0},s')\left(\frac{F_2(q^2=0)}{2m_\mu}\frac{i}{2}[\gamma_i,\gamma_j]\right)u(\vec{0},s)=\overline{u}(\vec{0},s')\frac{\partial}{\partial q_j}\mathcal{M}_i(\vec{q})|_{\vec{q}=\vec{0}}u(\vec{0},s)$$

multiply both sides by $\frac{1}{2}\epsilon_{ijk}$, sum over i and j,

$$\frac{F_2(0)}{m}\bar{u}_{s'}(\vec{0})\frac{\vec{\Sigma}}{2}u_s(\vec{0}) \quad = \quad \sum_r\left[\sum_{z,x_{\text{op}}}\frac{1}{2}\vec{x}_{\text{op}}\times\bar{u}_{s'}(\vec{0})i\vec{\mathcal{F}}^C\left(\vec{0};x=-\frac{r}{2},y=+\frac{r}{2},z,x_{\text{op}}\right)u_s(\vec{0})\right]$$

where $\Sigma_i=\frac{1}{4i}\epsilon_{ijk}[\gamma_j,\gamma_k]$.

Lattice setup

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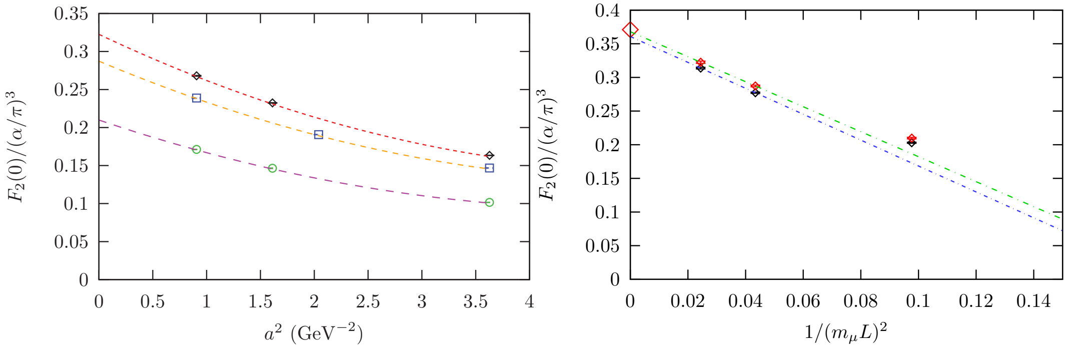
a (fm)	0.114	0.084
L (fm)	5.47	5.38
L_s	24	12
m_π (MeV)	139	135
m_μ (MeV)	106	106

Continuum and ∞ volume limits in QED [Blum et al., 2016]

Test method in pure QED

QED systematics large, $O(a^4)$, $O(1/L^2)$, but under control

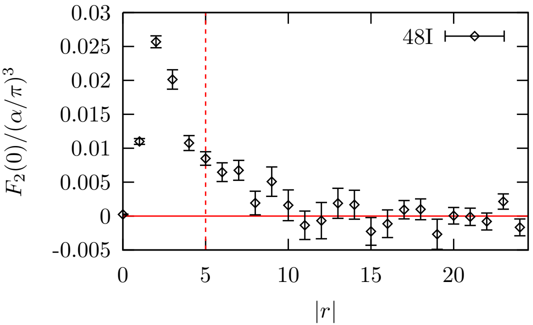


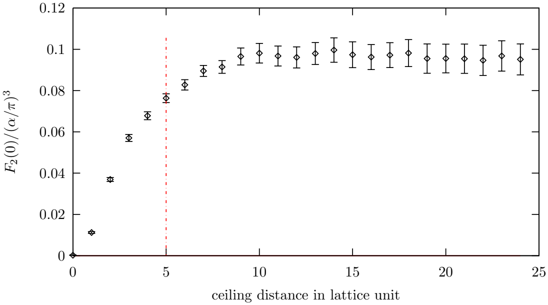
Limits quite consistent with well known PT result

Outline I

Physical point cHLbL contribution, 48^3, 1.73 GeV lattice [Blum et al., 2017a]

- Measurements on 65 configurations, separated by 20 trajectories
- ignore strange quark contribution (down by 1/17 plus mass suppressed)
- exponentially suppressed with distance
- most of contribution by about 1 fm

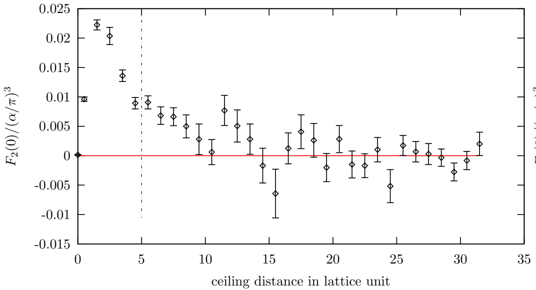


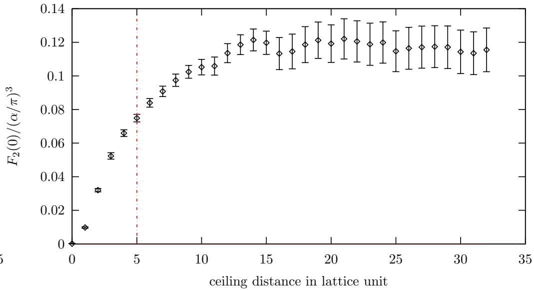


$$a_\mu^{\mathrm{cHLbL}} = 11.60 \pm 0.96 \times 10^{-10}$$

Physical point cHLbL contribution, 64^3, 2.36 GeV lattice (preliminary)

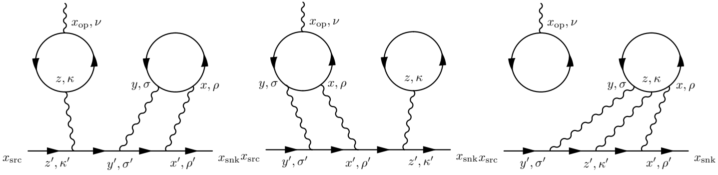
- Measurements as before, but 43 configurations
- exponentially suppressed with distance
- most of contribution by about 1 fm



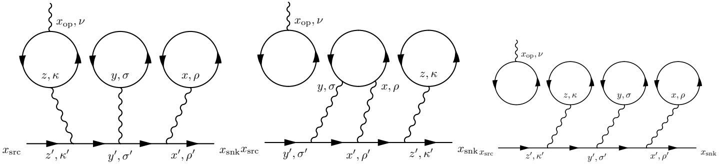


Disconnected contributions

SU(3) flavor:



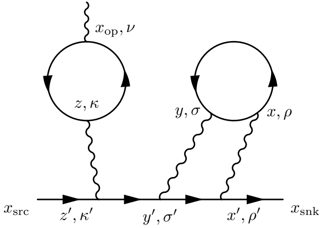
Leading $\qquad\qquad\qquad\qquad$ $O(m_s - m_{u,d})$



$O(m_s - m_{u,d})^2$ $\qquad\qquad$ and higher

- Gluons within and connecting quark loops have not been drawn
- To ensure loops are connected by gluons, explicit "vacuum" subtraction is required

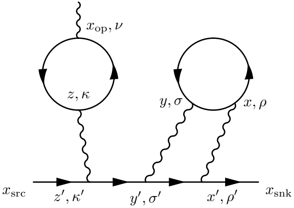
Leading disconnected contribution



- We use two point sources at y and z, chosen randomly. The points sinks x_{op} and x are summed over exactly on lattice.
- Only point source quark propagators are needed. We compute M point source propagators and all M^2 combinations are used to perform the stochastic sum over $r = z - y$ (M^2 trick).

$$
\begin{aligned}
\mathcal{F}^D_\nu(x, y, z, x_{\text{op}}) &= (-ie)^6 \mathcal{G}_{\rho,\sigma,\kappa}(x, y, z)\mathcal{H}^D_{\rho,\sigma,\kappa,\nu}(x, y, z, x_{\text{op}}) \\
\mathcal{H}^D_{\rho,\sigma,\kappa,\nu}(x, y, z, x_{\text{op}}) &= \left\langle \frac{1}{2}\Pi_{\nu,\kappa}(x_{\text{op}}, z)\left[\Pi_{\rho,\sigma}(x, y) - \Pi^{\text{avg}}_{\rho,\sigma}(x - y)\right]\right\rangle_{\text{QCD}} \\
\Pi_{\rho,\sigma}(x, y) &= -\sum_q (e_q/e)^2 \,\text{Tr}[\gamma_\rho S_q(x, y)\gamma_\sigma S_q(y, x)].
\end{aligned}
$$

Leading disconnected contribution



$$\frac{F_2^{\mathrm{dHLbL}}(0)}{m}\frac{(\sigma_{s',s})_i}{2} = \sum_{r,x}\sum_{x_{\mathrm{op}}}\frac{1}{2}\epsilon_{i,j,k}\left(x_{\mathrm{op}}\right)_j \cdot i\bar{u}_{s'}(\vec{0})\mathcal{F}_k^D\left(x, y=r, z=0, x_{\mathrm{op}}\right)u_s(\vec{0})$$

$$\mathcal{H}^D_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\mathrm{op}}) = \left\langle \frac{1}{2}\Pi_{\nu,\kappa}\left(x_{\mathrm{op}}, z\right)\left[\Pi_{\rho,\sigma}(x,y) - \Pi^{\mathrm{avg}}_{\rho,\sigma}(x-y)\right]\right\rangle_{\mathrm{QCD}}$$

$$\sum_{x_{\mathrm{op}}}\frac{1}{2}\epsilon_{i,j,k}\left(x_{\mathrm{op}}\right)_j\left\langle\Pi_{\rho,\sigma}\left(x_{\mathrm{op}},0\right)\right\rangle_{\mathrm{QCD}} = \sum_{x_{\mathrm{op}}}\frac{1}{2}\epsilon_{i,j,k}\left(-x_{\mathrm{op}}\right)_j\left\langle\Pi_{\rho,\sigma}\left(-x_{\mathrm{op}},0\right)\right\rangle_{\mathrm{QCD}} = 0$$

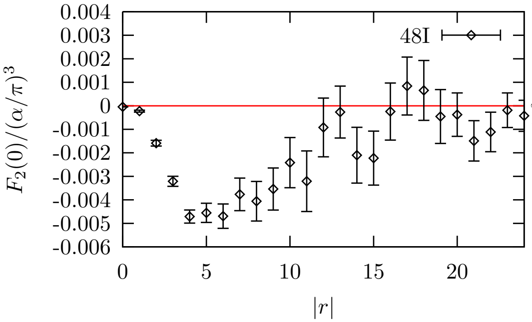
- Because of parity, the expectation value for the (moment of) left loop averages to zero.
- $\left[\Pi_{\rho,\sigma}(x,y) - \Pi^{\mathrm{avg}}_{\rho,\sigma}(x-y)\right]$ is only a noise reduction technique. $\Pi^{\mathrm{avg}}_{\rho,\sigma}(x-y)$ should remain constant through out the entire calculation.

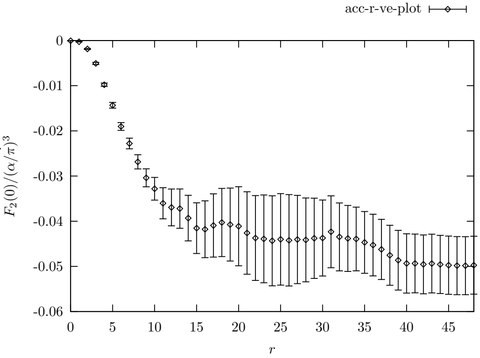
Physical point dHLbL contribution [Blum et al., 2017a]

- Use AMA with 2000 low-modes of the Dirac operator and
- randomly choose 256 “spheres” of radius 6 lattice units
- Uniformly sample 4 (unique) points in each
- do half as many strange quark props
- Construct $(1024 + 512)^2$ point-pairs per configuration

Physical point dHLbL contribution, 48^3, 1.73 GeV lattice [Blum et al., 2017a]

- strange contributes less than 5 %



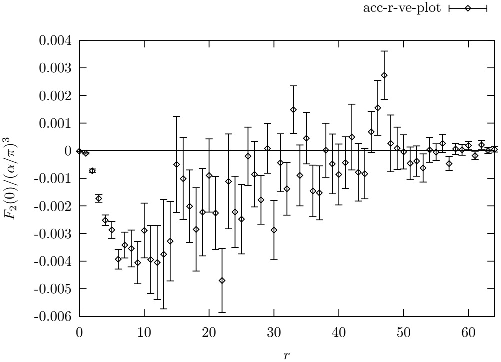


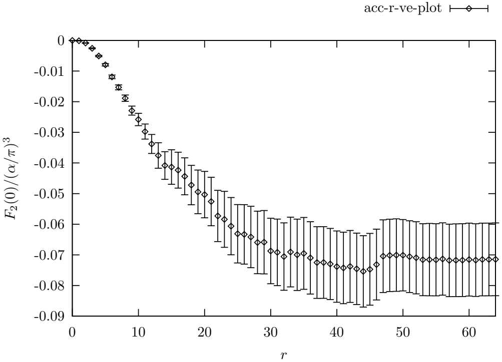
$$a_\mu^{\mathrm{dHLbL}} = -6.25 \pm 0.80 \times 10^{-10}$$

$$a_\mu^{\mathrm{cHLbL}} + a_\mu^{\mathrm{dHLbL}} = 5.35 \pm 1.35 \times 10^{-10}$$

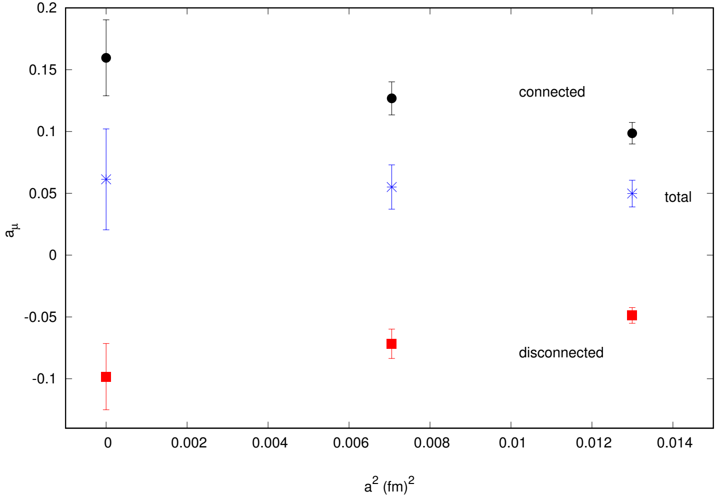
Physical point dHLbL contribution, 64^3, 2.36 GeV lattice (preliminary)

- 44 configurations





Continuum extrapolation (preliminary)



- linear in $a^2 \to 0$ extrapolation
- Effects tend to cancel between cHLbL and dHLbL contributions
- Collecting more statistics

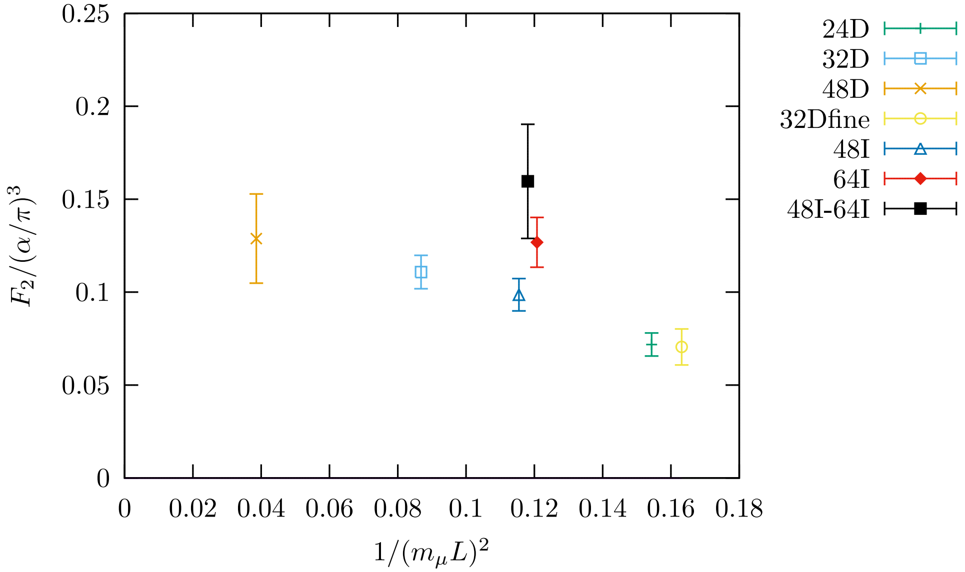
Outline I

New QCD ensembles for finite volume study

2+1f Möbius-DWF, I-DSDR physical point ensembles (RBC/UKQCD)

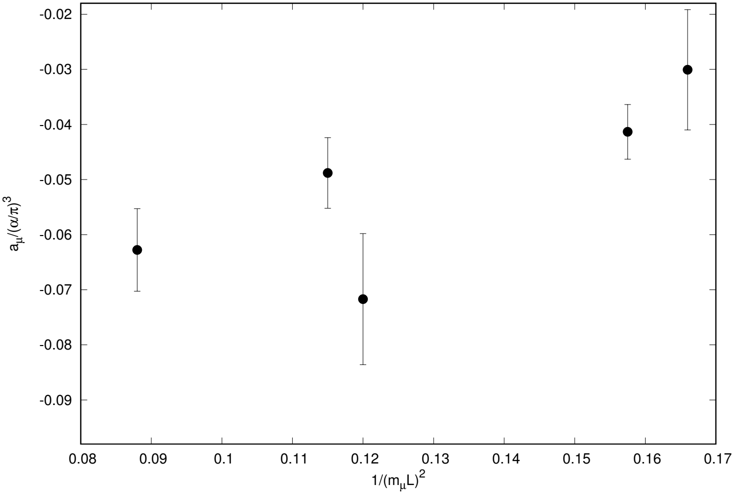
	$24^3 \times 64$	$32^3 \times 64$	$32^3 \times 64$	$48^3 \times 96$
a^{-1} (GeV)	1.0	1.0	1.38	1.0
a (fm)	0.2	0.2	0.14	0.2
L (fm)	4.8	6.4	4.6	9.6
L_s	24	24	24	24
m_π (MeV)	140	140	140	140
m_μ (MeV)	106	106	106	106

QED$_L$, connected diagram



(all particles with physical masses)

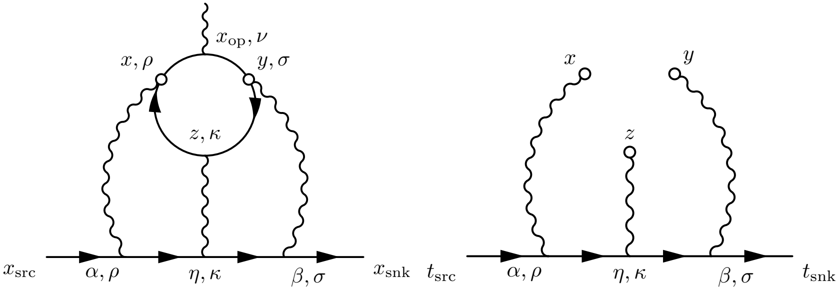
QED$_L$, disconnected diagram



- Left to right: 32D, 48I, 64I, 24D, 32Df
- Effects tend to cancel with cHLbL
- Collecting more statistics

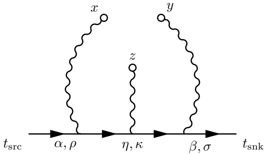
(all particles with physical masses)

QED$_\infty$ [Green et al., 2015, Asmussen et al., 2016, Lehner and Izubuchi, 2015, Jin et al., 2015, Blum et al., 2017b]



- Mainz group made first concrete proposal for QED$_\infty$
- QED$_\infty$: muon, photons computed in infinite volume (*c.f.* HVP)
- QCD mass gap: $\mathcal{H}^{C}_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op}) \sim \exp -m_\pi \times \mathrm{dist}(x,y,z,x_{\rm op})$
- QED weight function does not grow exponentially
- So leading FV error is exponentially suppressed (*c.f.* HVP) instead of $O(1/L^2)$

QED_∞ weighting function [Blum et al., 2017b]



$$= \quad \mathfrak{G}_{\rho,\sigma,\kappa}(x, y, z) + 5 \text{ perms.}$$

- Note Hermitian part gives same F_2 but is infrared finite,

$$\mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x, y, z) = \frac{1}{2}\mathfrak{G}_{\rho,\sigma,\kappa}(x, y, z) + \frac{1}{2}\mathfrak{G}_{\rho,\sigma,\kappa}(x, y, z)^\dagger$$

- In units of the muon mass m_μ,

$$\begin{aligned} \mathfrak{G}^{(1)}_{\sigma,\kappa,\rho}(y, z, x) &= \frac{\gamma_0 + 1}{2} i\gamma_\sigma(-\partial\!\!\!/_y + \gamma_0 + 1) i\gamma_\kappa(\partial\!\!\!/_x + \gamma_0 + 1) i\gamma_\rho \frac{\gamma_0 + 1}{2} \\ &\quad \times \frac{1}{4\pi^2} \int d^4\eta \frac{1}{(\eta - z)^2} f(\eta - y) f(x - \eta) \end{aligned}$$

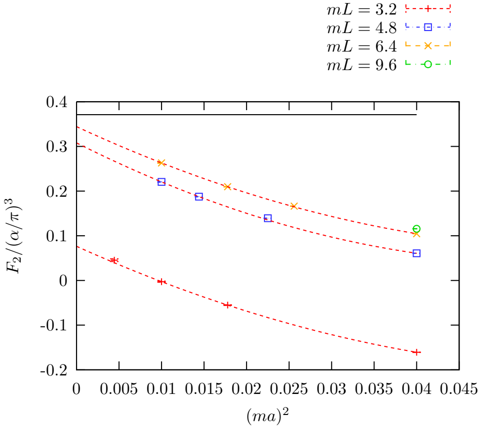
QED_∞ subtraction [Blum et al., 2017b]

- Current conservation implies $\sum_x \mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\mathrm{op}}) = 0$ ($V \to \infty$ and $a \to 0$)
- Subtract terms that vanish as $a, V \to 0$
$\mathfrak{G}^{(2)}_{\rho,\sigma,\kappa}(x,y,z) = \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,z) - \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(y,y,z) - \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,y) + \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(y,y,y)$
- subtraction changes (may reduce) a and V systematic errors ($c.f.$ HVP)
- Further, $\mathfrak{G}^{(2)}_{\rho,\sigma,\kappa}(z,z,x) = 0$ so short distance $O(a^2)$ effects suppressed.

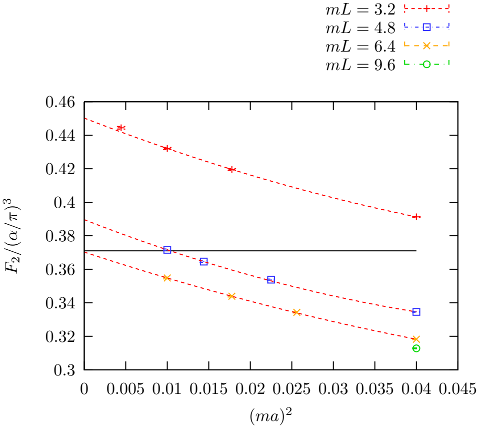
- The 4-dim integral is (pre-)calculated numerically with CUBA library (cubature rules).
- Translation/rotation symmetry: parametrize (x,y,z) by 5 parameters on N^5 grid points (Mainz uses 3 params by averaging over muon time direction).
- (linearly) Interpolate grid in stochastic integral over (x,y)

QED$_\infty$ results- pure QED, lattice-spacing error [Blum et al., 2017b]

- lattice spacing error $\approx$const for $mL \gtrsim 4.8$
- FV effect $\lesssim 1\%$ for $mL = 9.6$
- fit: $F_2(L,a) = F_2(L) + k_1 a^2 + k_2 a^4$



$\mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,z)$

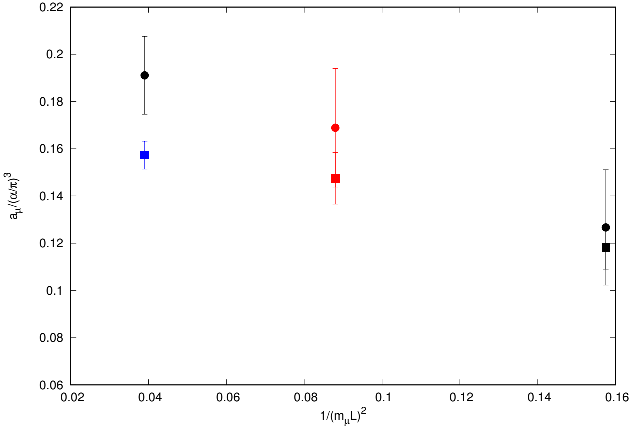


$\mathfrak{G}^{(2)}_{\rho,\sigma,\kappa}(x,y,z)$

QED$_\infty$ results- pure QED, finite volume error [Blum et al., 2017b]

- Take $F_2(\infty) \approx F_2(mL = 9.6)$
- results for $m_{\rm loop} = m_{\rm line}$ (a_e) and $m_{\rm loop} = 2m_{\rm line}$
- $F_2/(\alpha/\pi)^3 =$ 0.3686(37)(35) and 0.1232(30)(28) compared to
- QED perturbation theory results : 0.371 and 0.120

QED$_\infty$, connected diagram, $a = 0.2$ fm (preliminary)



(all particles with physical masses)

- QED$_\infty$ noisier than QED$_L$
- make distance cuts to enhance signal, suppress noise
  - Upper: 'short' cut = 0.16 fm
  - Lower: 'short' cut = 0.10 fm
- Collecting more statistics

Outline I

Hadronic vacuum polarization

- Much progress recently (many groups)
- Lattice QCD(+QED) calculations done with physical masses, large boxes + improved measurement algorithms
- disconnected contributions computed by 2 groups
- Included NLO QED and Strong Isospin breaking corrections
- Lattice and R-ratio results cross-checked and combined with window method
- Window method allows further error reduction over R-ratio alone
  most accurate determination to date RBC/UKQCD [Blum et al., 2018]
- improved bounding+inclusive channel methods being developed: significant stat error reduction, improved systematics

Hadronic Light-by-Light

- Lattice QCD(+QED) calculations done with physical masses, large boxes + improved measurement algorithms
- Physical point calculations complete at $a = 0.114$ fm [Blum et al., 2017a]
- Physical point nearly complete at $a = 0.084$ fm (increasing statistics)
- together, good control of non-zero a systematic error.
- FV corrections: QED_∞ (QED_L) + large 9.5 fm QCD box (underway)
- Need non-leading disconnected diagrams (underway)
- Lattice: unlikely that HLbL contribution will rescue standard model

On track for solid SM result in time for E989 result: Muon g-2 Theory Initiative 2nd plenary meeting in June (Mainz), WP by the end of the year

Acknowledgments

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Outline I

Aoyama, T., Hayakawa, M., Kinoshita, T., and Nio, M. (2012).

Complete Tenth-Order QED Contribution to the Muon g-2.

Phys.Rev.Lett., 109:111808.

Asmussen, N., Green, J., Meyer, H. B., and Nyffeler, A. (2016).

Position-space approach to hadronic light-by-light scattering in the muon $g-2$ on the lattice.

PoS, LATTICE2016:164.

Benayoun, M., Bijnens, J., Blum, T., Caprini, I., Colangelo, G., et al. (2014).

Hadronic contributions to the muon anomalous magnetic moment Workshop. $(g-2)_\mu$: Quo vadis? Workshop. Mini proceedings.

Bennett, G. et al. (2006).

Final Report of the Muon E821 Anomalous Magnetic Moment Measurement at BNL.

Phys.Rev., D73:072003.

Blum, T. (2003).

Lattice calculation of the lowest order hadronic contribution to the muon anomalous magnetic moment.
Phys.Rev.Lett., 91:052001.

Blum, T., Boyle, P. A., Glpers, V., Izubuchi, T., Jin, L., Jung, C., Jttner, A., Lehner, C., Portelli, A., and Tsang, J. T. (2018).
Calculation of the hadronic vacuum polarization contribution to the muon anomalous magnetic moment.
to be published, Phys. Rev. Lett.

Blum, T., Chowdhury, S., Hayakawa, M., and Izubuchi, T. (2015).
Hadronic light-by-light scattering contribution to the muon anomalous magnetic moment from lattice QCD.
Phys.Rev.Lett., 114(1):012001.

Blum, T., Christ, N., Hayakawa, M., Izubuchi, T., Jin, L., Jung, C., and Lehner, C. (2017a).
Connected and Leading Disconnected Hadronic Light-by-Light Contribution to the Muon Anomalous Magnetic Moment with a Physical Pion Mass.
Phys. Rev. Lett., 118(2):022005.

Blum, T., Christ, N., Hayakawa, M., Izubuchi, T., Jin, L., Jung, C., and Lehner, C. (2017b).

Using infinite volume, continuum QED and lattice QCD for the hadronic light-by-light contribution to the muon anomalous magnetic moment.

Blum, T., Christ, N., Hayakawa, M., Izubuchi, T., Jin, L., and Lehner, C. (2016).

Lattice Calculation of Hadronic Light-by-Light Contribution to the Muon Anomalous Magnetic Moment.

Phys. Rev., D93(1):014503.

Blum, T. et al. (2014).

Domain wall QCD with physical quark masses.

Colangelo, G., Hoferichter, M., Nyffeler, A., Passera, M., and Stoffer, P. (2014a).

Remarks on higher-order hadronic corrections to the muon g?2.

Phys. Lett., B735:90–91.

Colangelo, G., Hoferichter, M., Procura, M., and Stoffer, P. (2014b).

Dispersive approach to hadronic light-by-light scattering.

JHEP, 1409:091.

Colangelo, G., Hoferichter, M., Procura, M., and Stoffer, P. (2015).
Dispersion relation for hadronic light-by-light scattering: theoretical foundations.

Colangelo, G., Hoferichter, M., Procura, M., and Stoffer, P. (2017).
Dispersion relation for hadronic light-by-light scattering: two-pion contributions.
JHEP, 04:161.

Davier, M., Hoecker, A., Malaescu, B., and Zhang, Z. (2011).
Reevaluation of the Hadronic Contributions to the Muon g-2 and to alpha(MZ).
Eur.Phys.J., C71:1515.

Gnendiger, C., Stckinger, D., and Stckinger-Kim, H. (2013).
The electroweak contributions to $(g-2)_\mu$ after the Higgs boson mass measurement.
Phys.Rev., D88:053005.

Green, J., Gryniuk, O., von Hippel, G., Meyer, H. B., and Pascalutsa, V. (2015).
Lattice QCD calculation of hadronic light-by-light scattering.
Phys. Rev. Lett., 115(22):222003.

Hagiwara, K., Liao, R., Martin, A. D., Nomura, D., and Teubner, T. (2011).
$(g-2)_\mu$ and alpha(M_Z^2) re-evaluated using new precise data.
J.Phys., G38:085003.

Hayakawa, M. and Uno, S. (2008).
QED in finite volume and finite size scaling effect on electromagnetic properties of hadrons.
Prog.Theor.Phys., 120:413–441.

Jegerlehner, F. and Nyffeler, A. (2009).
The Muon g-2.
Phys. Rept., 477:1–110.

Jin, L., Blum, T., Christ, N., Hayakawa, M., Izubuchi, T., and Lehner, C. (2015).
Lattice Calculation of the Connected Hadronic Light-by-Light Contribution to the Muon Anomalous Magnetic Moment.
In *Proceedings, 12th Conference on the Intersections of Particle and Nuclear Physics (CIPANP 2015): Vail, Colorado, USA, May 19-24, 2015*.

Kurz, A., Liu, T., Marquard, P., and Steinhauser, M. (2014).
Hadronic contribution to the muon anomalous magnetic moment to next-to-next-to-leading order.
Phys.Lett., B734:144–147.

Lautrup, B., Peterman, A., and De Rafael, E. (1971).
On sixth-order radiative corrections to a(mu)-a(e).
Nuovo Cim., A1:238–242.

Lehner, C. and Izubuchi, T. (2015).
Towards the large volume limit - A method for lattice QCD + QED simulations.
PoS, LATTICE2014:164.

Pauk, V. and Vanderhaeghen, M. (2014).
Two-loop massive scalar three-point function in a dispersive approach.

Prades, J., de Rafael, E., and Vainshtein, A. (2009).
Hadronic Light-by-Light Scattering Contribution to the Muon Anomalous Magnetic Moment.