ELECTROMAGNETIC SHOWERS

Lecture 2

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WAPP 2014
Ooty 21th december 2014

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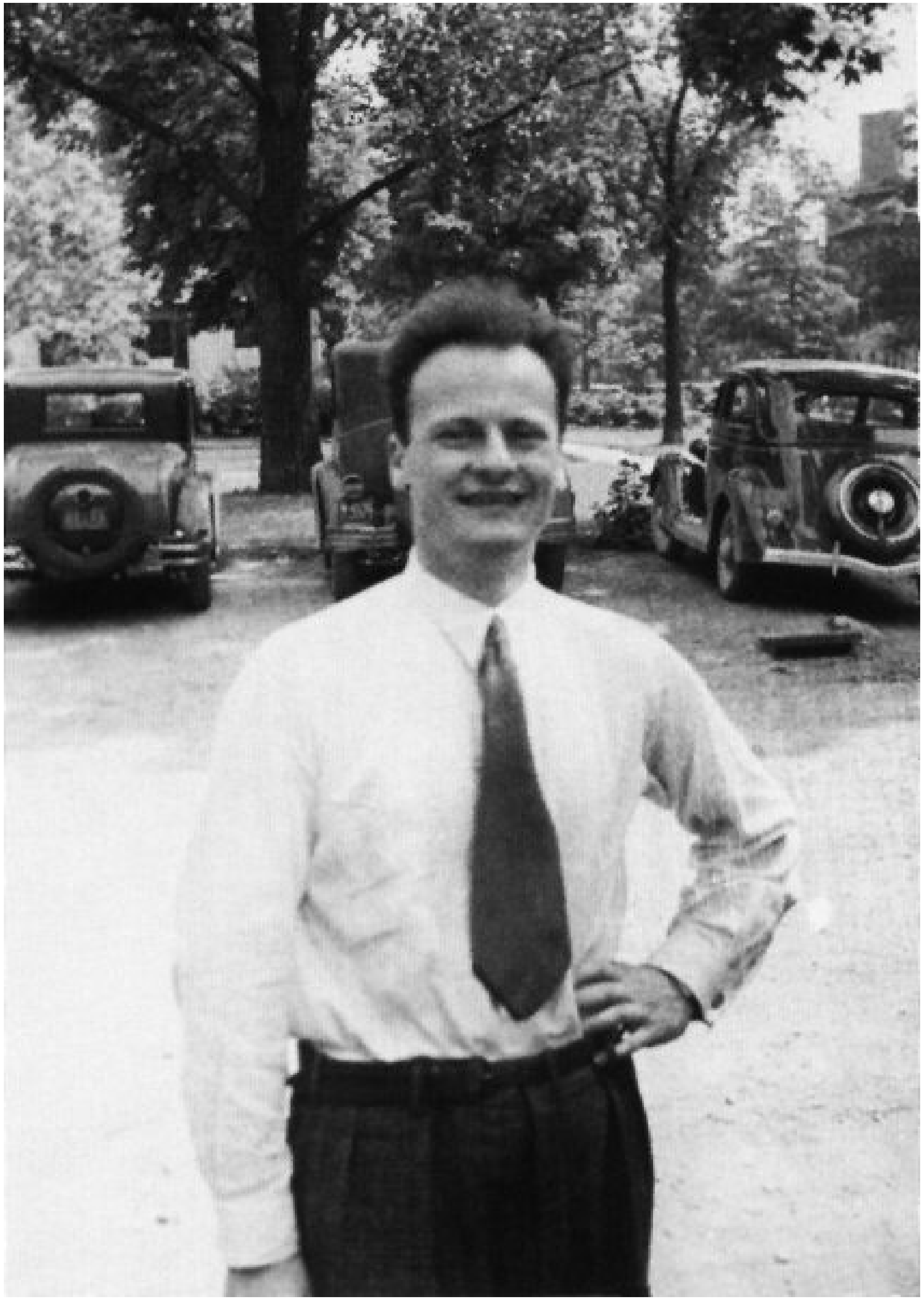
The Passage of Fast Electrons and the Theory of Cosmic Showers

By H. J. Bhabha, *Gonville and Caius College, Cambridge*
and W. Heitler, *Wills Physical Laboratory, University of Bristol*

(*Communicated by N. F. Mott, F.R.S.—Received* 11 *December* 1936)

We have used relativistic quantum mechanics to calculate, subject to some simplifying assumptions (§ 1), the number of secondary positive and negative electrons produced by a fast primary electron with energy E_0 passing through a layer of matter of thickness l. The process in question is the following: The primary electron in the field of a nucleus has a large probability of emitting a hard light quantum which then creates a pair. The pair electrons emit light quanta again which create pairs and so on.



Hans BETHE

Walter HEITLER

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Bhabha Chakrabarthy (3 papers)

FEBRUARY 15, 1937 PHYSICAL REVIEW VOLUME 51

On Multiplicative Showers

J. F. CARLSON AND J. R. OPPENHEIMER
University of California, Berkeley, California

(Received December 8, 1936)

In I we discuss the status of the quantum theoretic formulae for pair production and radiation in the domain of cosmic-ray energies, and the relevance of these processes to an understanding of showers and bursts. In II we give a qualitative estimate of the course implied by the theory for a shower or burst built up by multiplication from a very energetic primary; we then set up the diffusion equations for the equilibrium of electrons and gamma-rays, and show how these can be simplified. In III we carry through the analytic solution of the diffusion equations, and find the distribution of electrons and gamma-rays as a function of their energy, the primary energy, and the thickness and atomic number of the matter traversed. We treat the effect of ionization losses on the shower, calculate the amount of radiation of low energy to be expected, and treat transition effects in passing from one substance to another. In IV we discuss the results of the calculations, and give a summary of the conclusions to which they lead, and the difficulties.

Independent work performed
at essentially exactly the same time
(3 days difference in the submission time)

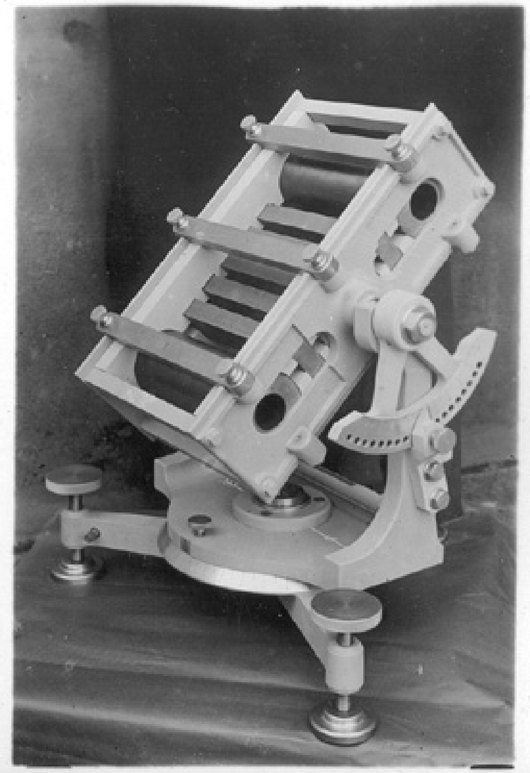
Similar work by L.LANDAU in the Soviet Union

OCTOBER, 1941 REVIEWS OF MODERN PHYSICS

Cosmic-Ray Theory

BRUNO ROSSI AND KENNETH GREISEN
Cornell University, Ithaca, New York

Bruno Rossi
1933 Eritrea
East West effect



$$\frac{1}{\lambda_{\rm rad}} = 4\alpha r_0^2 \, \frac{Z^2 \, N_A}{A} \, \log\left[183 \, Z^{-1/3}\right]$$

$$+ \text{ Electron Contribution.}$$

RADIATION LENGTH Meaning:

Length where the energy
of an electron is reduced to E/e

7/9 of the mean free path of photons

From Particle Data Book

27.4. Photon and electron interactions in matter

27.4.1. *Radiation length* : High-energy electrons predominantly lose energy in matter by bremsstrahlung, and high-energy photons by e^+e^- pair production. The characteristic amount of matter traversed for these related interactions is called the radiation length X_0, usually measured in g cm^{-2}. It is both (a) the mean distance over which a high-energy electron loses all but $1/e$ of its energy by bremsstrahlung, and (b) $\frac{7}{9}$ of the mean free path for pair production by a high-energy photon [35]. It is also the appropriate scale length for describing high-energy electromagnetic cascades. X_0 has been calculated and tabulated by Y.S. Tsai [36]:

$$\frac{1}{X_0} = 4\alpha r_e^2 \frac{N_A}{A} \left\{ Z^2 \left[L_{\rm rad} - f(Z) \right] + Z\, L'_{\rm rad} \right\} . \qquad (27.20)$$

For $A = 1$ g mol^{-1}, $4\alpha r_e^2 N_A/A = (716.408 \text{ g cm}^{-2})^{-1}$. $L_{\rm rad}$ and $L'_{\rm rad}$ are given in Table 27.2. The function $f(Z)$ is an infinite sum, but for elements up to uranium can be represented to 4-place accuracy by

$$\begin{aligned} f(Z) = a^2 \big[&(1+a^2)^{-1} + 0.20206 \\ &-0.0369\, a^2 + 0.0083\, a^4 - 0.002\, a^6 \big] , \end{aligned} \qquad (27.21)$$

where $a = \alpha Z$ [37].

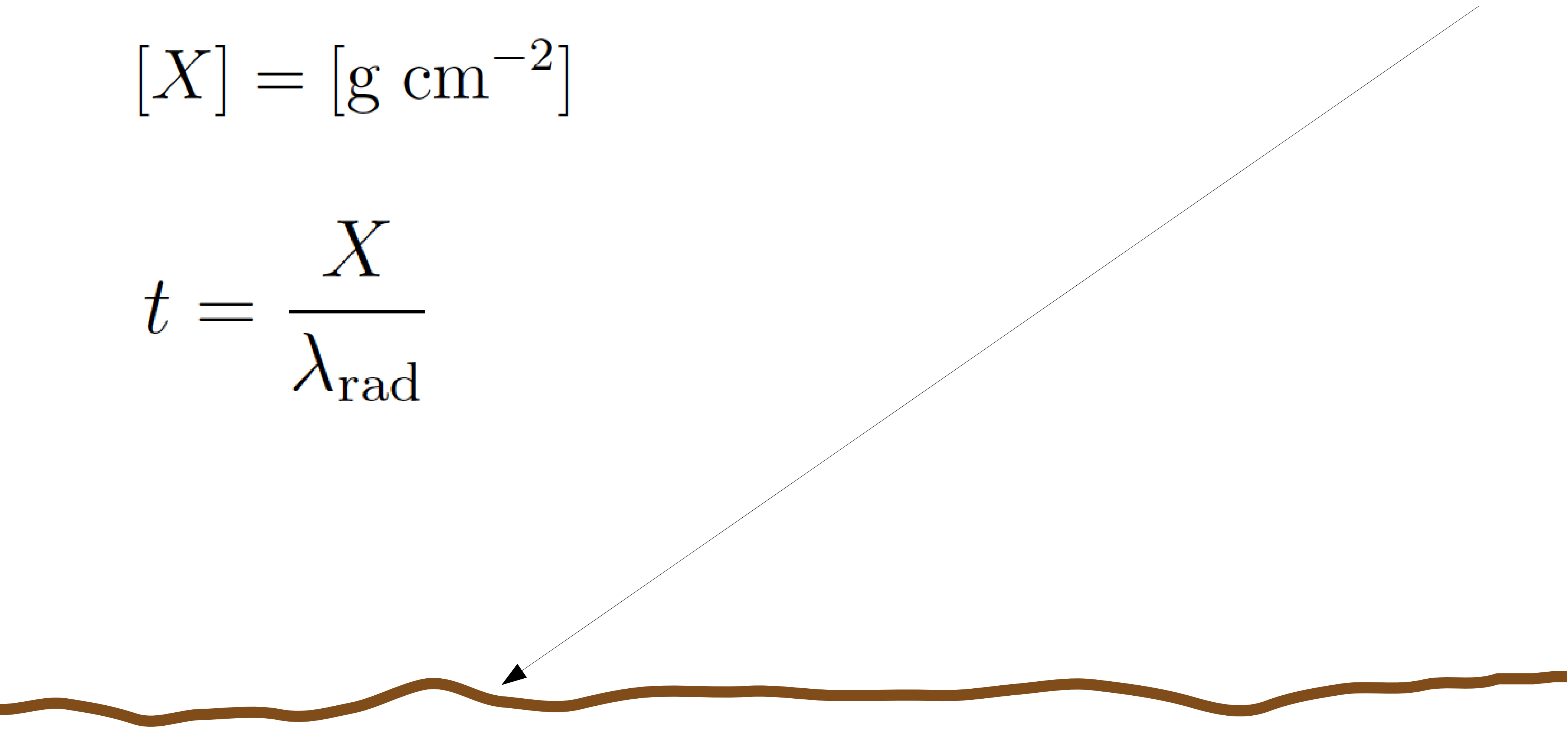
Longitudinal development of a shower.

$$X(z) = \int_0^z dz' \; \rho(z')$$

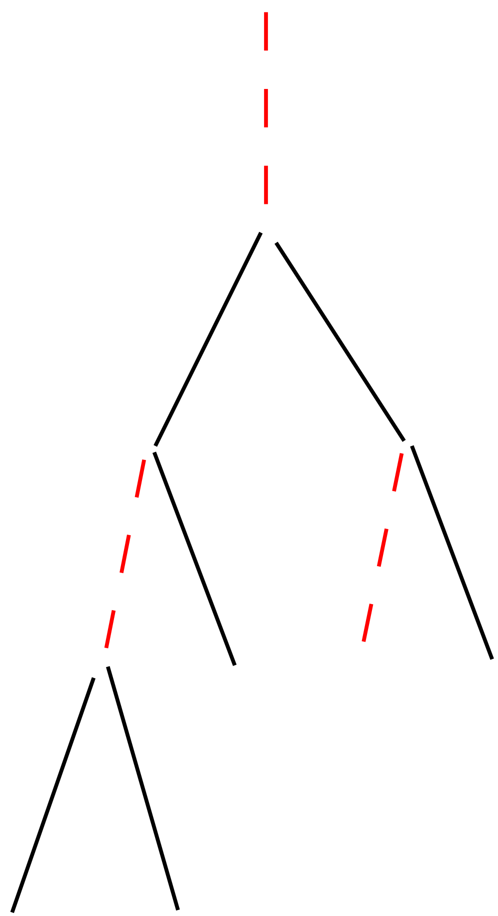
"Column density"

$$[X] = [\mathrm{g\ cm}^{-2}]$$

$$t = \frac{X}{\lambda_{\mathrm{rad}}}$$



ELECTROMAGNETIC SHOWERS



$\psi(u)$ Pair Production

$$e + Z \to e + \gamma + Z$$

$\varphi(v)$ Brems-strahlung

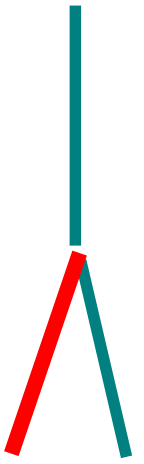
$$\gamma + Z \to e^+ + e^- + Z$$

Iteration of
2 fundamental processes
Pair Production
Bremsstrahlung

The “SPLITTING FUNCTIONS”

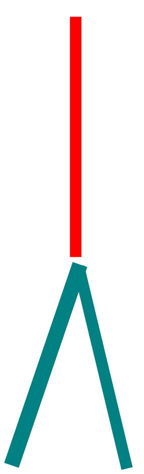
$$\varphi(v) = \left[\frac{d\sigma}{dv}(v)\right]_{\text{brems}} \left(\frac{N_A}{A}\,\lambda_{\text{rad}}\right)$$

$$\varphi(v) = \frac{1}{v}\,\left[1 - \left(\frac{2}{3} - 2\,b\right)\,(1-v) + (1-v)^2\right]$$



$$\psi(u) = \left[\frac{d\sigma}{du}(u)\right]_{\text{pair}} \left(\frac{N_A}{A}\,\lambda_{\text{rad}}\right)$$

$$\psi(u) = (1-u)^2 + \left(\frac{2}{3} - 2\,b\right)\,(1-u)\;u + u^2$$



$$\varphi(v)\ dv\ dt$$

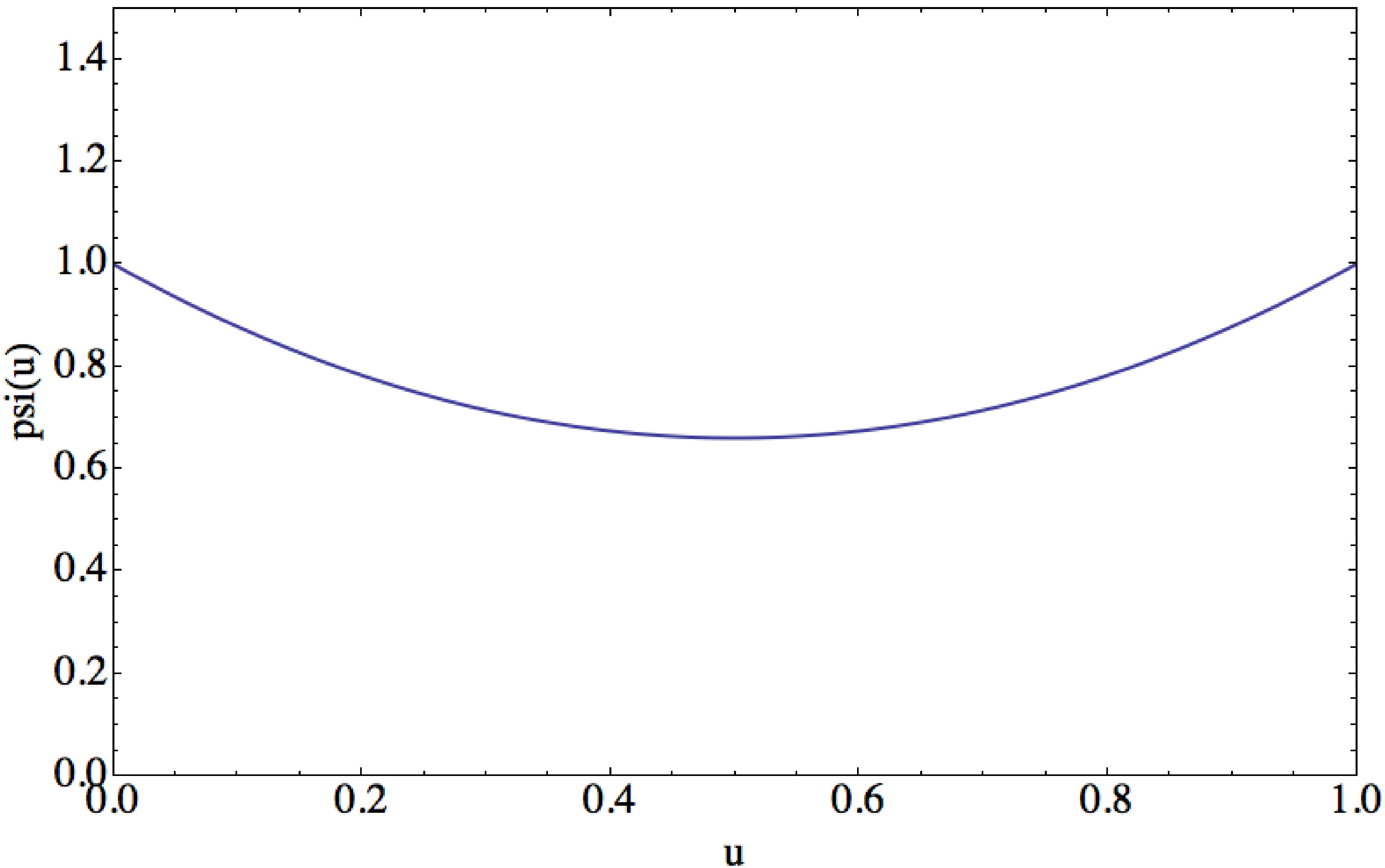
Probability for a photon of (any) energy E_γ
to generate one positron with fractional energy
$u = E_e/E_\gamma$ in the interval $[u, u + du]$
when traversing a layer of material of thickness dt

$$\psi(u)\ dv\ dt$$

Probability for an electron of (any) energy E_e
to generate one photon with fractional energy
$v = E_\gamma/E_e$ in the interval $[v, v + dv]$,
when traversing a layer of material of thickness dt

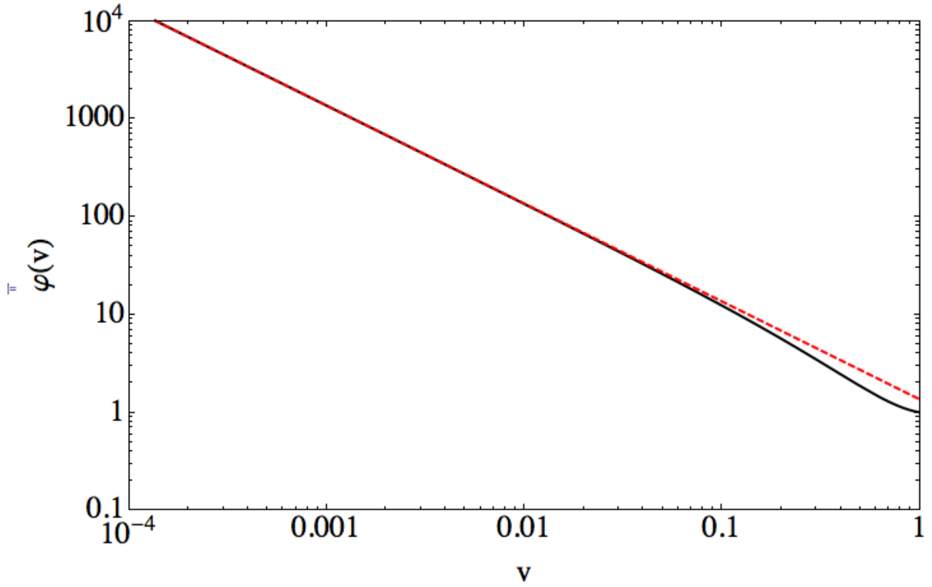
$\psi(u)$

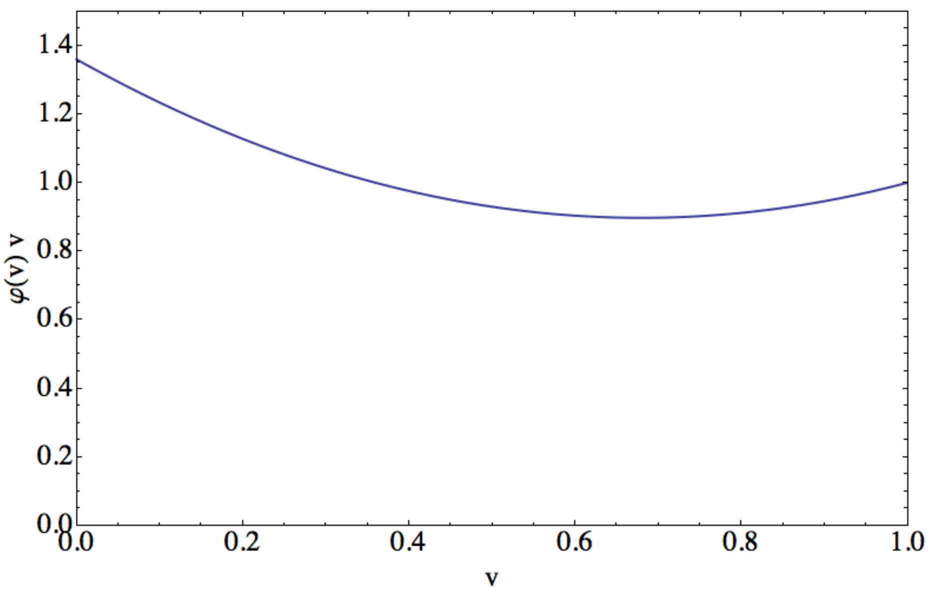
Pair Production



$\varphi(v)$

Bremsstrahlung





Energy loss for electrons and positrons

$$\frac{dE}{dX} = \left.\frac{dE}{dX}\right|_{\text{collisions}} + \left.\frac{dE}{dX}\right|_{\text{bremsstrahlung}}$$

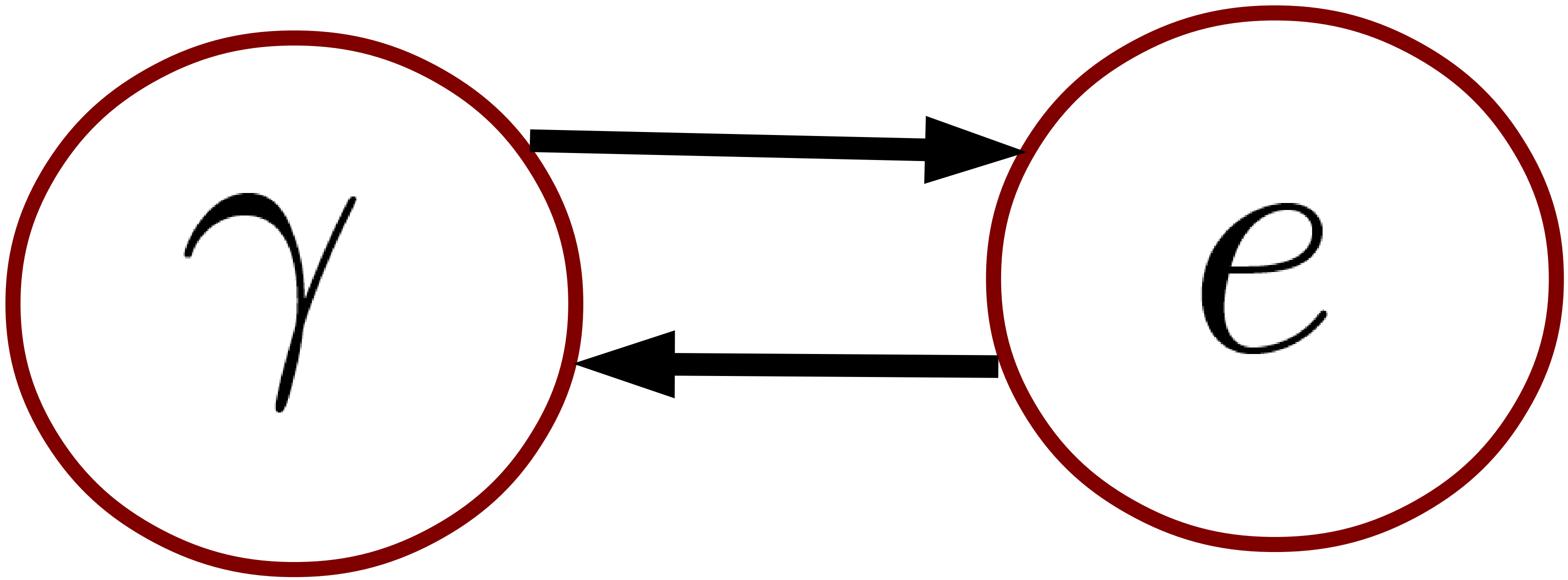
$$= \epsilon_{\text{coll}} + \frac{E}{\lambda_{\text{rad}}}$$

$$\left.\frac{dE}{dX}\right|_{\text{collisions}} = \left.\frac{dE}{dX}\right|_{\text{bremsstrahlung}}$$

at the "critical energy"

$$E^* \equiv \varepsilon = \epsilon_{\text{coll}} \times \lambda_{\text{rad}}$$

$$2.2\ \frac{\text{MeV}}{\text{g cm}^{-2}} \times 37\ \text{g}\,\text{cm}^{-2} \simeq 81\ \text{MeV}$$



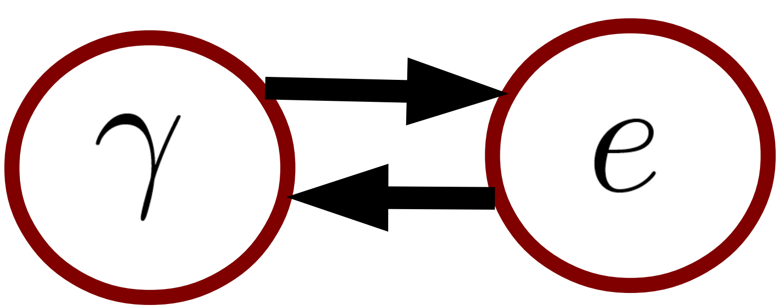
γ
e

AVERAGE LONGITUDINAL EVOLUTION for a PURELY ELECTROMAGNETIC SHOWER

$$n_e(E,t)$$

$$n_\gamma(E,t)$$

Two functions of energy and depth



In what follows we shall call "*approximation A*" the approximation in which collision processes and Compton effect are neglected, and the asymptotic formulae are used to describe radiation processes and pair production.

A

We shall call "*approximation B*" the approximation in which the Compton effect is neglected, the collision loss is described as a constant energy dissipation and the asymptotic formulae for radiation processes and pair production are used.

B

Possible Generalizations:

3-Dimensional treatment.

$$n_{e,\gamma}(E, x, \theta_x, y, \theta_y, \ t)$$

Hadronic Showers: add other components

$$n_{p,n}(E, t) \qquad n_{\mu^\pm}(E, t)$$

$$n_{\pi^\pm}(E, t) \qquad n_\nu(E, t)$$

SYSTEM of
INTEGRO-DIFFERENTIAL EQUATIONS

that describe the evolution with **t** of

$$n_e(E,t) \qquad\qquad n_\gamma(E,t)$$

for a given initial condition.

Variation with t of the number of photons with energy E

$$\frac{\partial n_\gamma}{\partial t}(E,t) = \quad -n_\gamma(E,t) \int_0^1 du \; \psi(u)$$

$$+ \int_E^\infty dE' \int_0^1 dv \; n_e(E',t) \; \varphi(v) \; \delta[E - v\,E']$$

Variation with t of the number of photons with energy E

$$\frac{\partial n_\gamma}{\partial t}(E,t) = -n_\gamma(E,t) \int_0^1 du\ \psi(u)$$

$$+ \int_E^\infty dE' \int_0^1 dv\ n_e(E',t)\ \varphi(v)\ \delta[E - v\,E']$$

$$= \boxed{-\sigma_0\ n_\gamma(E,t) + \int_0^1 \frac{dv}{v}\ n_e\left(\frac{E}{v}, t\right)\ \varphi(v)}$$

$$\boxed{(e \to \gamma)}$$

Electrons

$$\frac{\partial n_e}{\partial t}(E,t) = -n_e(E,t)\int_0^1 dv\ \varphi(v)$$

$$+\int_E^\infty dE'\int_0^1 dv\ n_e(E',t)\ \varphi(v)\ \delta[E-(1-v)\,E']$$

$$+\int_E^\infty dE'\int_0^1 du\ n_\gamma(E',t)\ \psi(u)\ \delta[E-u\,E']$$

Electrons

$$\frac{\partial n_e}{\partial t}(E,t) = \quad -n_e(E,t) \int_0^1 dv\ \varphi(v)$$

$$+\int_E^\infty dE' \int_0^1 dv\ n_e(E',t)\ \varphi(v)\ \delta[E-(1-v)\,E']$$

$$\boxed{(\gamma \to e)} \qquad +\int_E^\infty dE' \int_0^1 du\ n_\gamma(E',t)\ \psi(u)\ \delta[E-u\,E']$$

$$2 \int_0^1 \frac{du}{u}\ n_\gamma\left(\frac{E}{u},t\right)\ \psi(u)$$

Electrons

$$\frac{\partial n_e}{\partial t}(E,t) = -n_e(E,t)\int_0^1 dv\ \varphi(v)$$

$(e \to e)$

$$+\int_E^\infty dE' \int_0^1 dv\ n_e(E',t)\ \varphi(v)\ \delta[E-(1-v)\,E']$$

$$\int_0^1 \frac{dv}{1-v}\ n_e\left(\frac{E}{1-v},t\right)\ \varphi(v)$$

$$+\int_E^\infty dE' \int_0^1 du\ n_\gamma(E',t)\ \psi(u)\ \delta[E-u\,E']$$

Electrons

$$\frac{\partial n_e}{\partial t}(E,t) = \quad -n_e(E,t) \int_0^1 dv\ \varphi(v)$$

$$(e \to e)$$

$$+ \int_E^\infty dE' \int_0^1 dv\ n_e(E',t)\ \varphi(v)\ \delta[E-(1-v)\,E']$$

$$\int_0^1 \frac{dv}{1-v}\ n_e\left(\frac{E}{1-v}, t\right)\ \varphi(v)$$

$$+ \int_E^\infty dE' \int_0^1 du\ n_\gamma(E',t)\ \psi(u)\ \delta[E-u\,E']$$

2 divergent e -> e contributions.
Their combination is finite.

$$\frac{\partial n_e(E,t)}{\partial t} = -\int_0^1 dv \;\; \varphi(v) \; \left[n_e(E,t) - \frac{1}{1-v} \, n_e\left(\frac{E}{1-v}, t \right) \right]$$

$$+2 \int_0^1 \frac{du}{u} \, \psi(u) \, n_\gamma\left(\frac{E}{u}, t \right)$$

$$\frac{\partial n_\gamma(E,t)}{\partial t} = \int_0^1 \frac{dv}{v} \, \varphi(v) \, n_e\left(\frac{E}{v}, t \right) - \sigma_0 \, n_\gamma(E,t) \; .$$

Approximation A

Insert the collision energy loss in the shower equations:
General treatment:

$$\frac{dE}{dt} = \beta(E)$$

Energy variation Law

$$n(E,t) \qquad n(E,t+dt)$$

$$\boxed{\begin{gathered} n(E,t+dt)\ dE = n(E',dt)\ dE' \\ E' = E - \beta(E)\ dt \qquad dE' = \left(1 - \frac{d\beta(E)}{dE}\right)\ dE \end{gathered}}$$

$$\boxed{n(E, t + dt)\ dE = n(E', dt)\ dE'}$$

$$\left[n(E,t) + \frac{\partial n(E,t)}{\partial t}\ dt\right]\ dE \quad = \quad \left[n(E,t) + \frac{\partial n(E,t)}{\partial E}\ \beta(E)\ dt\right] \left(1 - \frac{d\beta(E)}{dE}\right)\ dE$$

$$n(E, t + dt)\ dE = n(E', dt)\ dE'$$

$$\left[n(E,t) + \frac{\partial n(E,t)}{\partial t}\ dt \right]\ dE \qquad =$$

$$\left[n(E,t) + \frac{\partial n(E,t)}{\partial E}\ \beta(E)\ dt \right] \left(1 - \frac{d\beta(E)}{dE} \right)\ dE$$

$$\frac{\partial n(E,t)}{\partial t} = -\frac{\partial}{\partial E}\ [n(E,t)\ \beta(E)]$$

$$\frac{\partial n_e(E,t)}{\partial t} = -\int_0^1 dv \;\; \varphi(v) \left[n_e(E,t) - \frac{1}{1-v} n_e\left(\frac{E}{1-v}, t\right) \right]$$

$$+2 \int_0^1 \frac{du}{u} \, \psi(u) \, n_\gamma\left(\frac{E}{u}, t\right)$$

$$\frac{\partial n_\gamma(E,t)}{\partial t} = \int_0^1 \frac{dv}{v} \, \varphi(v) \, n_e\left(\frac{E}{v}, t\right) - \sigma_0 \, n_\gamma(E,t)$$

Approximation A

$$\frac{\partial n_e(E,t)}{\partial t} = -\int_0^1 dv \;\; \varphi(v) \left[n_e(E,t) - \frac{1}{1-v} n_e\left(\frac{E}{1-v}, t\right)\right]$$

$$+2 \int_0^1 \frac{du}{u} \psi(u) \, n_\gamma \left(\frac{E}{u}, t\right)$$

$$+ \varepsilon \frac{\partial n_e(E,t)}{\partial E}$$

$$\frac{\partial n_\gamma(E,t)}{\partial t} = \int_0^1 \frac{dv}{v} \varphi(v) \, n_e \left(\frac{E}{v}, t\right) - \sigma_0 \, n_\gamma(E,t)$$

$$\boxed{\text{Approximation B}}$$

Shower Equations in "Approximation A"

$$n_e(E,t)$$
$$n_\gamma(E,t)$$

(neglect electron ionization losses)

$$\frac{\partial n_e(E,t)}{\partial t} = -\int_0^1 dv\ \ \varphi_0(v)\ \left[n_e(E,t) - \frac{1}{1-v}\ \ n_e\left(\frac{E}{1-v},t\right)\right]$$

$$+\,2\int_0^1 \frac{du}{u}\,\psi(u)\,n_\gamma\left(\frac{E}{u},t\right)$$

No Parameters with the Dimension of Energy

$$\frac{\partial n_\gamma(E,t)}{\partial t} = \int_0^1 \frac{dv}{v}\,\varphi(v)\,n_e\left(\frac{E}{v},t\right) - \sigma_0\,n_\gamma(E,t)$$

Solutions to the shower equations.

Initial Condition:

$$\begin{cases} n_e(E,0) & = & 0 \\ n_\gamma(E,0) & = & \delta[E-E_0] \end{cases}$$

Photon of energy E_0

$$\begin{cases} n_e(E,0) & = & \delta[E-E_0] \\ n_\gamma(E,0) & = & 0 \end{cases}$$

Electron of energy E_0

Let us consider an electron population that is has the spectral shape of an unbroken power law and no photons:

$$\left\{ \begin{array}{lcl} n_e(E,0) & = & K\ E^{-(s+1)} \\ n_\gamma(E,0) & = & 0 \end{array} \right.$$

Study the Shower evolution using approximation A

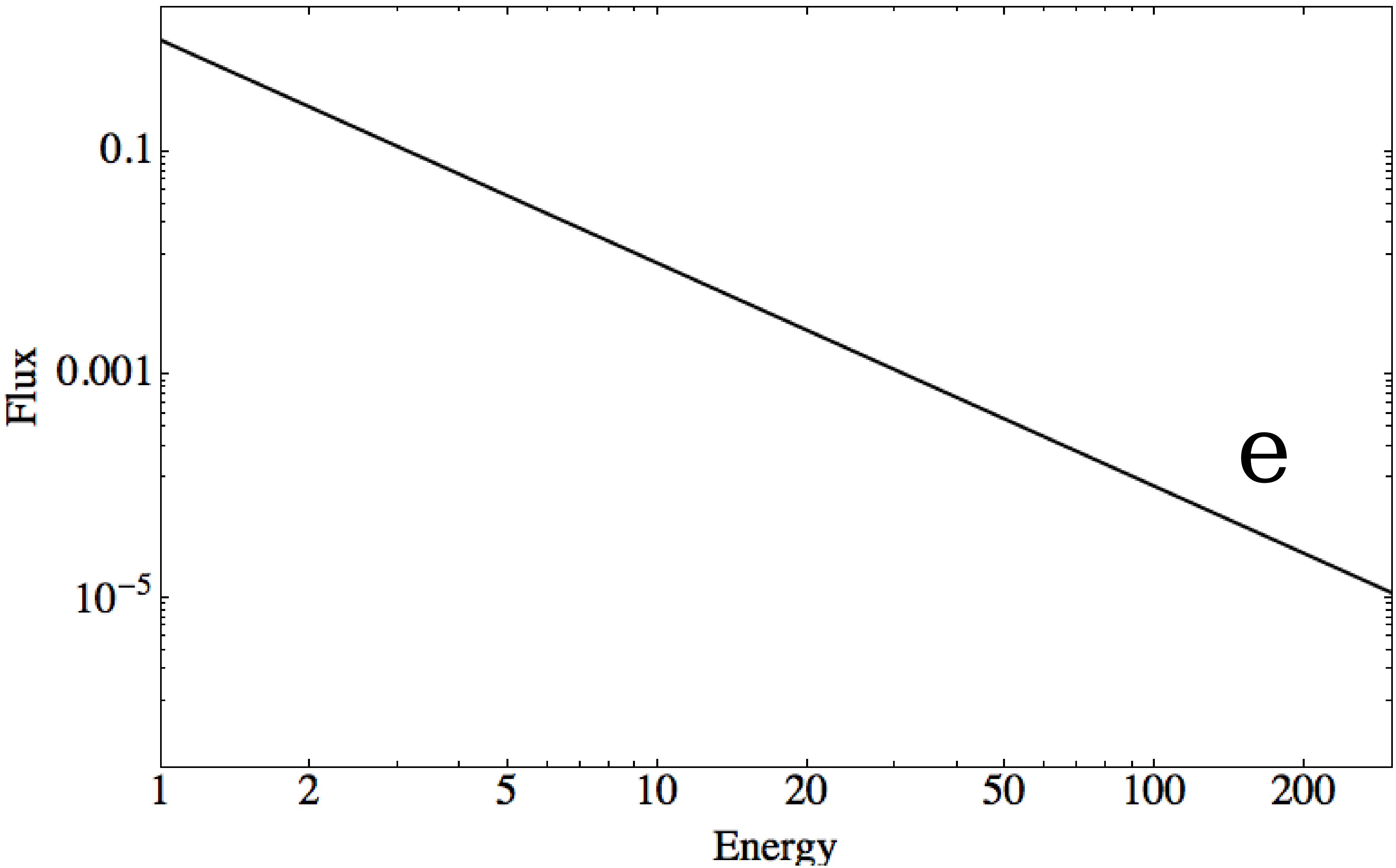
$$\begin{cases} n_e(E,0) &= K\ E^{-(s+1)} \\ n_\gamma(E,0) &= 0 \end{cases}$$

Initial condition

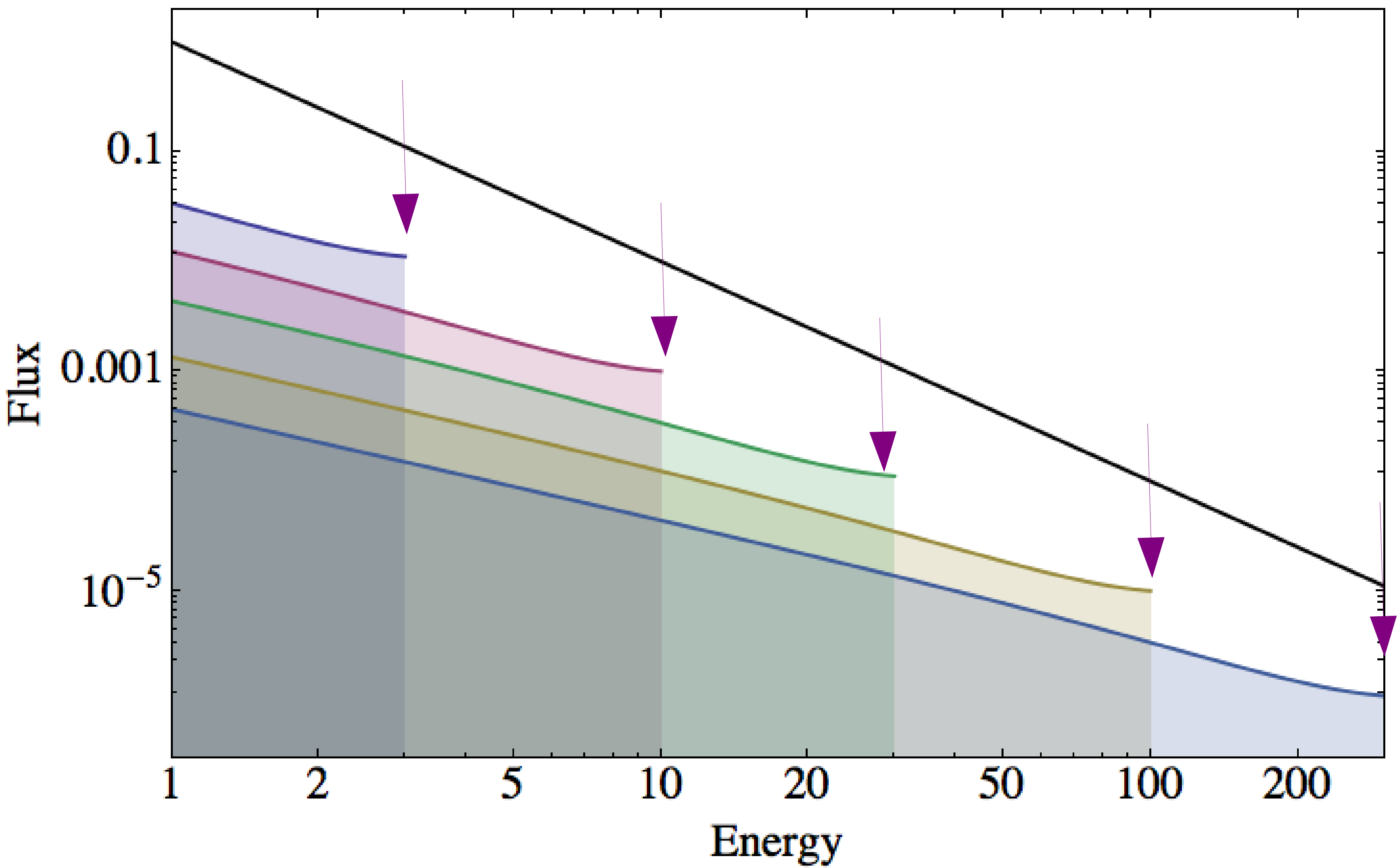
Electron and Photon population
remain a power law of same slope
Only the normalizations are a function of the depth t

$$\begin{cases} n_e(E,t) &= K_e(t)\ E^{-(s+1)} \\ n_\gamma(E,t) &= K_\gamma(t)\ E^{-(s+1)} \end{cases}$$

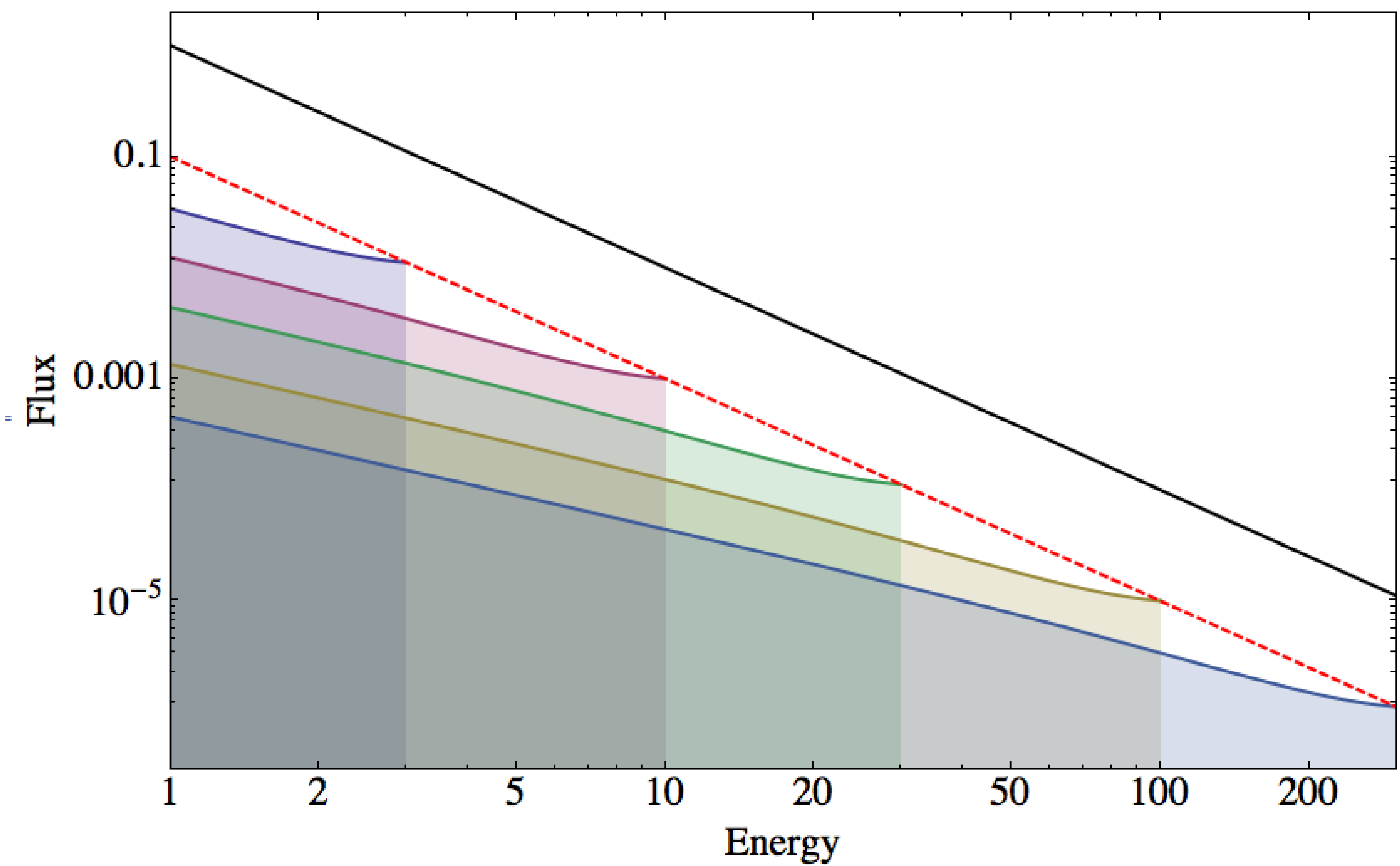
Depth evolution



0.1
0.001
10^{-5}
Flux
1
2
5
10
20
50
100
200
Energy
e



Flux
0.1
0.001
10^{-5}
1
2
5
10
20
50
100
200
Energy



0.1
0.001
10^{-5}
Flux
1
2
5
10
20
50
100
200
Energy

$$\begin{cases} n_e(E,t) = K_e(t)\ E^{-(s+1)} \\ n_\gamma(E,t) = K_\gamma(t)\ E^{-(s+1)} \end{cases}$$

Coefficients $K_{e,g}(t)$ are linear combinations of two exponential

$$K_{e,\gamma}(t) = a_{e,\gamma}\ e^{\lambda_1(s)\,t} + b_{e,\gamma}\ e^{\lambda_2(s)\,t}$$

$$\begin{cases} n_e(E,t) = K_e(t)\ E^{-(s+1)} \\ n_\gamma(E,t) = K_\gamma(t)\ E^{-(s+1)} \end{cases}$$

$$K_{e,\gamma}(t) = a_{e,\gamma}\ e^{\lambda_1(s)\,t} + b_{e,\gamma}\ e^{\lambda_2(s)\,t}$$

One controls the (faster) convergence to an s-dependent gamma/e ratio (large and negative)

$$\lambda_2(s)$$

A second exponential describes the (slower) evolution of the two population with a constant ratio.

$$\lambda_1(s)$$

$$\begin{cases} n_e(E,t) = K_e\ E^{-2} \\ n_\gamma(E,t) = K_\gamma\ E^{-2} \end{cases}$$

$\boxed{S = 1}$

Special spectrum

Equal amount of energy per decade of E

$$\begin{cases} n_e(E,t) = K_e\ E^{-2} \\ n_\gamma(E,t) = K_\gamma\ E^{-2} \end{cases}$$

S = 1

Special spectrum
Equal amount of energy per decade of E

$$t \to t + dt$$

$$dn_e = -dn_\gamma = \left(-n_e\ \langle v \rangle + n_\gamma\ \sigma_0\right)\ dt$$

$$\frac{n_\gamma}{n_e} = \frac{\langle v \rangle}{\sigma_0} \iff dn_e = dn_\gamma = 0$$

depth-independent solution

What can we say about: $\lambda_1(s)$

Without explicit calculation ?

$$s = 1 \iff \lambda_1(s) = 0$$

Spectrum E^{-2} equal power per decade of E

Pair Production and Bremsstrahlung
“redistribute the energy”
but “nothing can change”

What can we say about: $\lambda_1(s)$

$$s = 1 \iff \lambda_1(s) = 0$$

$$s < 1 \iff \lambda_1(s) > 0$$

$$s > 1 \iff \lambda_1(s) < 0$$

Spectrum steeper than E^{-2}

power per decade of E decreases with E

Insert functional form of the solution in the shower equation.

$$\begin{cases} n_e(E,t) &= K_e\ E^{-(s+1)}\ e^{\lambda t} \\ n_\gamma(E,t) &= K_\gamma\ E^{-(s+1)}\ e^{\lambda t} \end{cases}$$

$$\begin{aligned} \frac{\partial n_e(E,t)}{\partial t} &= -\int_0^1 dv\ \ \varphi(v)\ \left[n_e(E,t) - \frac{1}{1-v}\, n_e\left(\frac{E}{1-v}, t\right)\right] \\ &\quad +2\int_0^1 \frac{du}{u}\, \psi(u)\, n_\gamma\left(\frac{E}{u}, t\right) \\ \frac{\partial n_\gamma(E,t)}{\partial t} &= \int_0^1 \frac{dv}{v}\, \varphi(v)\, n_e\left(\frac{E}{v}, t\right) - \sigma_0\, n_\gamma(E,t)\ . \end{aligned}$$

Obtain simple quadratic equation connecting

s λ K_e/K_γ

$$\frac{\partial n_e(E,t)}{\partial t} \rightarrow \lambda\, n_e(E,t)$$

Time derivative

$$2 \int_0^1 \frac{du}{u}\, \psi(u)\, n_\gamma\left(\frac{E}{u}, t\right)$$

Example of one term

$$2 \int_0^1 \frac{du}{u}\, \psi(u) \left[K_\gamma \left(\frac{E}{u}\right)^{-(s+1)} e^{\lambda t}\right]$$

$$2\, K_\gamma\, E^{-(s+1)}\, e^{\lambda t)} \boxed{\int_0^1 \frac{du}{u}\, \psi(u)\, u^{(s+1)}}$$

$$2 K_\gamma\, E^{-(s+1)}\, e^{\lambda t} \boxed{B(s)}$$

$$\frac{\partial n_e(E,t)}{\partial t} = -\int_0^1 dv \;\; \varphi(v) \; \left[n_e(E,t) - \frac{1}{1-v} \, n_e \left(\frac{E}{1-v}, t \right) \right]$$

$$+2 \; \int_0^1 \frac{du}{u} \, \psi(u) \, n_\gamma \left(\frac{E}{u}, t \right)$$

$$\frac{\partial n_\gamma(E,t)}{\partial t} = \int_0^1 \frac{dv}{v} \, \varphi(v) \, n_e \left(\frac{E}{v}, t \right) - \sigma_0 \, n_\gamma(E,t)$$

Approximation A

$$\frac{\partial n_e(E,t)}{\partial t} \;=\; -\int_0^1 dv \;\; \varphi(v) \; \left[n_e(E,t) - \frac{1}{1-v}\, n_e\left(\frac{E}{1-v}, t\right)\right]$$

$$+2 \; \int_0^1 \frac{du}{u}\, \psi(u)\, n_\gamma\left(\frac{E}{u}, t\right)$$

$$+\, \varepsilon\, \frac{\partial n_e(E,t)}{\partial E}$$

$$\frac{\partial n_\gamma(E,t)}{\partial t} \;=\; \int_0^1 \frac{dv}{v}\, \varphi(v)\, n_e\left(\frac{E}{v}, t\right) - \sigma_0\, n_\gamma(E,t)$$

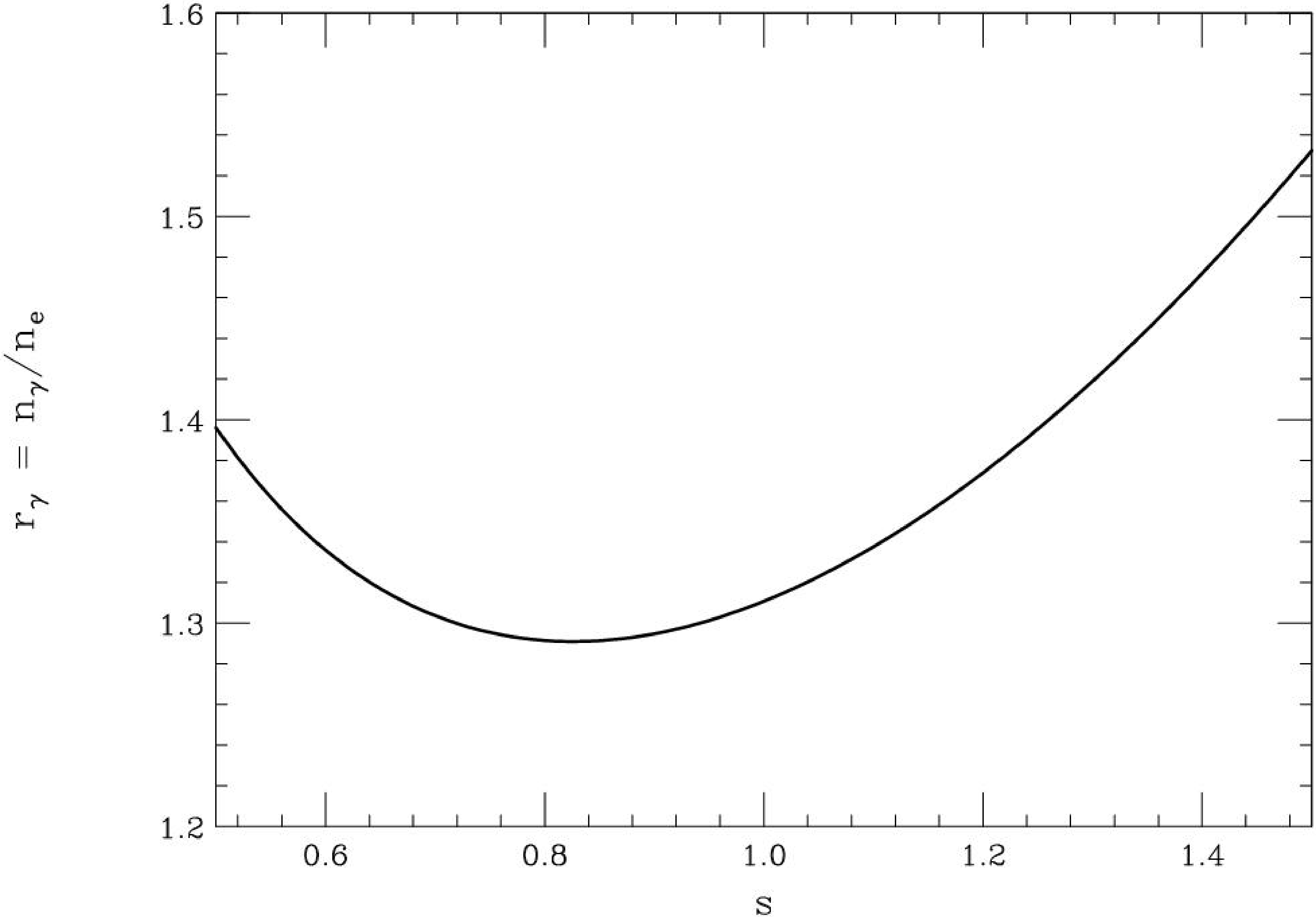
Approximation B

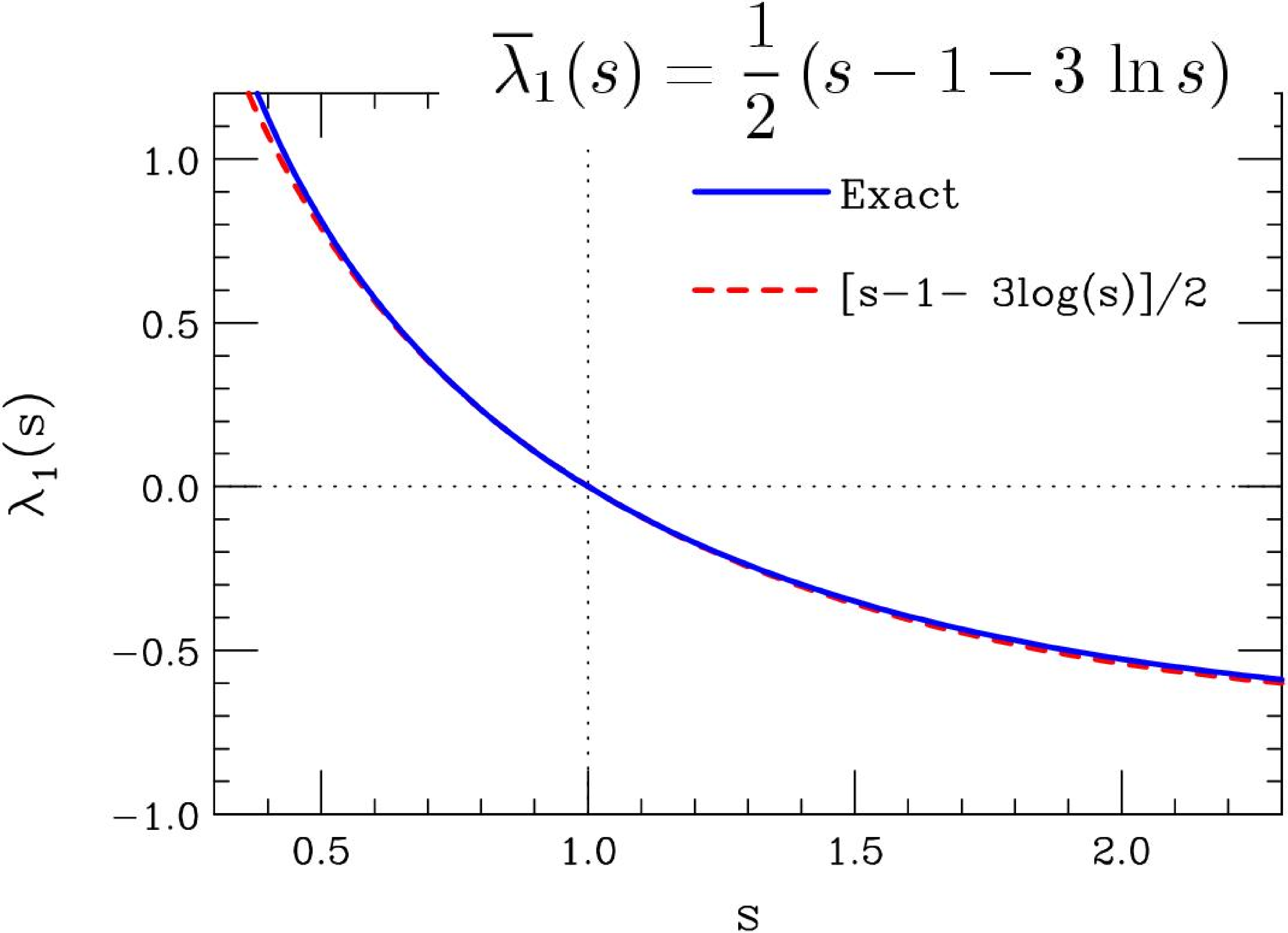
“Elementary Solutions”
Approximation A:

$$\boxed{\begin{cases} n_e(E,t) &= K\ E^{-(s+1)}\ e^{\lambda_1(s)\,t} \\ n_\gamma(E,t) &= K\ r_\gamma(s)\ E^{-(s+1)}\ e^{\lambda_1(s)\,t} \end{cases}}$$

$$\begin{cases} n_e(E,t) &= K\ E^{-(s+1)}\ e^{\lambda_2(s)\,t} \\ n_\gamma(E,t) &= K\ r_{\gamma 2}(s)\ E^{-(s+1)}\ e^{\lambda_2(s)\,t} \end{cases}$$

“Stability Ratio” for the Photon/Electron Ratio





$\overline{\lambda}_1(s) = \frac{1}{2}\,(s - 1 - 3\,\ln s)$
Exact
[s−1− 3log(s)]/2
1.0
0.5
0.0
−0.5
−1.0
0.5
1.0
1.5
2.0
s
$\lambda_1(s)$

$$\boxed{\lambda_{1,2}(s) = -\frac{1}{2}\left(A(s) + \sigma_0\right) \pm \frac{1}{2}\sqrt{\left(A(s) - \sigma_0\right)^2 + 4\,B(s)\,C(s)}}$$

$$\begin{aligned}
A(s) &= \int_0^1 dv\ \varphi(v)\ \left[1 - (1-v)^s\right] \\
&= \left(\frac{4}{3} + 2\,b\right)\left(\frac{\Gamma'(1+s)}{\Gamma(1+s)} + \gamma\right) + \frac{s\ \left(7 + 5\,s + 12\,b\ (2+s)\right)}{6\ (1+s)\ (2+s)}
\end{aligned}$$

$$B(s) = 2\int_0^1 du\ u^s\ \psi(u) = \frac{2\ \left(14 + 11\,s + 3\,s^2 - 6\,b\ (1+s)\right)}{3\ (1+s)\ (2+s)\ (3+s)}$$

$$C(s) = \int_0^1 dv\ v^s\ \varphi(v) = \frac{8 + 7\,s + 3\,s^2 + 6\,b\ (2+s)}{3\,s\ \left(2 + 3\,s + s^2\right)}$$

$$\boxed{\frac{dN_\gamma}{dE} = n_\gamma(E)}$$

$$\boxed{\frac{d\mathcal{E}_\gamma}{dE} = n_\gamma(E)\ E}$$

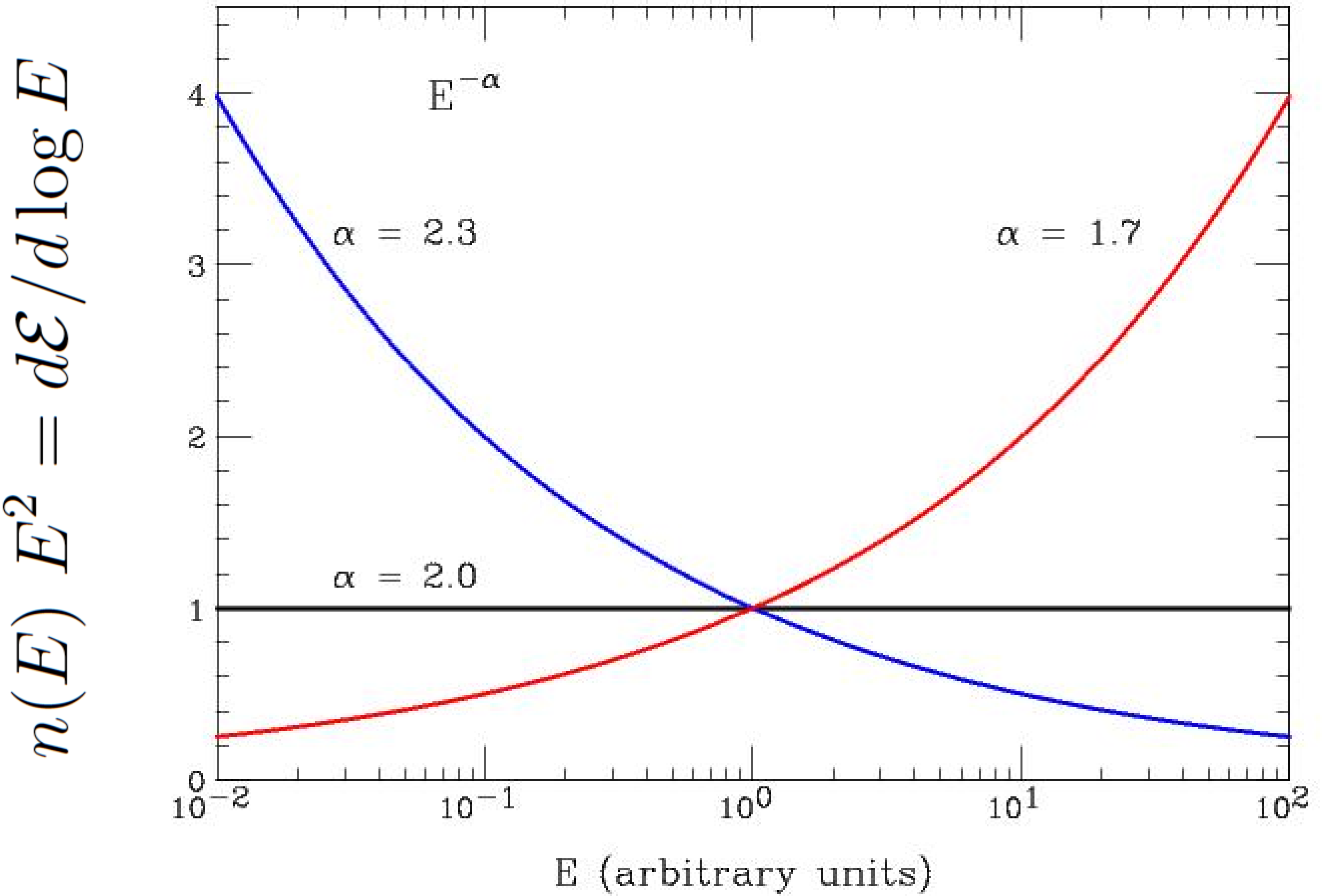
$$\boxed{\begin{aligned}\frac{d\mathcal{E}_\gamma}{d\ln E} &= n_\gamma(E)\ E\ \frac{dE}{d\ln E} \\ &= n_\gamma(E)\ E^2\end{aligned}}$$

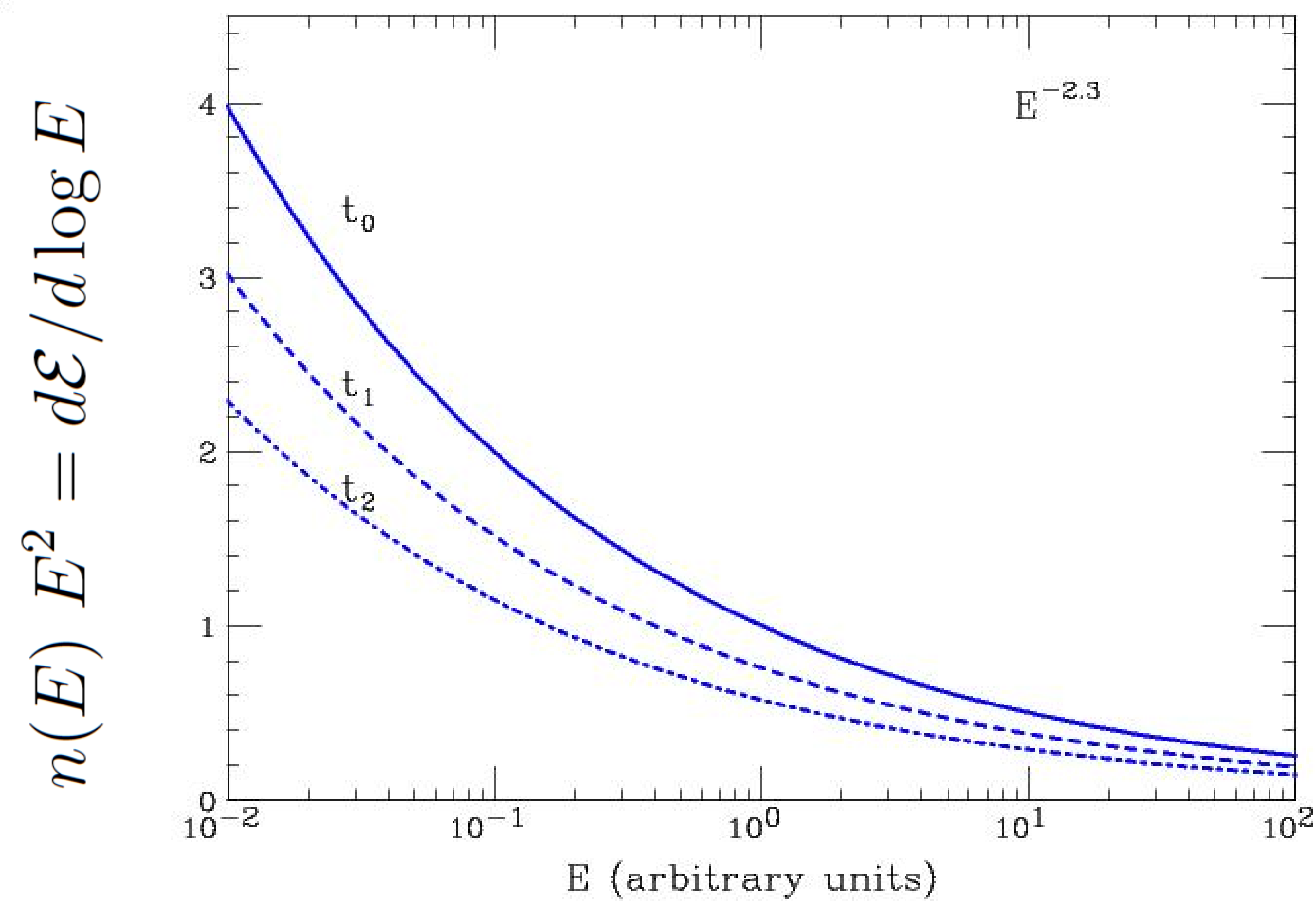
Concept of SPECTRAL ENERGY DISTRIBUTION:

$$\frac{d\mathcal{E}_\gamma}{d\log_{10} E} = n_\gamma(E)\; E^2 \;\ln 10$$

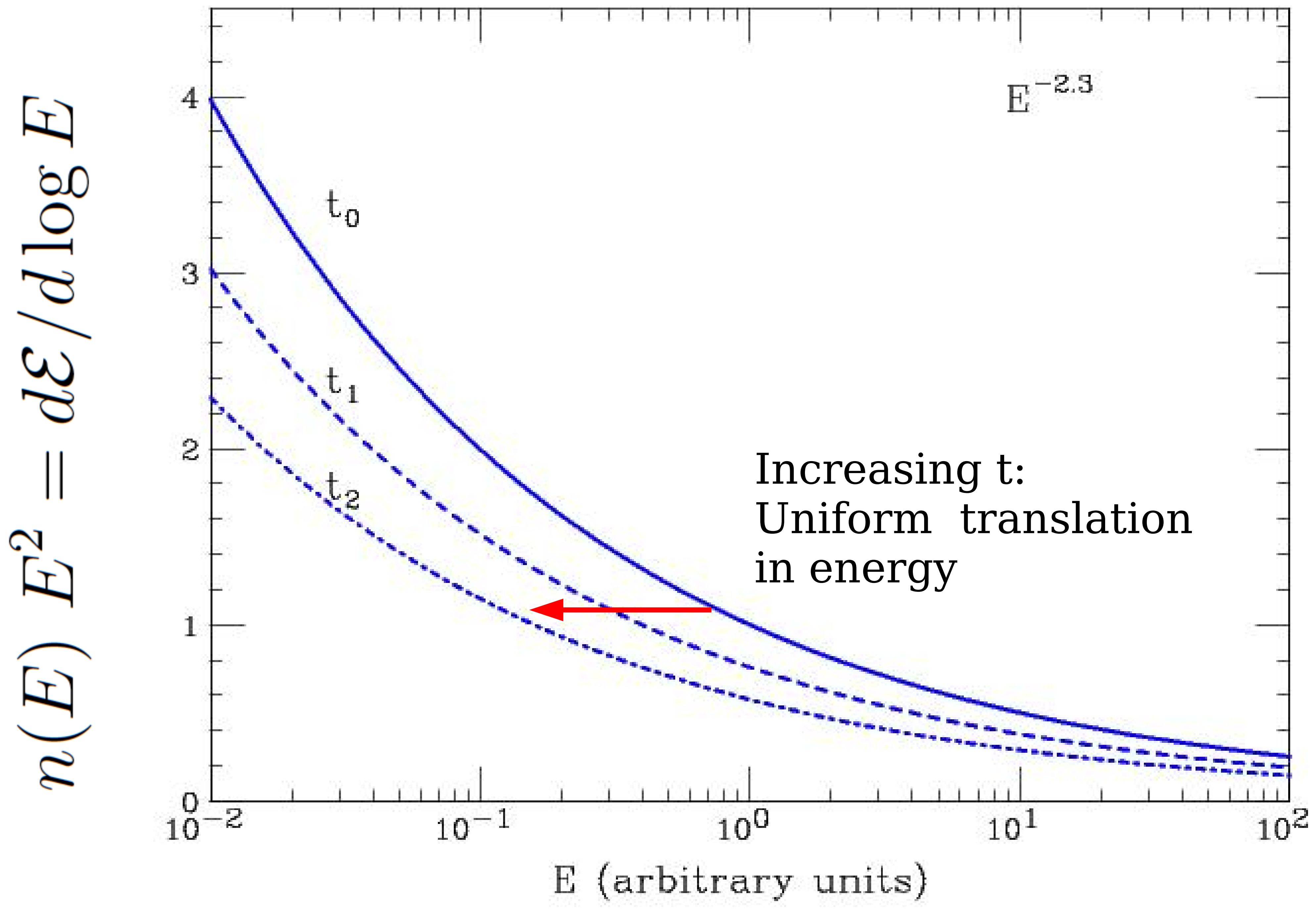
Amount of energy
carried by photons
per decade of energy

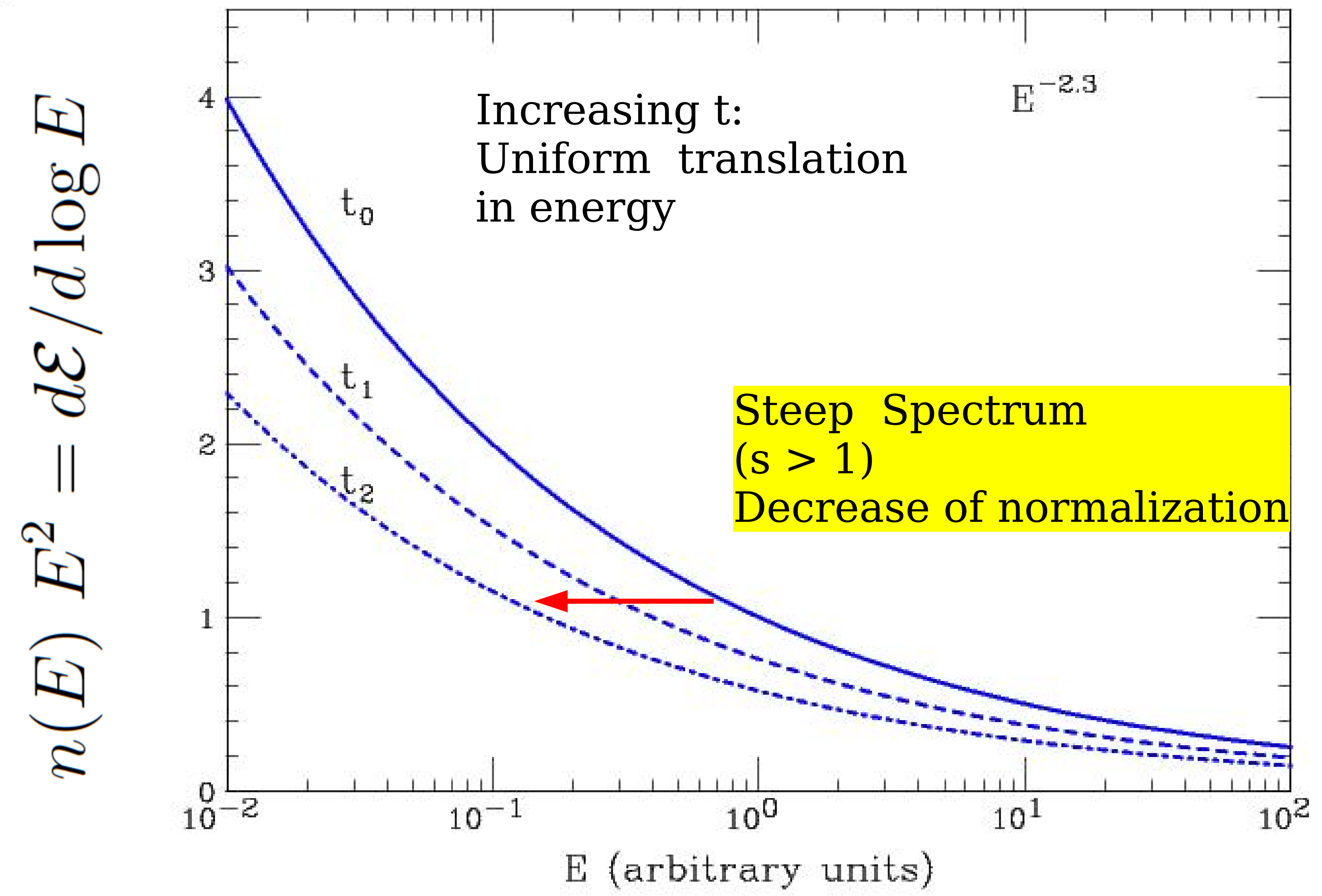
Power Law Solutions : Spectral Energy Distribution

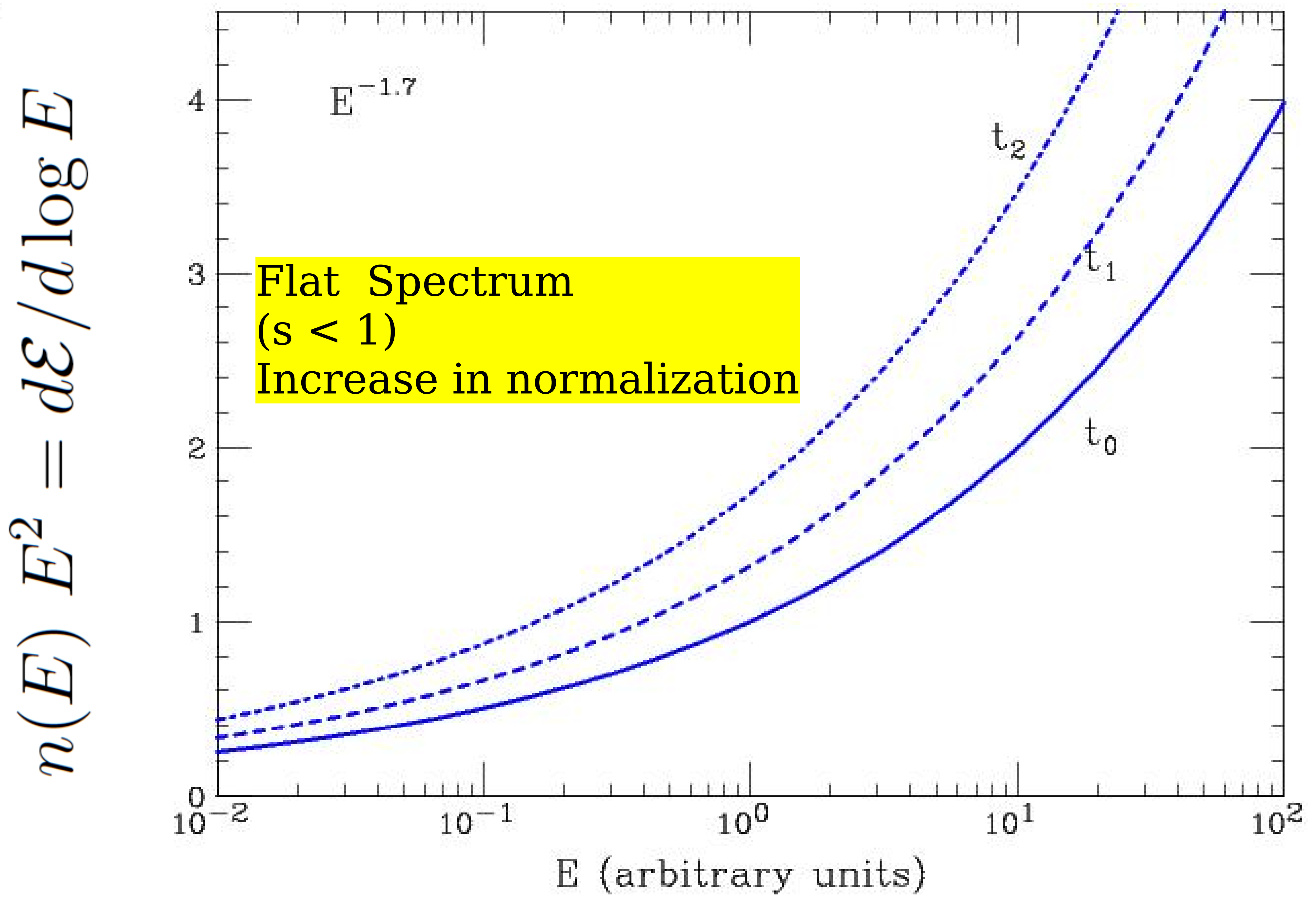




$n(E)\ E^2 = d\mathcal{E}/d\log E$
E (arbitrary units)
$E^{-2.3}$
t_0
t_1
t_2
0
1
2
3
4
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}


$n(E)\ E^2 = d\mathcal{E}/d\log E$
$E^{-2.3}$
t_0
t_1
t_2
Increasing t:
Uniform translation
in energy
E (arbitrary units)
0
1
2
3
4
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}


$n(E)\ E^2 = d\mathcal{E}/d\log E$
Increasing t:
Uniform translation
in energy
$E^{-2.3}$
t_0
t_1
t_2
Steep Spectrum
(s > 1)
Decrease of normalization
E (arbitrary units)


$n(E)\ E^2 = d\mathcal{E}/d\log E$
$E^{-1.7}$
t_2
t_1
t_0
Flat Spectrum
(s < 1)
Increase in normalization
0
1
2
3
4
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
E (arbitrary units)

“Elementary Solutions

Approximation A

$$\begin{cases} n_e(E,t) & = & K \; E^{-(s+1)} \; e^{\lambda_1(s)\,t} \\ n_\gamma(E,t) & = & K \; r_\gamma(s) \; E^{-(s+1)} \; e^{\lambda_1(s)\,t} \end{cases}$$

Approximation B

$$E/\varepsilon \to 0$$

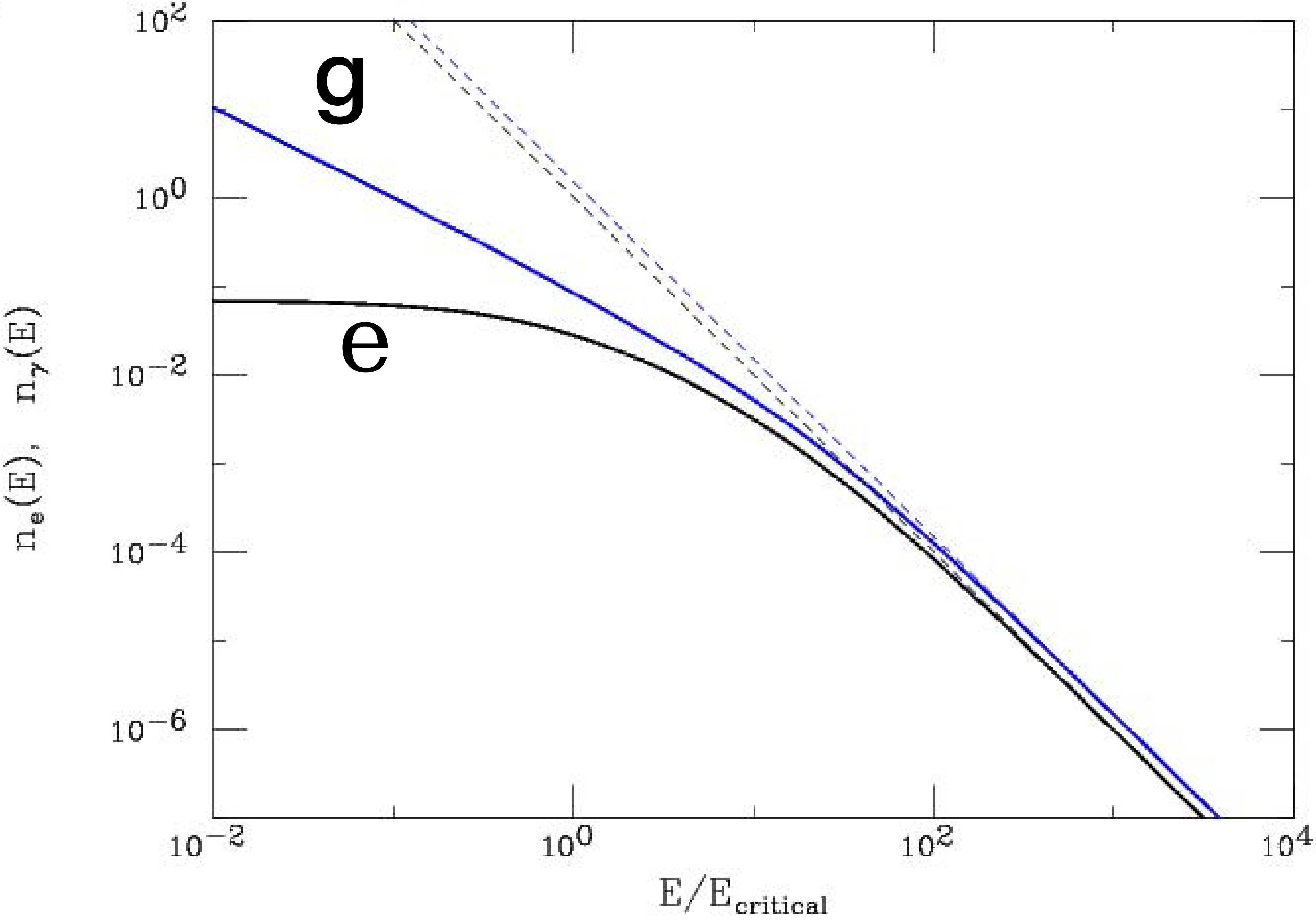
$$E \gg \varepsilon \qquad n_e(E) \to \text{constant}$$

= Approximation A

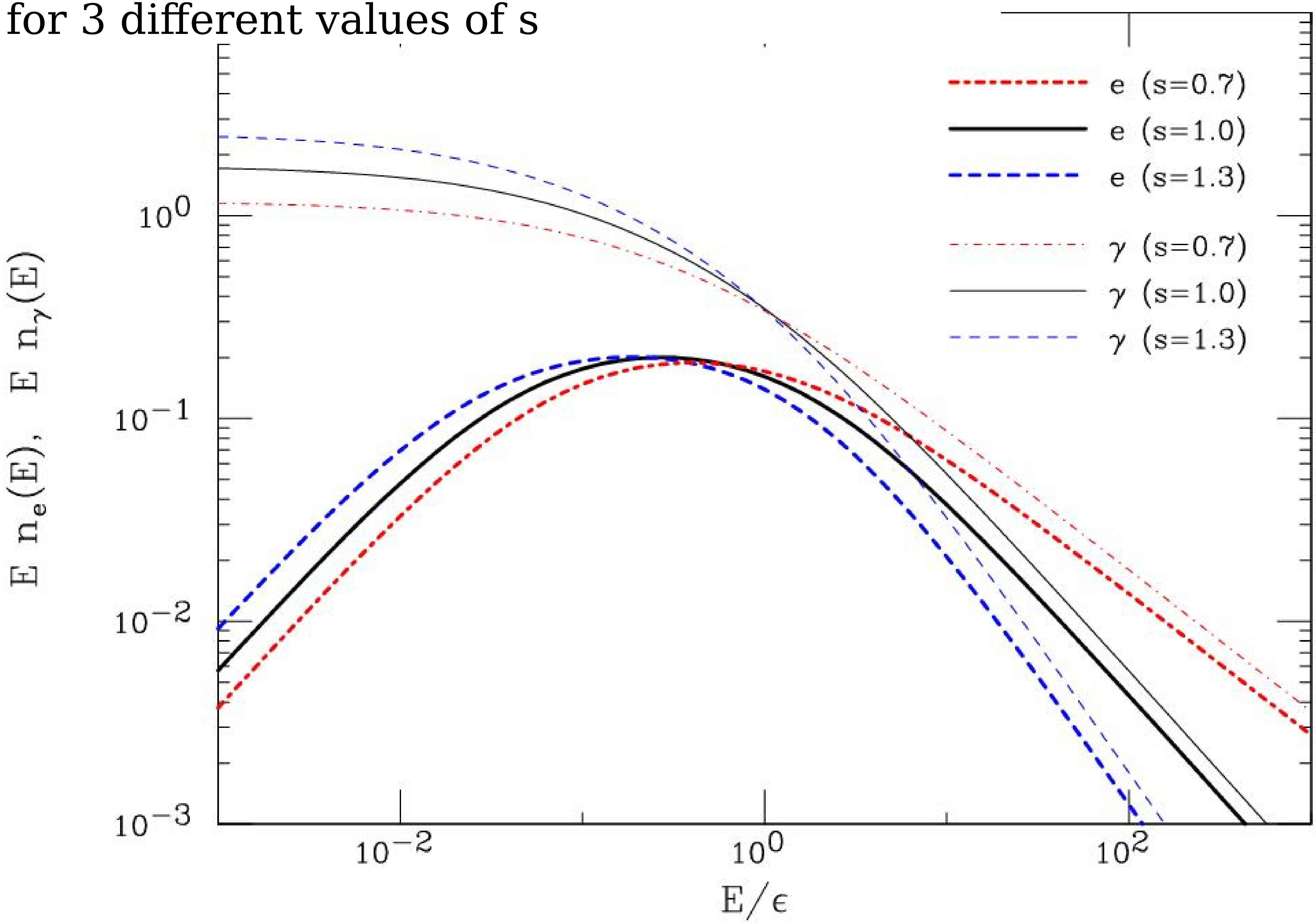
$$n_\gamma(E) \to E^{-1}$$

s=1

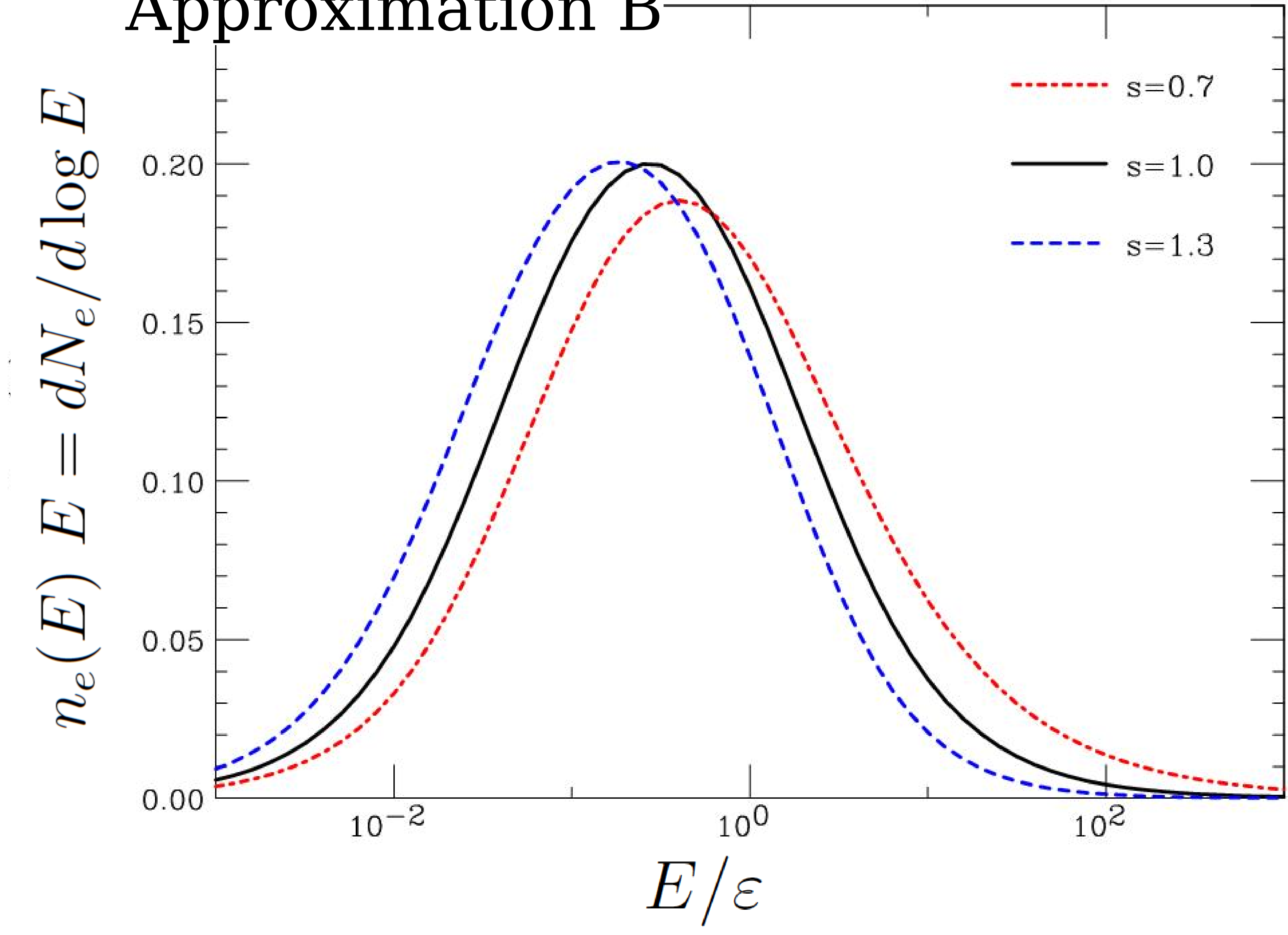
Approximation B “elementary solution”



Electron/photon spectra (elementary solution) for 3 different values of s



Approximation B



Solutions to the shower equations for the "real case".

Initial Condition:

$$\begin{cases} n_e(E,0) & = & 0 \\ n_\gamma(E,0) & = & \delta[E-E_0] \end{cases}$$

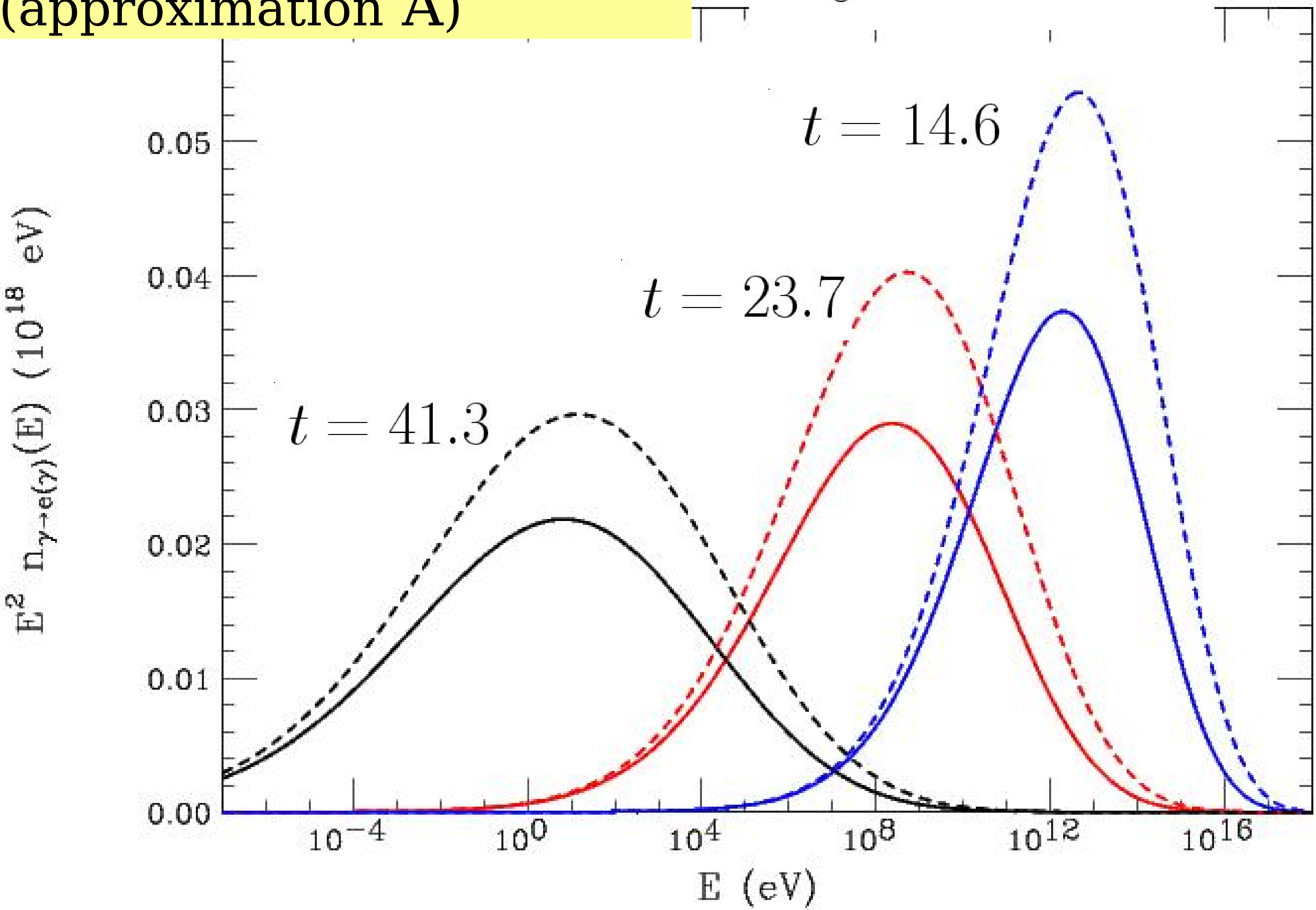
Photon of energy E_0

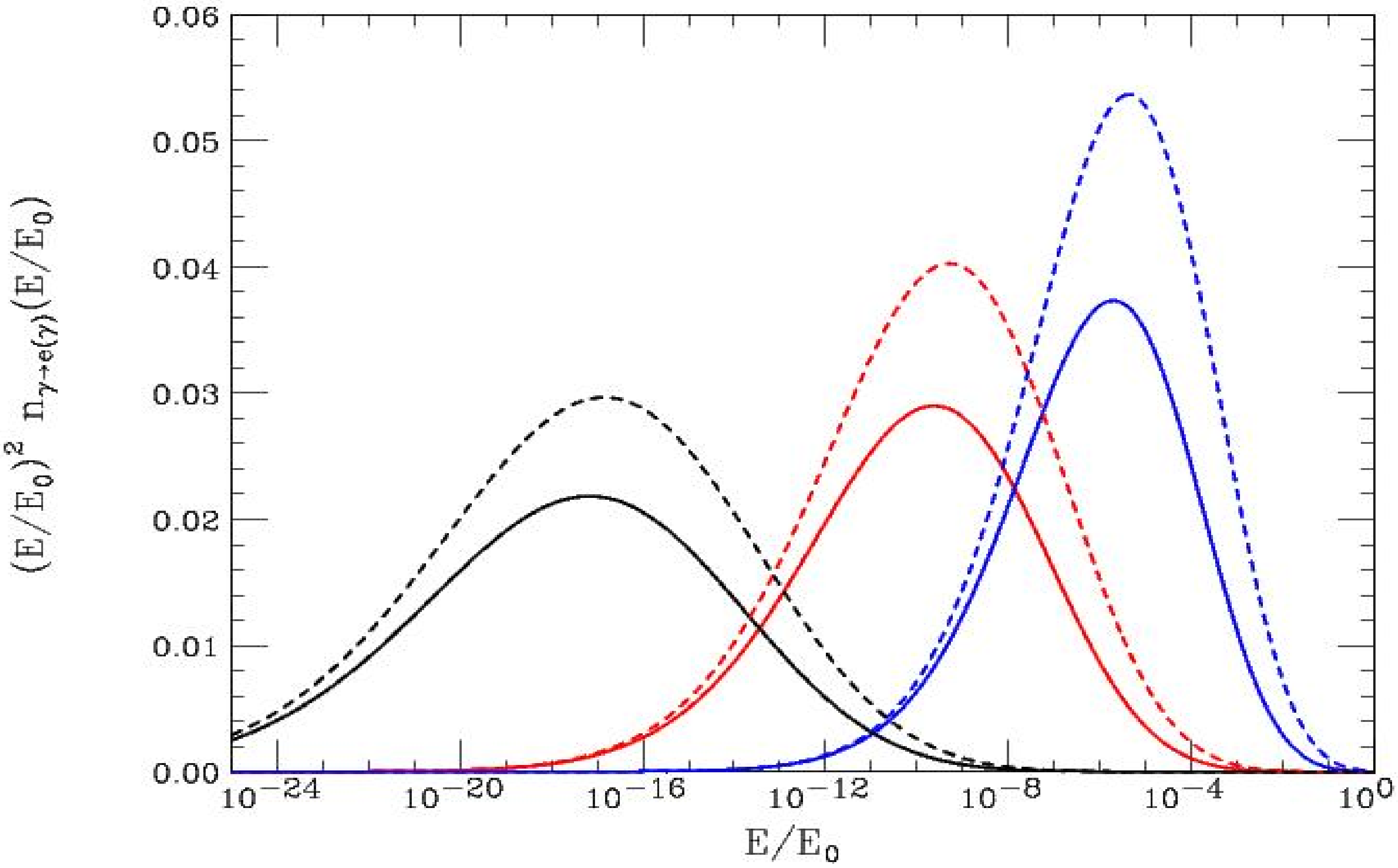
$$\begin{cases} n_e(E,0) & = & \delta[E-E_0] \\ n_\gamma(E,0) & = & 0 \end{cases}$$

Electron of energy E_0

Monochromatic Photon
(approximation A)

$E_0 = 10^{18}$ eV





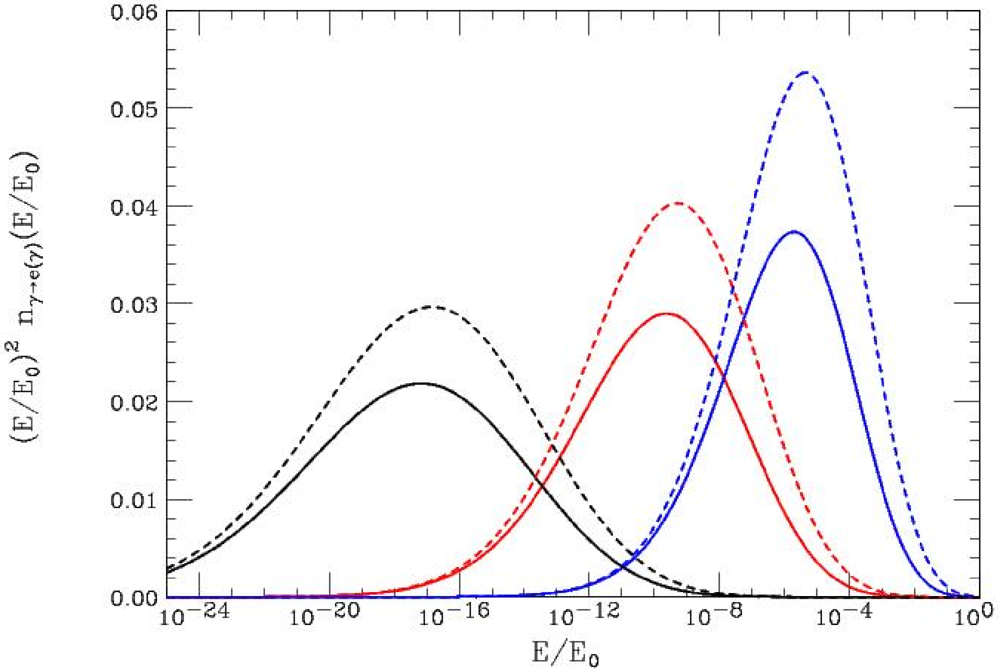
Solution valid for any initial energy

Function of E/E_0

$$n_\alpha(E_0, E, t) = \frac{1}{E_0}\, f_\alpha\left(\frac{E}{E_0}, t\right)$$

$\gamma \to e$ $\quad e \to e$

$\gamma \to \gamma$ $\quad e \to \gamma$



1. Energy Conservation

Area below
the curves constant
with t.

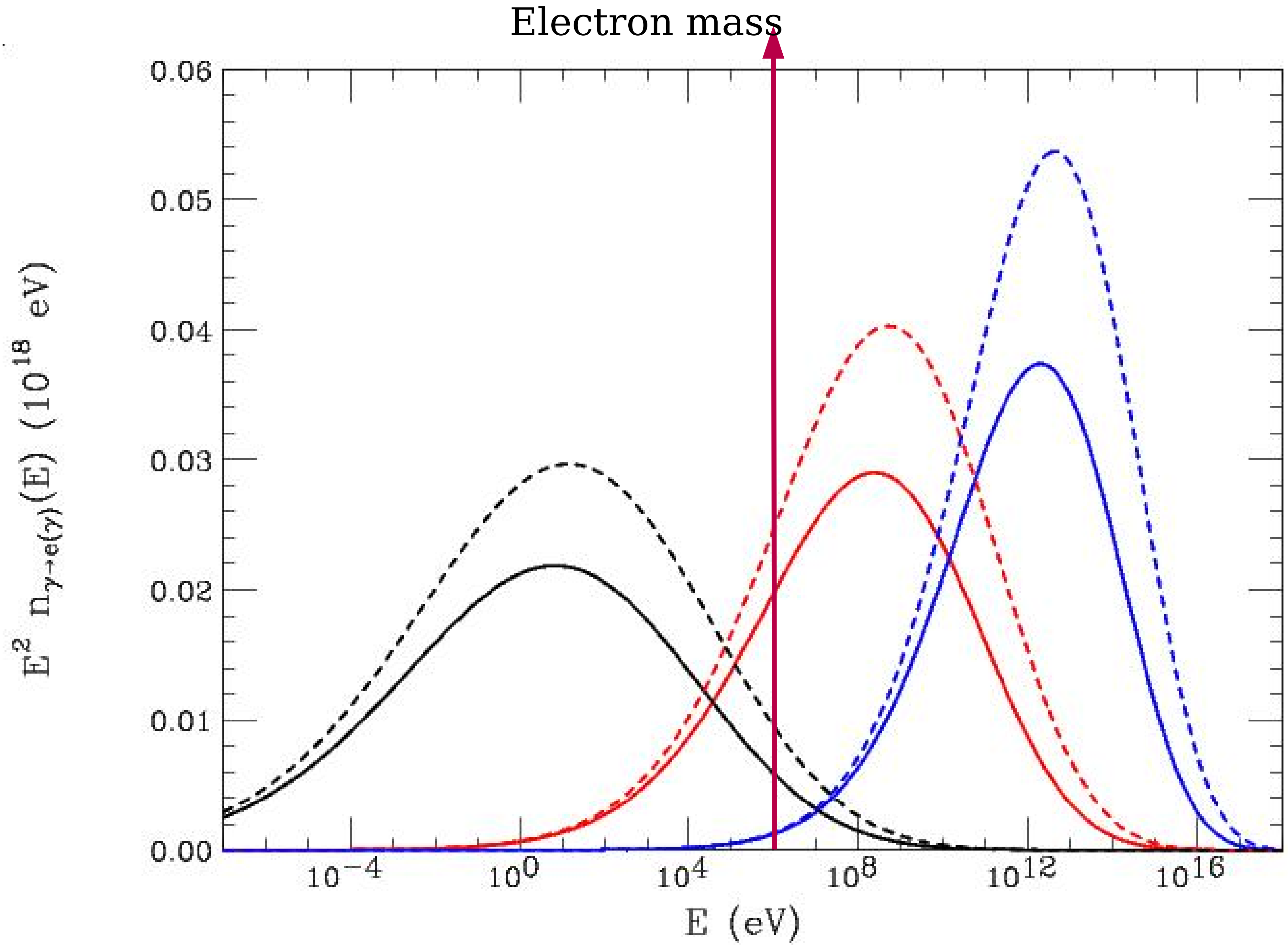
2. Electron and Photon
Spectra have very similar
shapes
The shapes are not
exactly identical
But the ratio gamma/e
is of order 1.3 , 1.4

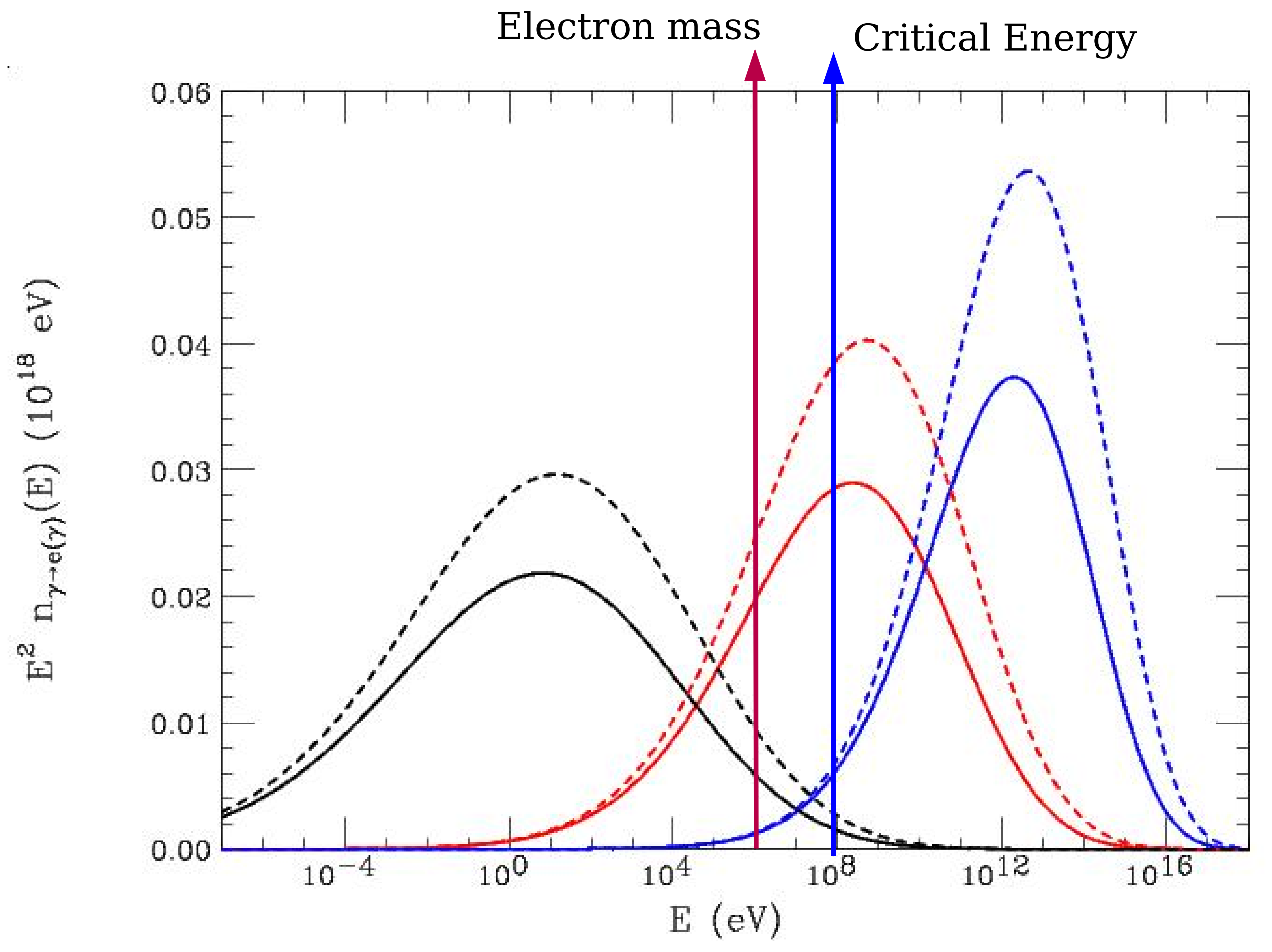
Total ENERGY in a Shower

$$\mathcal{E}_{\text{shower}}(t) = \mathcal{E}_{\text{electrons}}(t) + \mathcal{E}_{\text{photons}}(t)$$

$$\int_0^\infty dE\ E\ n_e(E,t) + \int_0^\infty dE\ E\ n_\gamma(E,t)$$

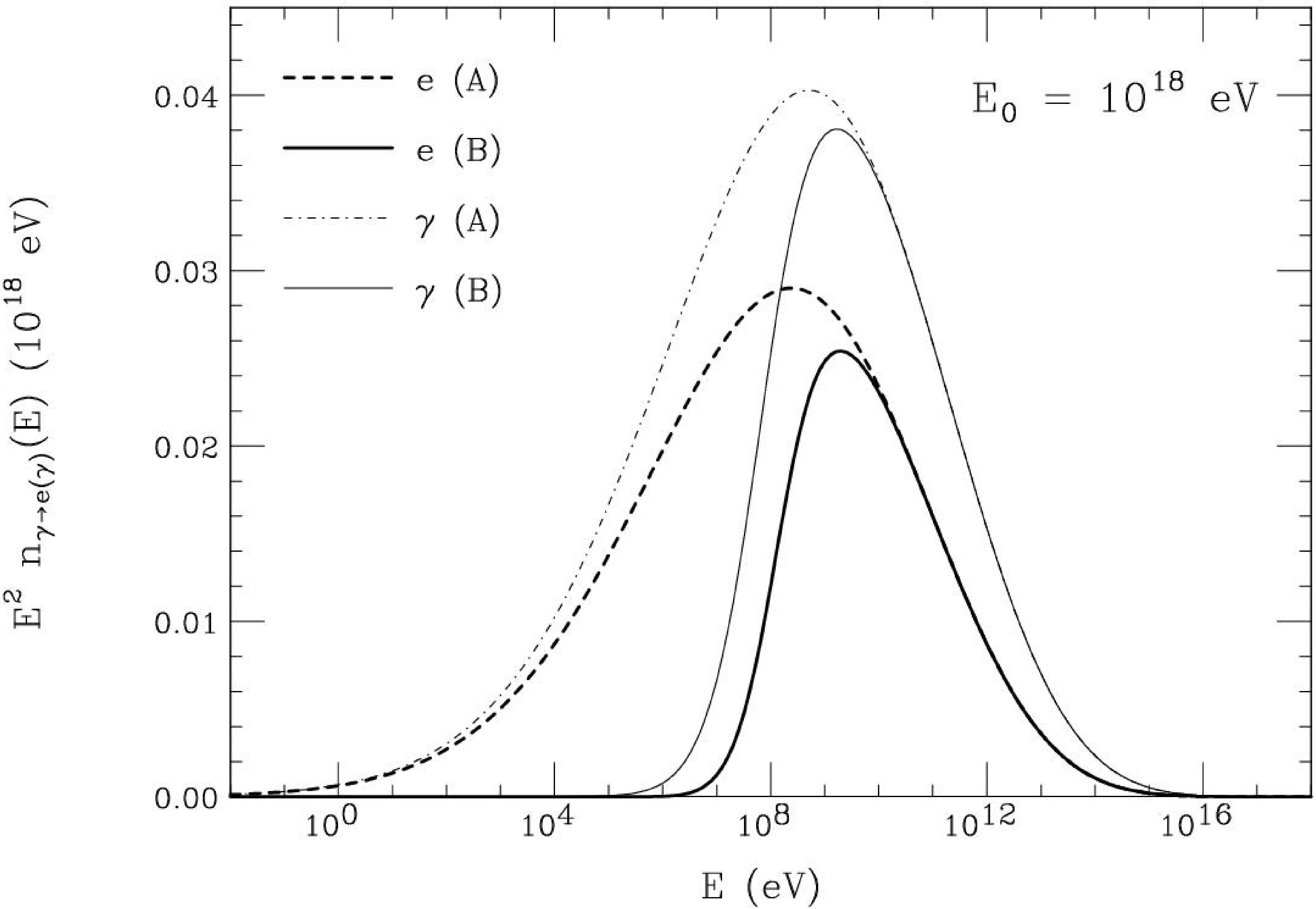
In Approximation A
the total Energy contained in Shower
is CONSTANT.

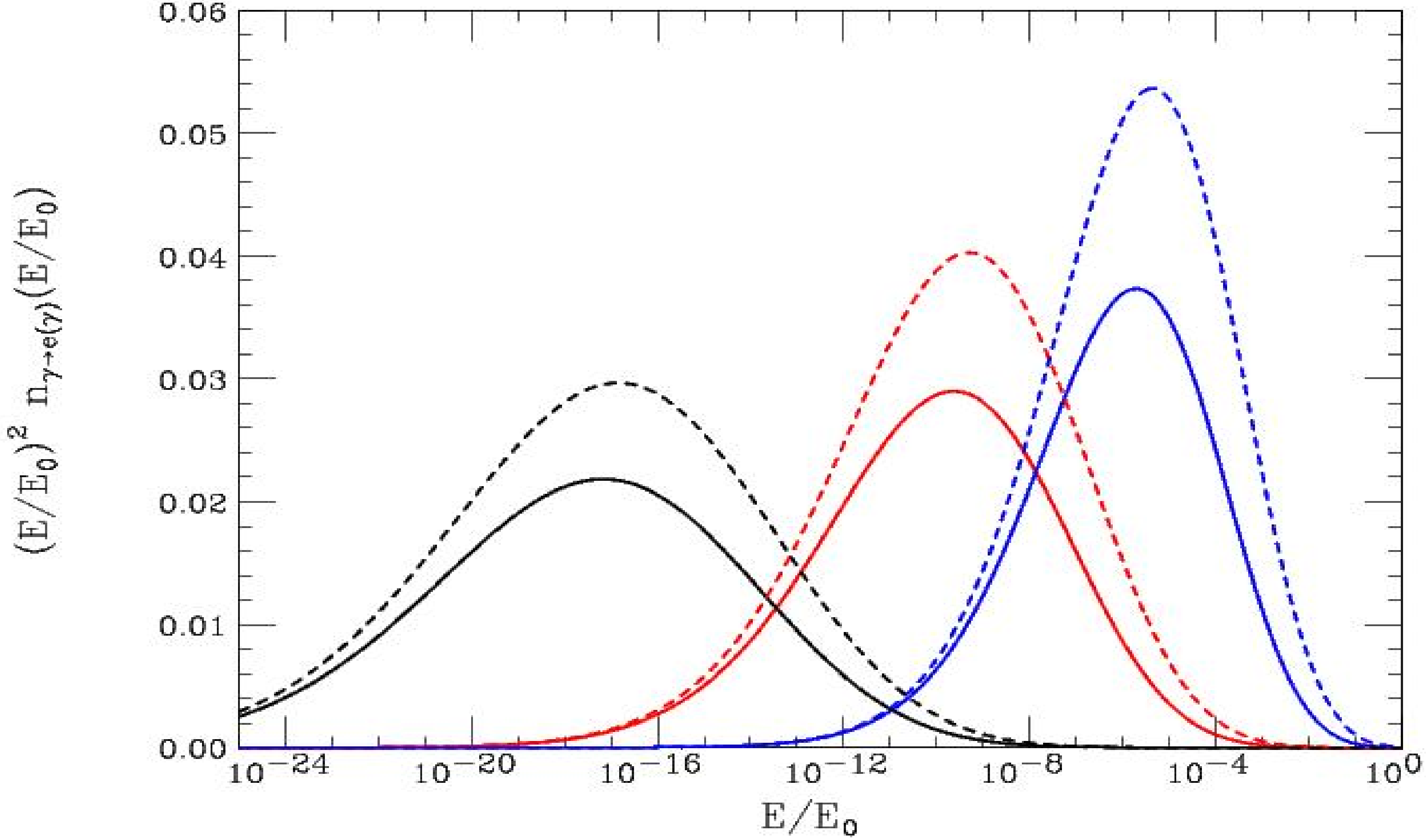

Electron mass
0.06
0.05
0.04
0.03
0.02
0.01
0.00
10^{-4}
10^{0}
10^{4}
10^{8}
10^{12}
10^{16}
E (eV)
$E^2\ n_{\gamma\to e(\gamma)}(E)\ (10^{18}\ eV)$



Electron mass
Critical Energy
0.06
0.05
0.04
0.03
0.02
0.01
0.00
$E^2\ n_{\gamma\to e(\gamma)}(E)\ (10^{18}\ eV)$
10^{-4}
10^{0}
10^{4}
10^{8}
10^{12}
10^{16}
E (eV)

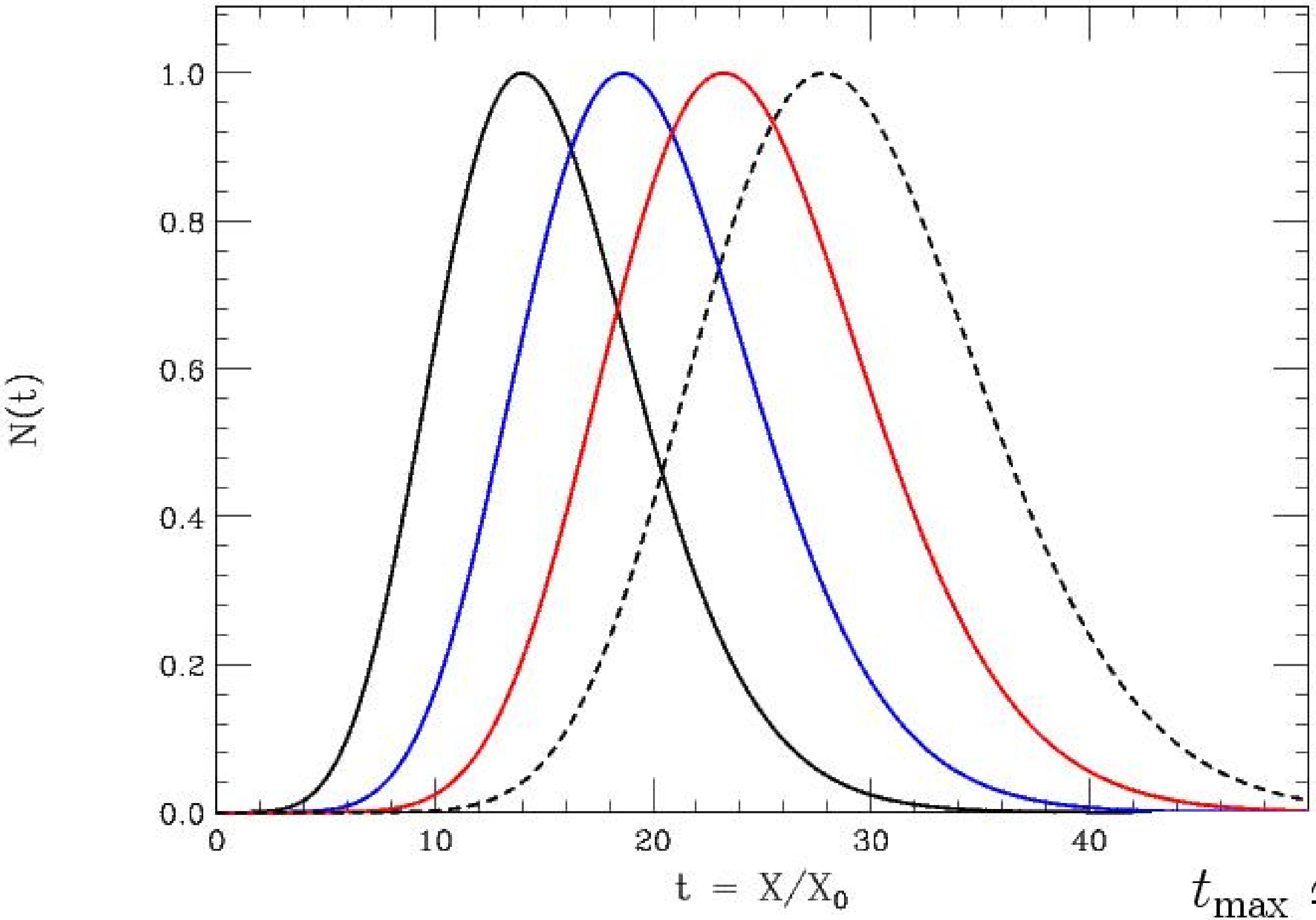
Monochromatic Photon. Approximation A,B





Choose one energy (any energy)
and study how the particle number varies
with t at that energy.

$$n_e(E, E_0, t) \propto \exp\left[t\left(1 - \frac{3}{2}\log\left(\frac{3t}{t + 2\,\ln(E_0/E)}\right)\right)\right]$$ (good approximation)



$$t_{\max} \simeq \ln\left(\frac{E_0}{E}\right)$$

$$n(E) = E^{-\alpha}$$

Power Law

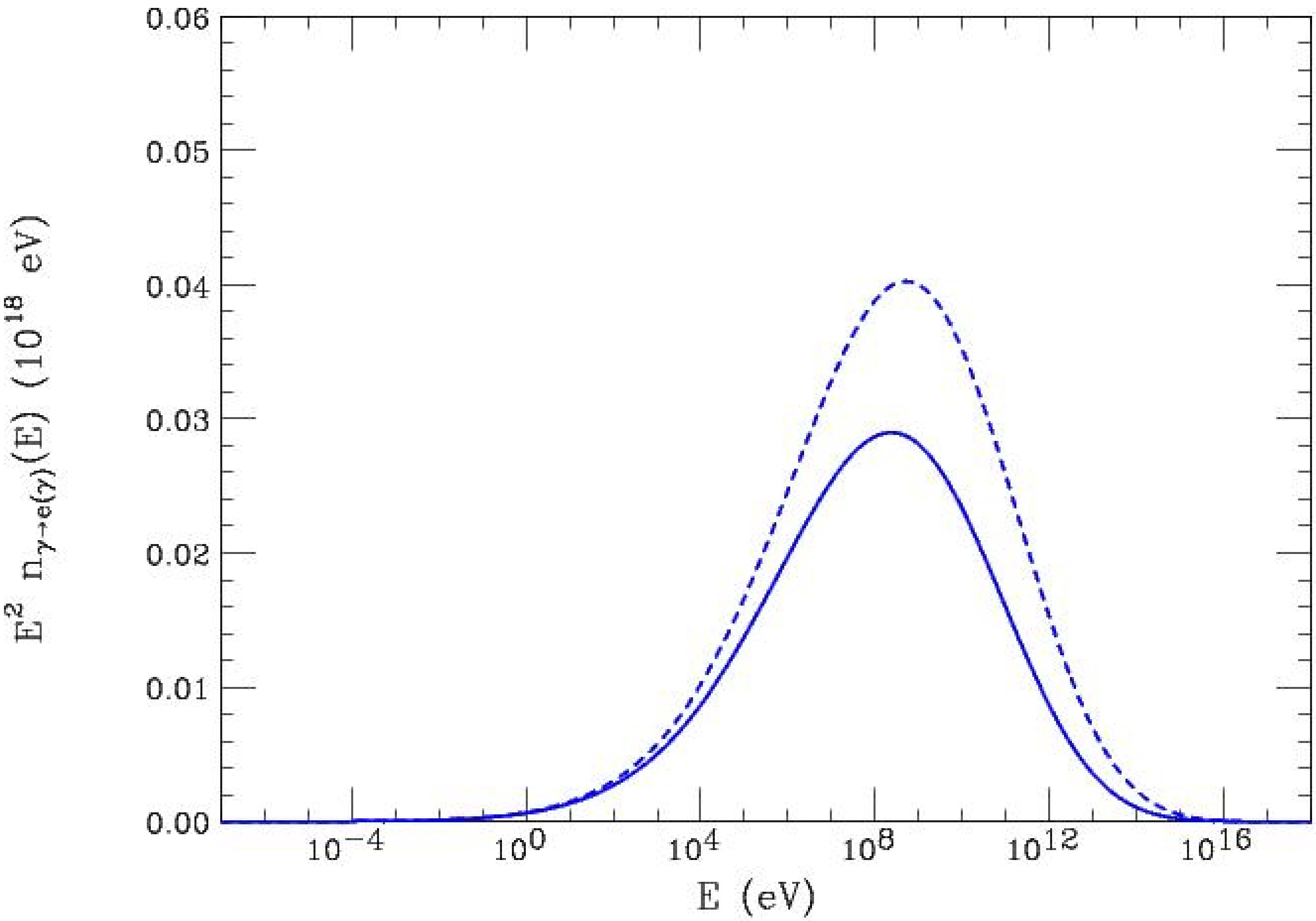
$$\alpha = -\left(\frac{E}{n}\right) \frac{dn(E)}{dE}$$

Slope

$n(E)$ Arbitrary shape

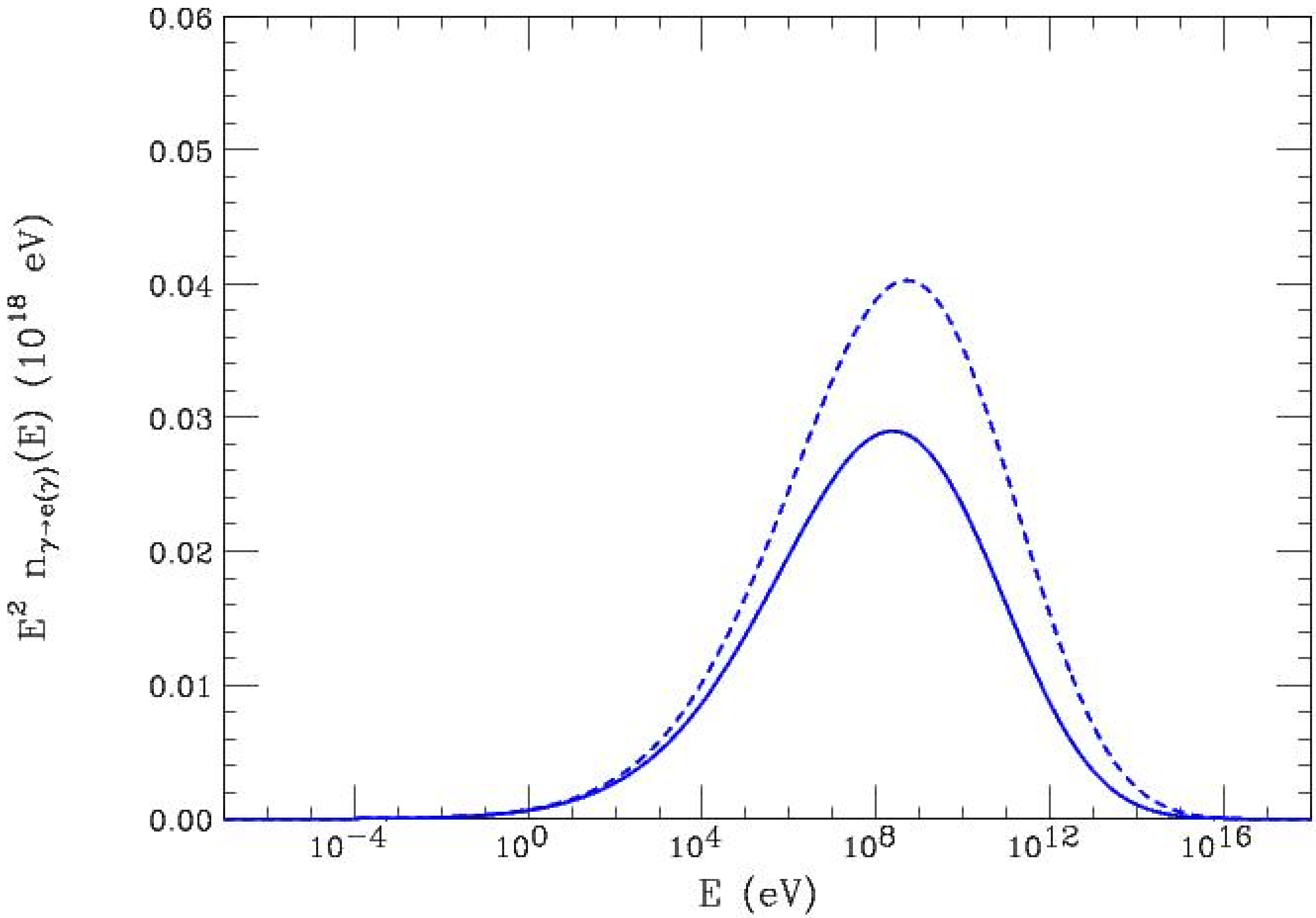
$$\alpha(E) = s(E) + 1 = -\left(\frac{E}{n}\right) \frac{dn(E)}{dE}$$

“Local (energy dependent) Slope”



Consider the shape of the spectra at a fixed t
It is a function of E/E0 and t.

$$s(E/E_0, t)$$ Local slope



$$s(E/E_0, t)$$

Consider the shape of the spectra at a fixed t
It is a function of E/E0 and t.

QUESTION : At what energy in this graph s(E) = 1 ?

t-slope and E-slope are connected

$$\lambda = \frac{1}{N(t)} \frac{dN(t)}{dt}$$

Integral Electron Spectrum Evolution

Can deduce the AGE (and spectral shape)

$$s = \lambda_1^{-1}(\lambda) = \lambda_1^{-1}\left(\frac{1}{N(t)} \frac{dN(t)}{dt}\right)$$

$$n_e(E) \sim n_\gamma(E) \sim E^{-(s+1)}$$

How does one solve the equations ?

We know how to solve the equations
for an initial condition that is a power law

“combine these 'elementary' solutions”
to obtain the solution for a
physically meaningful initial condition.

$$n_e(E,0) = \delta[E - E_0]$$

Write initial condition
as a superposition of power law
component
Inverse Mellin transform

$$f(E) = \frac{1}{2\pi i} \int_C ds\; E^{-(s+1)}\; M_f(s)$$

$$M_f(s) = \int_0^\infty dE\; E^s\; f(E)$$

C Arbitrary path
in the complex plane
that connects the points

$$s_{x_1} - i\,\infty$$
$$s_{x_2} + i\,\infty$$

$$n_e(E,0) = \frac{1}{2\,\pi\, i} \int_C ds\; E_0^s\; E^{-(s+1)}$$

The parameter s
takes complex values

$$n_e(E,0) = \delta[E - E_0]$$

Write initial condition
as a superposition of power law
component
Inverse Mellin transform

$$n_e(E,0) = \frac{1}{2\pi i} \int_C ds\ E_0^s\ E^{-(s+1)}$$

Depth Evolution

$$n_e(E,t) \simeq \frac{1}{2\pi i} \int_C ds\ E_0^s\ E^{-(s+1)}\ e^{\lambda_1(s)\,t}$$

For a given E_0, E, t

what is the parameter s that dominate ?

SADDLE point Approximation

$$n_e(E,t) \simeq \frac{1}{2\pi i} \int_C ds\, E_0^s\, E^{-(s+1)}\, e^{\lambda_1(s)\,t}$$

$$\frac{d}{ds}\left[\left(\frac{E}{E_0}\right)^{-s} e^{\lambda_1(s)\,t}\right] = 0$$

$$\lambda'(s)\,t + \ln\left(\frac{E_0}{E}\right) = 0$$

Solution of this equation

For a given E_0, E, t

what is the parameter s
that dominate ?

$$n_e(E,t) \simeq \frac{1}{2\,\pi\, i} \int_C ds\; E_0^s \; E^{-(s+1)} \; e^{\lambda_1(s)\, t}$$

$$\frac{d}{ds}\left[\left(\frac{E}{E_0}\right)^{-s} \; e^{\lambda_1(s)\, t}\right] = 0$$

$$\lambda'(s)\, t + \ln\left(\frac{E_0}{E}\right) = 0$$

Solution of this equation

$$\overline{\lambda}_1(s) = \frac{1}{2}\,(s - 1 - 3\,\ln s)$$

$$s \simeq \overline{s}\left(\frac{E}{E_0}, t\right) = \frac{3\,t}{t - 2\,\ln(E/E_0)}$$

Age and Longitudinal Development

$$\frac{dN_e(t)}{dt} = \lambda_1(s)\ N_e(t)$$

Age and Longitudinal Developmen t

$$\frac{dN_e(t)}{dt} = \lambda_1(s)\ N_e(t)$$

$$s = \frac{3\,t}{t + 2\,t_{\max}}$$

$$\overline{\lambda}_1(s) = \frac{1}{2}\,(s - 1 - 3\,\ln s)$$

Age and Longitudinal Developmen t

$$\frac{dN_e(t)}{dt} = \lambda_1(s)\ N_e(t)$$

$$\boxed{\begin{gathered} \mathrm{S} = \frac{3\,t}{t + 2\,t_{\max}} \\[2em] \overline{\lambda}_1(s) = \frac{1}{2}\,(s - 1 - 3\,\ln s) \end{gathered}}$$

$$= \frac{1}{2}\left[\frac{3t}{t+2\,t_{\max}} - 1 - 3\,\log\left(\frac{3t}{t+2\,t_{\max}}\right)\right]\ N(t)$$

Differential Equation

Differential Equation

$$\frac{dN_e(t)}{dt} = \lambda_1(s)\ N_e(t)$$

$$= \frac{1}{2}\left[\frac{3t}{t+2\,t_{\max}} - 1 - 3\log\left(\frac{3t}{t+2\,t_{\max}}\right)\right]\ N(t)$$

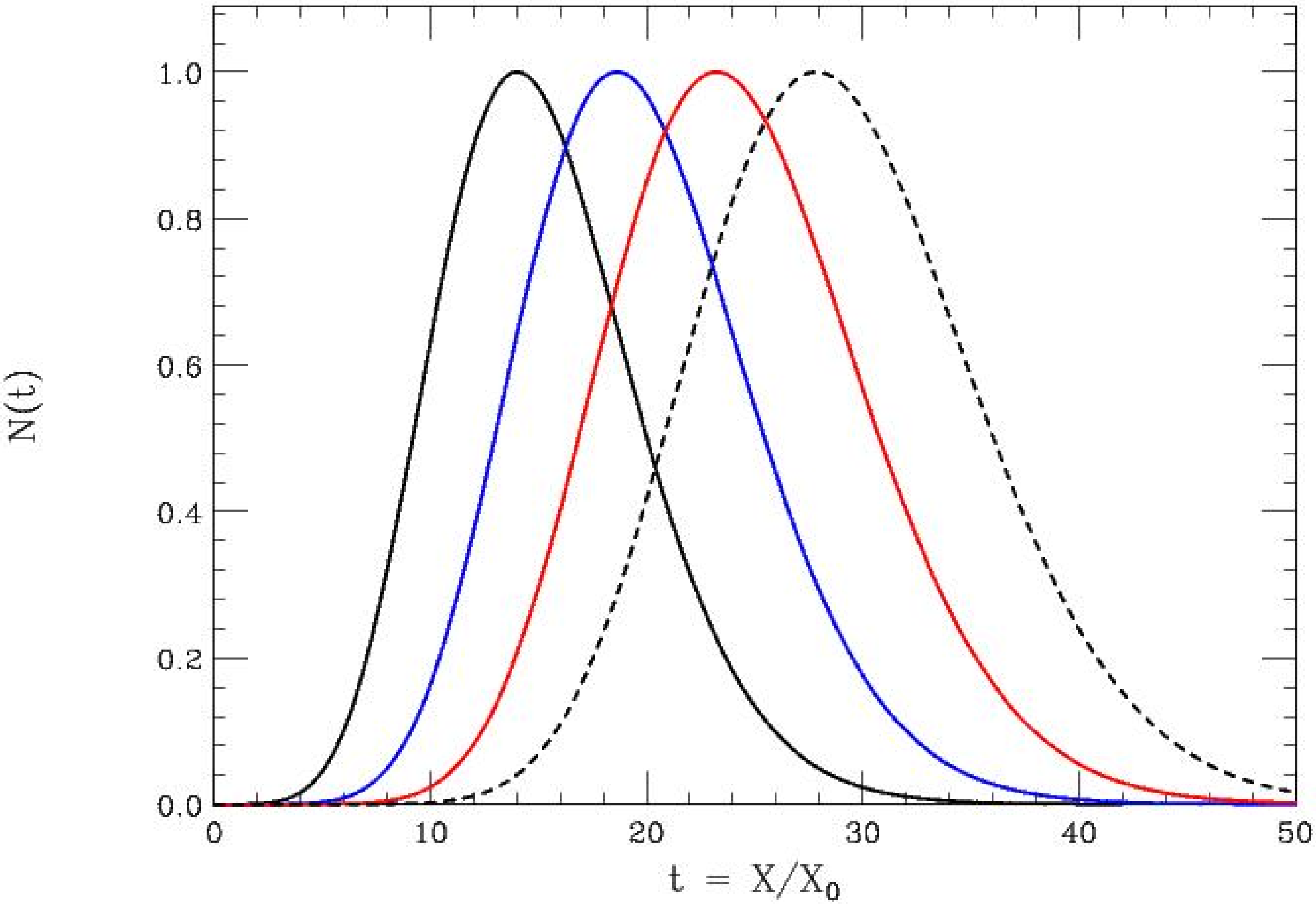
Boundary Condition

$$N(t_{\max}) = N_{\max}$$

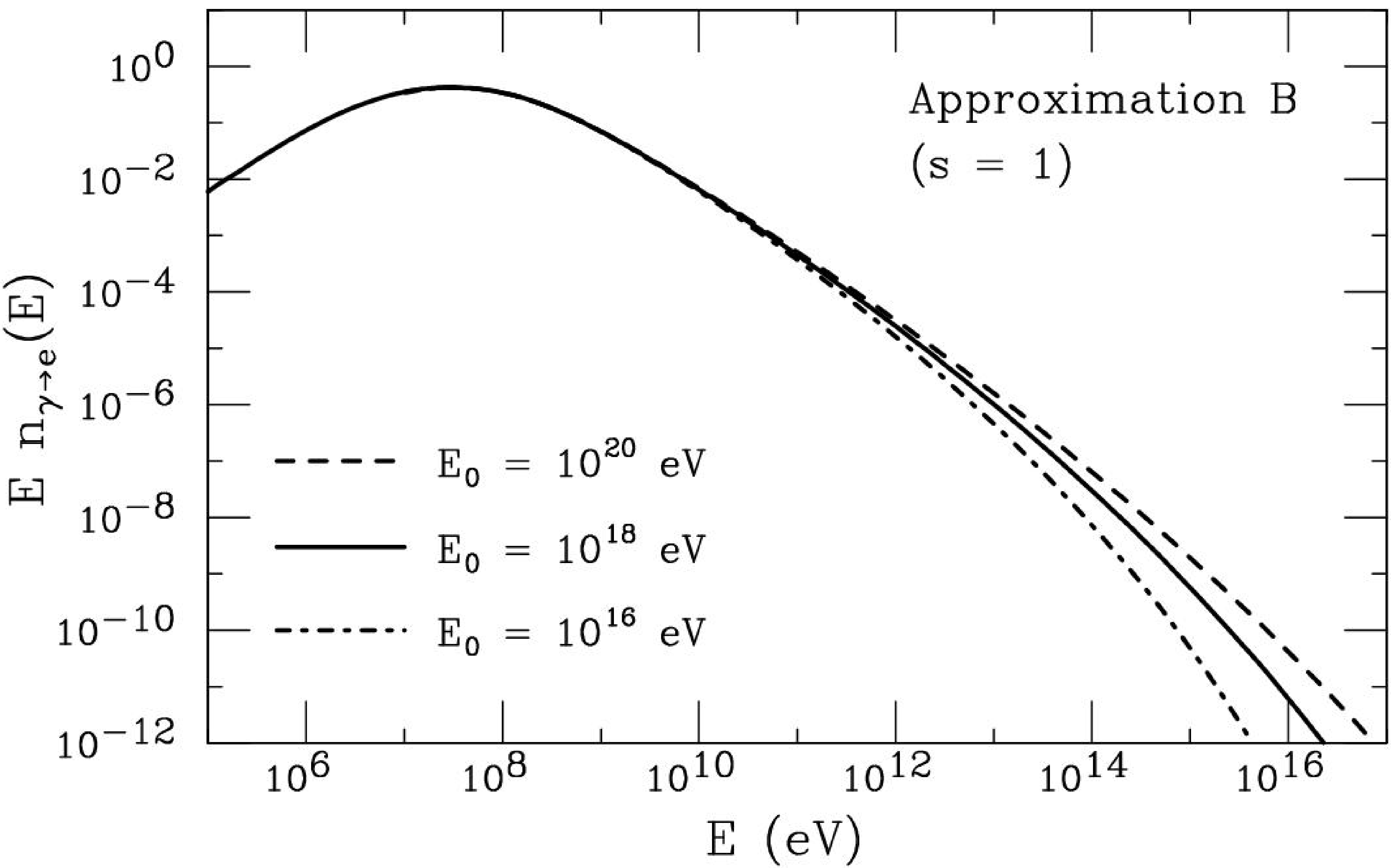
Solution : Greisen Profile

$$N_e(t) = N_{\max}\ e^{-t_{\max}}\ \exp\left[t\left(1-\frac{3}{2}\log\left(\frac{3t}{t+2\,t_{\max}}\right)\right)\right]$$

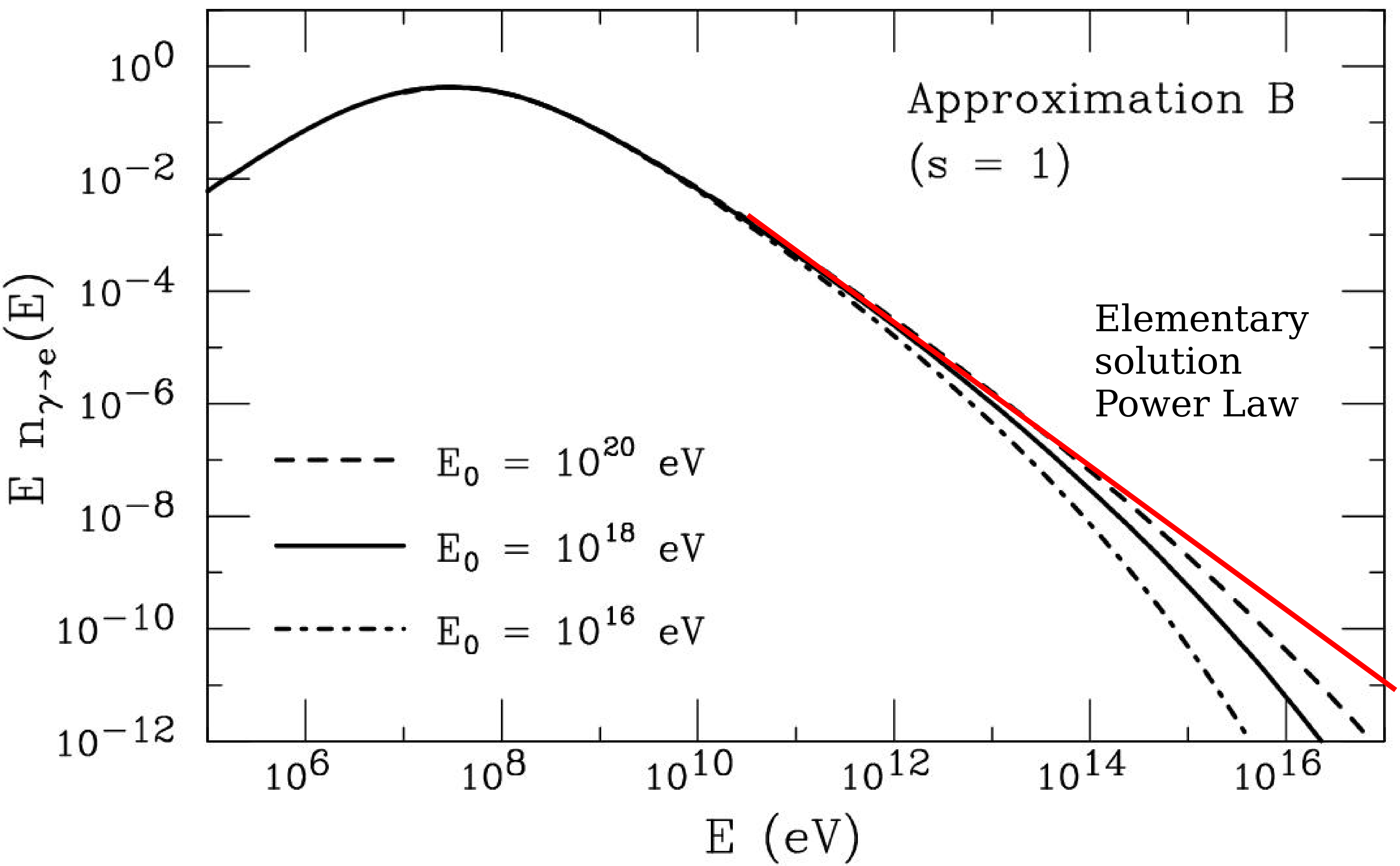
$$N_{\mathrm{Greisen}}(E_0, t) = \frac{0.31}{\sqrt{\ln(E_0/\varepsilon)}} \exp\left[t \left(1 - \frac{3}{2} \log\left(\frac{3t}{t + 2\ln(E_0/\varepsilon)} \right) \right) \right]$$

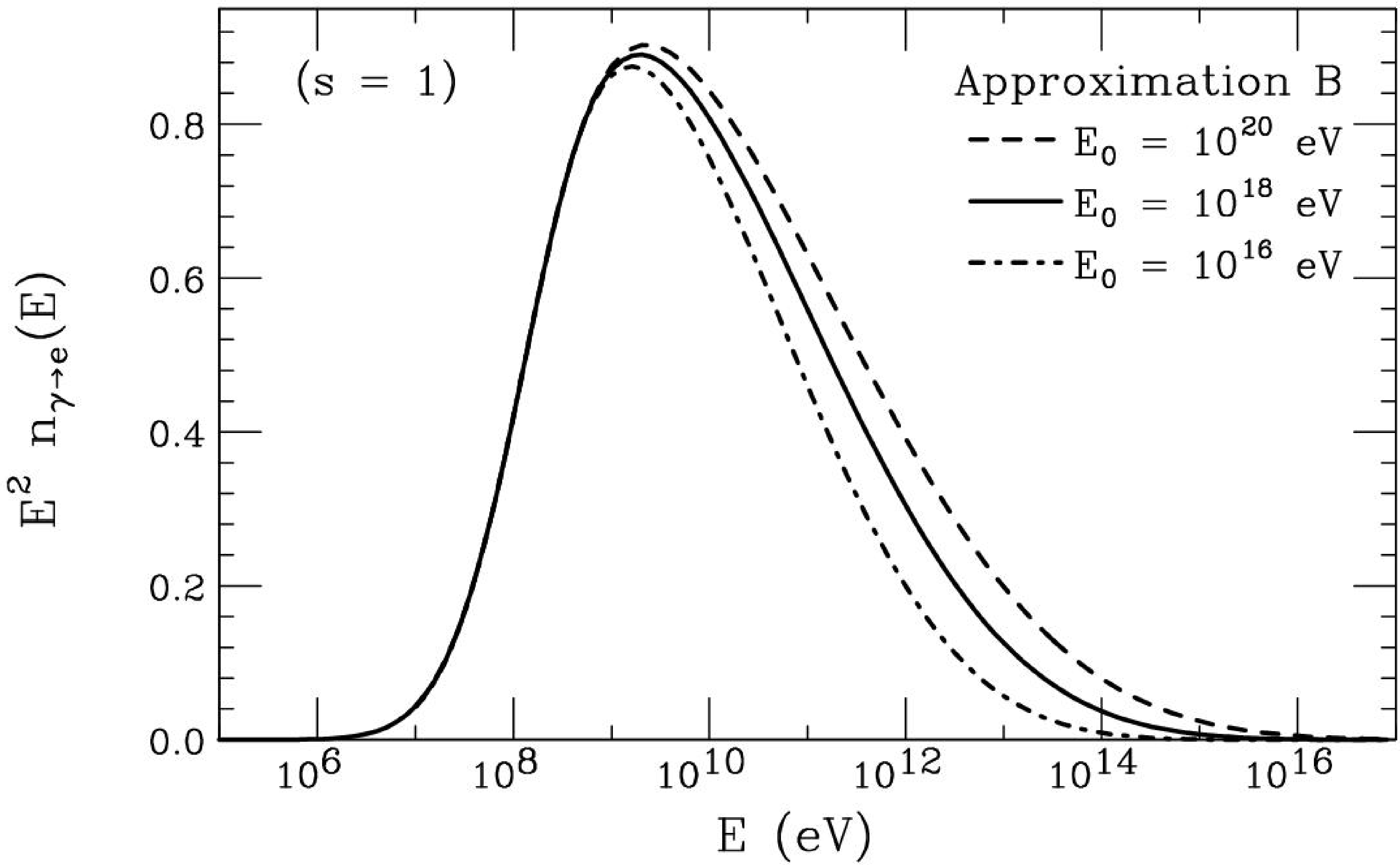


Different Energy : Same Age (Shower Maximum)



Different Energy : Same Age (Shower Maximum)





(s = 1)
Approximation B
$E_0 = 10^{20}$ eV
$E_0 = 10^{18}$ eV
$E_0 = 10^{16}$ eV
0.0
0.2
0.4
0.6
0.8
$E^2\ n_{\gamma\to e}(E)$
10^6
10^8
10^{10}
10^{12}
10^{14}
10^{16}
E (eV)

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Universality of electron distributions in high-energy air showers—Description of Cherenkov light production

F. Nerling [a,*], J. Blümer [a,b], R. Engel [a], M. Risse [a]

Abstract

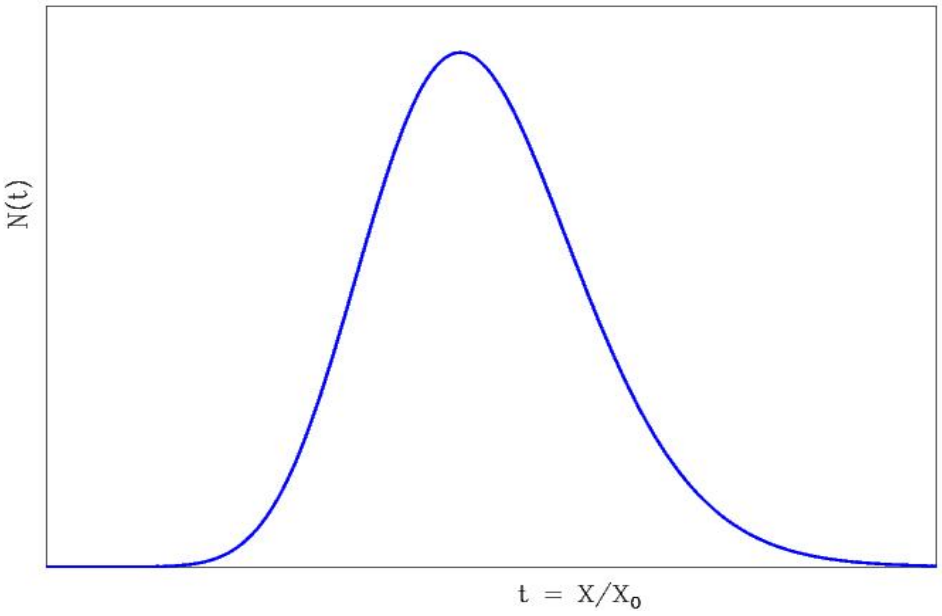
The shower simulation code CORSIKA has been used to investigate the electron energy and angular distributions in high-energy showers. Based on the universality of both distributions, we develop an analytical description of Cherenkov light emission in extensive air showers, which provides the total number and angular distribution of photons. The parameterisation can be used e.g. to calculate the contribution of direct and scattered Cherenkov light to shower profiles measured with the air fluorescence technique.

Earlier results

M. Giller et al., J. Phys. G: Nucl. Part. Phys. 30 (2004) 97; M. Giller, in: Proc. 28th Int. Cos. Ray Conf., Tsukuba, Japan, vol. 2, 2003, p. 619. Note: The set of parameters

Concept of : Shower AGE

Shower
Longitudinal Development



Often used
but (in my view)
unsatisfactory definition

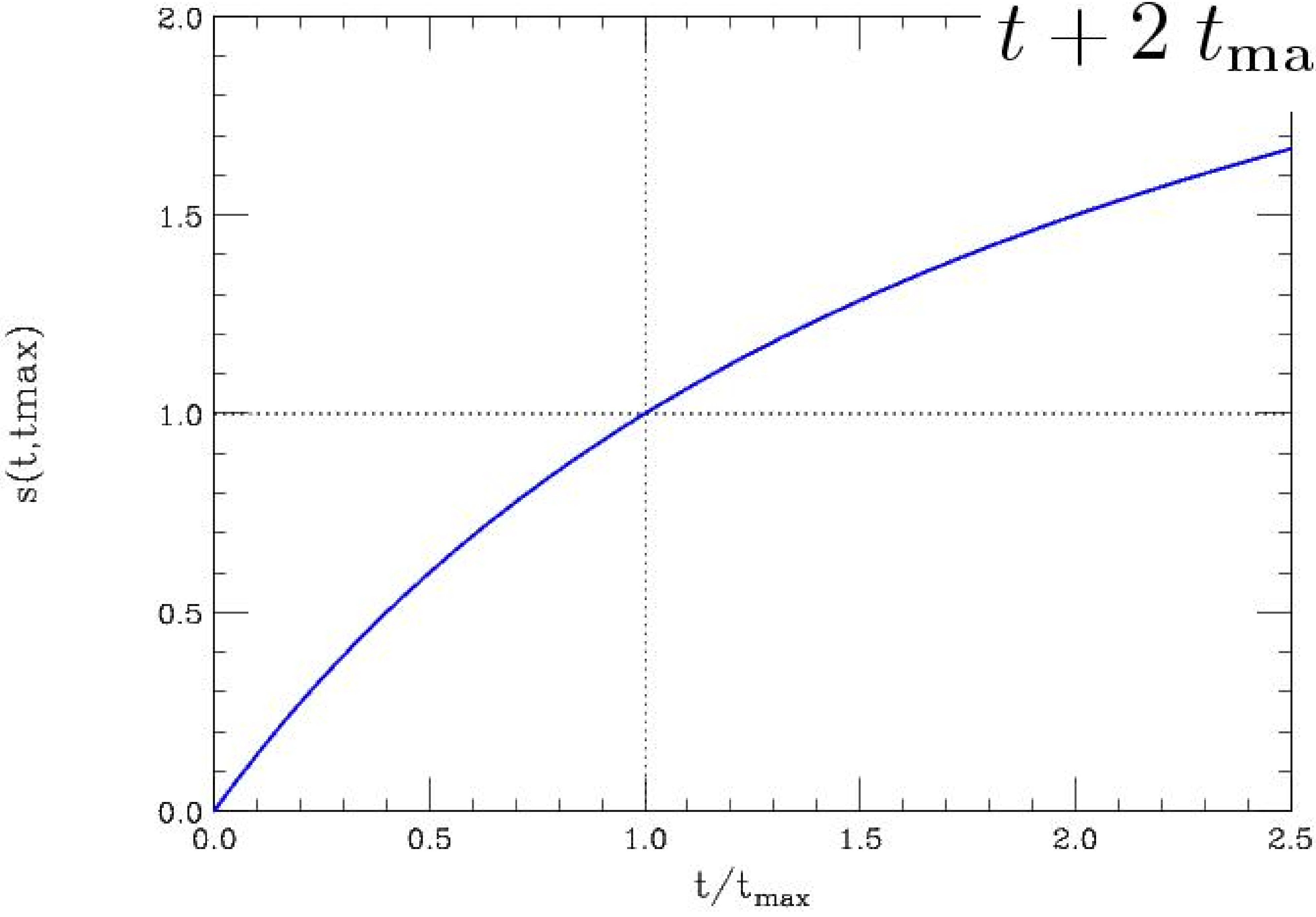
Shower at maximum: s = 1

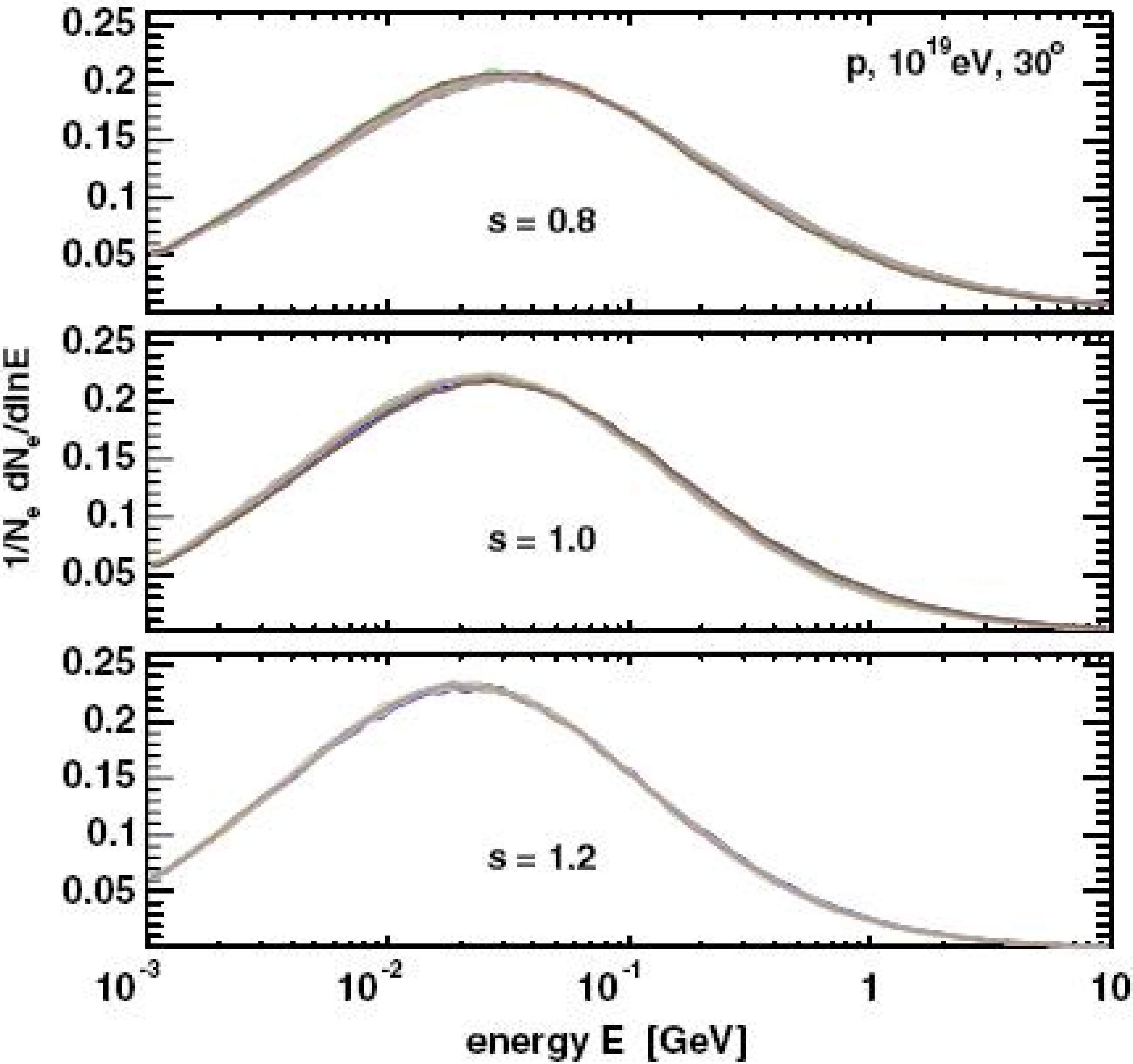
Shower before maximum s < 1
Shower after maximum s > 1

$$\boxed{S = \frac{3\,t}{t + 2\,t_{\max}}}$$

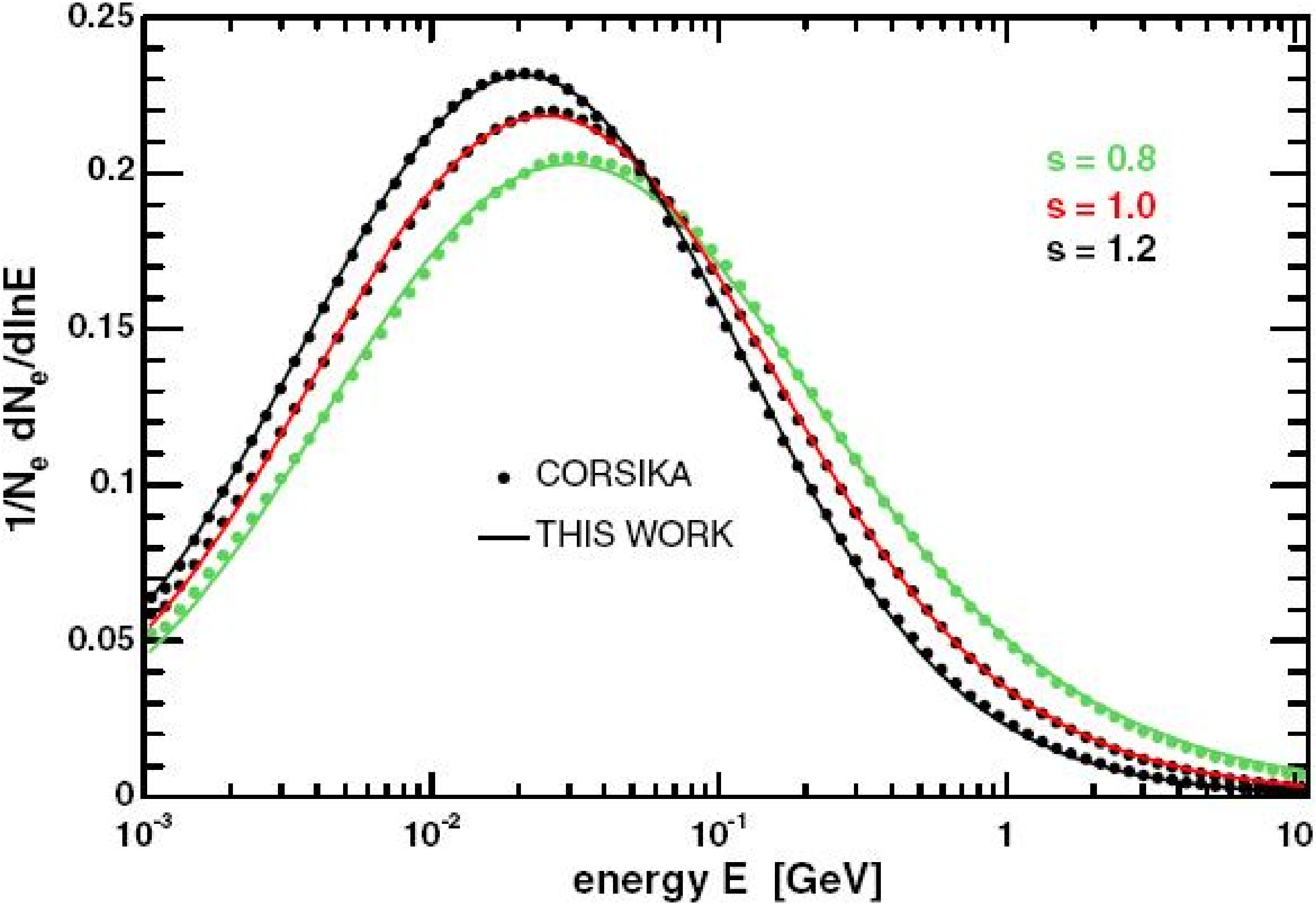
Age as a function of t/t_{max}

$$\frac{3\,t}{t + 2\,t_{\mathbf{max}}}$$





p, 10^19 eV, 30°
s = 0.8
s = 1.0
s = 1.2
1/N_e dN_e/dlnE
energy E [GeV]
0.25
0.2
0.15
0.1
0.05
10^-3
10^-2
10^-1
1
10



$$f_e(E,s) = a_0 \cdot \frac{E}{(E+a_1)(E+a_2)^s}$$

$$a_1 = 6.42522 - 1.53183 \cdot s$$

$$a_2 = 168.168 - 42.1368 \cdot s \qquad \text{with } E \text{ in MeV}$$

- The shape of the electron energy spectrum is determined (in good approximation) by the "shower Age"

- The Photon spectral shape is (in good Approximation) also determined by the shower Age

Calculated first by Rossi, Greisen in 1941

- The Ratio photon/Electron is determined by the shower Age

"Model Independent " Definition of AGE

$$\lambda = \frac{1}{N(t)} \frac{dN(t)}{dt} \qquad s = \lambda_1^{-1}(\lambda)$$

For real showers the longitudinal development is not identical to the “Greisen Profile” and fluctuates from shower to shower

Violations of the “Universality”

For real showers the longitudinal development is not identical to the "Greisen Profile" and fluctuates from shower to shower

Violations of the "Universality"

$$\lambda = \frac{1}{N(t)} \frac{dN(t)}{dt}$$

$$s = \lambda_1^{-1}(\lambda)$$

General Model Independent Definition of Age

Possible Generalizations:

3-Dimensional treatment.

$$n_{e,\gamma}(E, x, \theta_x, y, \theta_y, \ t)$$

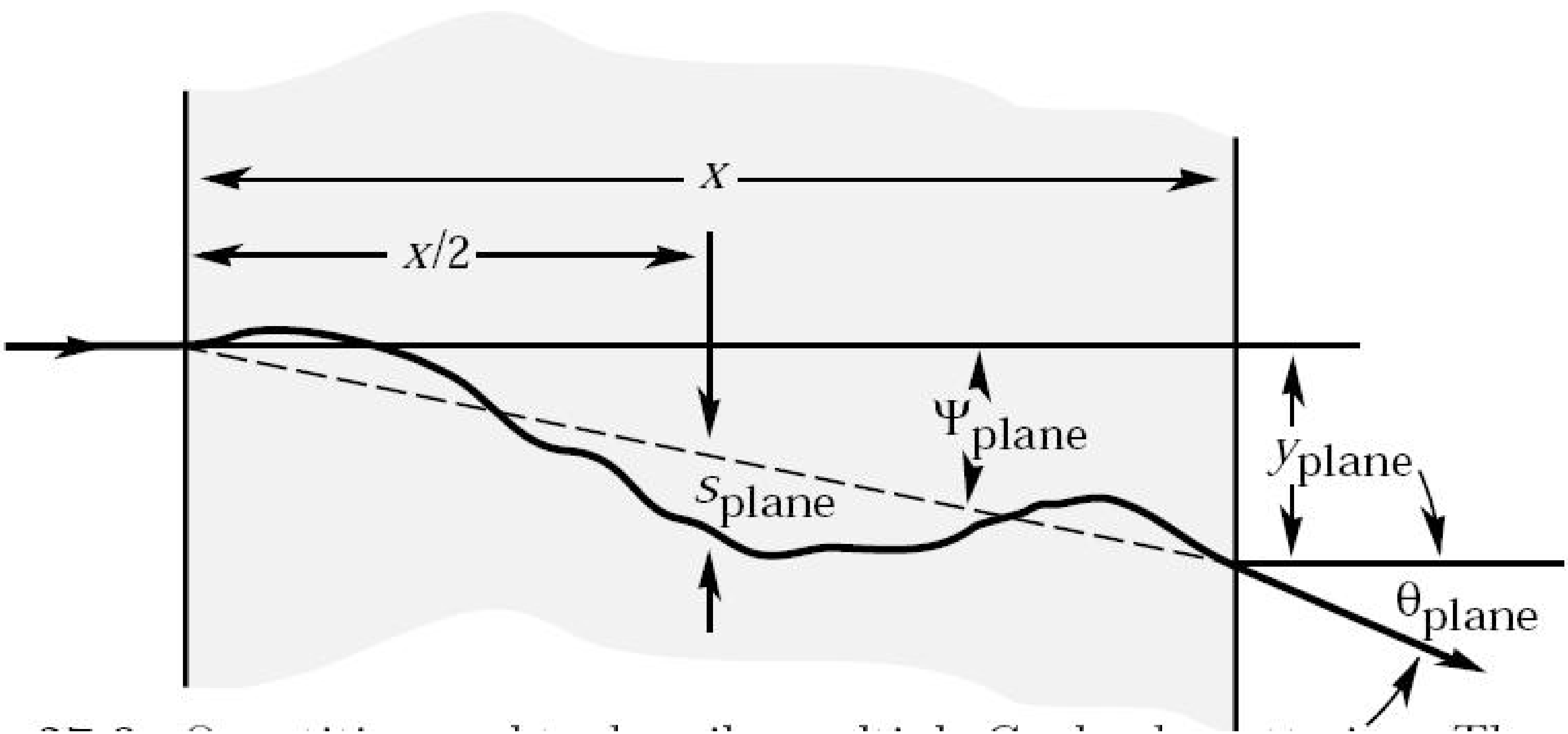
Nishimura
Kamata

Hadronic Showers: add other components

$$n_{p,n}(E, t) \qquad n_{\mu^\pm}(E, t)$$

$$n_{\pi^\pm}(E, t) \qquad n_{\nu}(E, t)$$

Multiple Scattering and LATERAL DISTRIBUTION



Progress of Theoretical Physics, Vol. 7, No. 2, February 1952

On the Theory of Cascade Showers, I

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and
Koichi KAMATA
Scientific Research Institute

(Received December 31, 1951)

The diffusion equation of the lateral and angular distribution function were given by Landau,[13] and they are

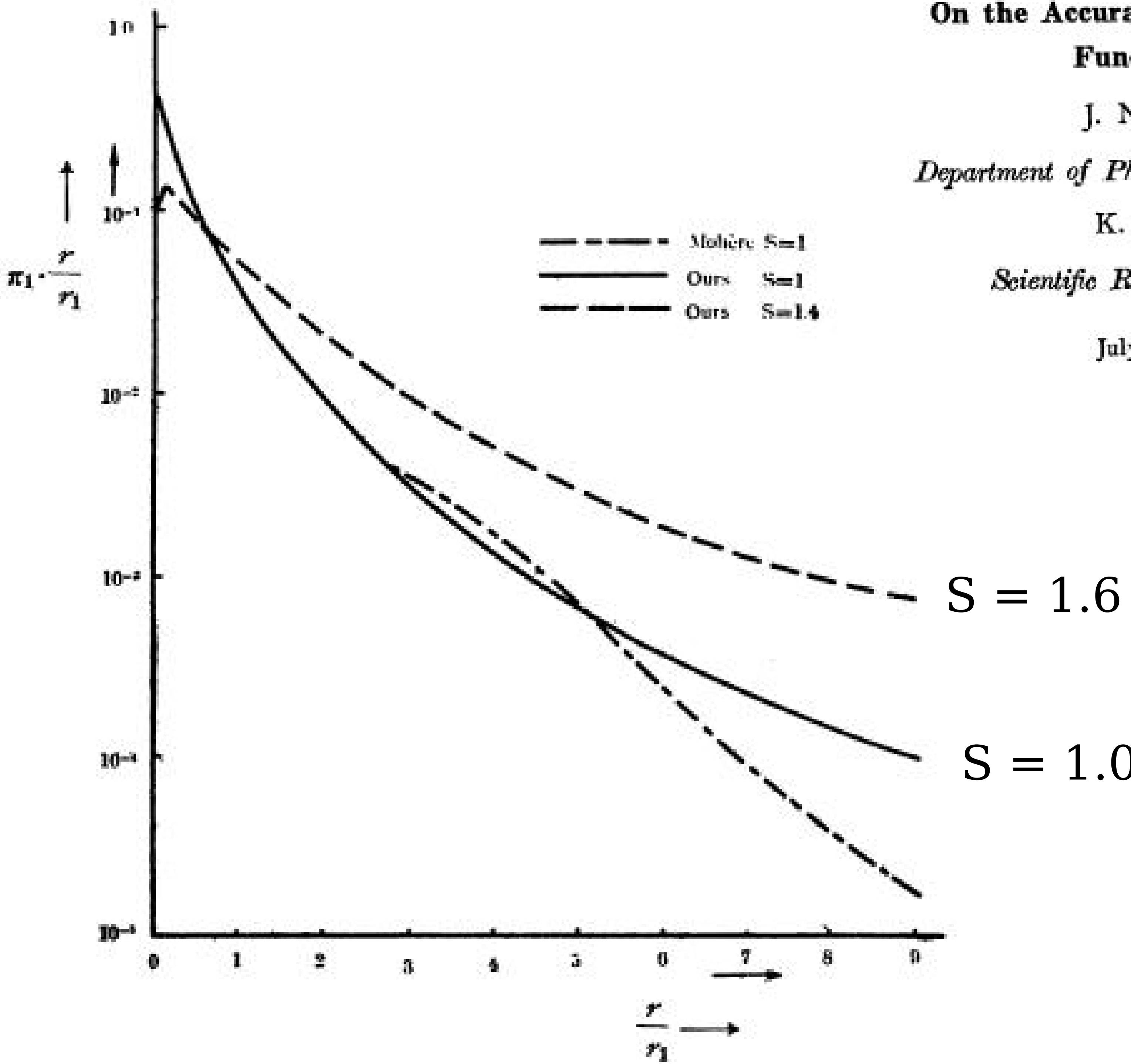
$$\frac{\partial \pi}{\partial t} = -A'\pi + B'\gamma + \frac{K^2}{4E^2}\left(\frac{\partial^2}{\partial \theta_1^2} + \frac{\partial^2}{\partial \theta_2^2}\right)\pi - \left(\theta_1 \frac{\partial}{\partial y_1} - \theta_2 \frac{\partial}{\partial y_2}\right)\pi + \epsilon \frac{\partial \pi}{\partial E} \qquad (16)$$

and

$$\frac{\partial \gamma}{\partial t} = C'\pi - \sigma_0 \gamma - \left(\theta_1 \frac{\partial}{\partial y_1} - \theta_2 \frac{\partial}{\partial y_2}\right)\gamma, \qquad (17)$$

where

y_1, y_2, θ_1, θ_2; Lateral and angular deviations of the shower particles from the shower axis.



On the Accuracy of the Molière Function, II

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July 9, 1951

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Kamata

Exact Montecarlo calculations
of the shower (electromagnetic and hadronic)
development are today possible
thanks to modern computer power

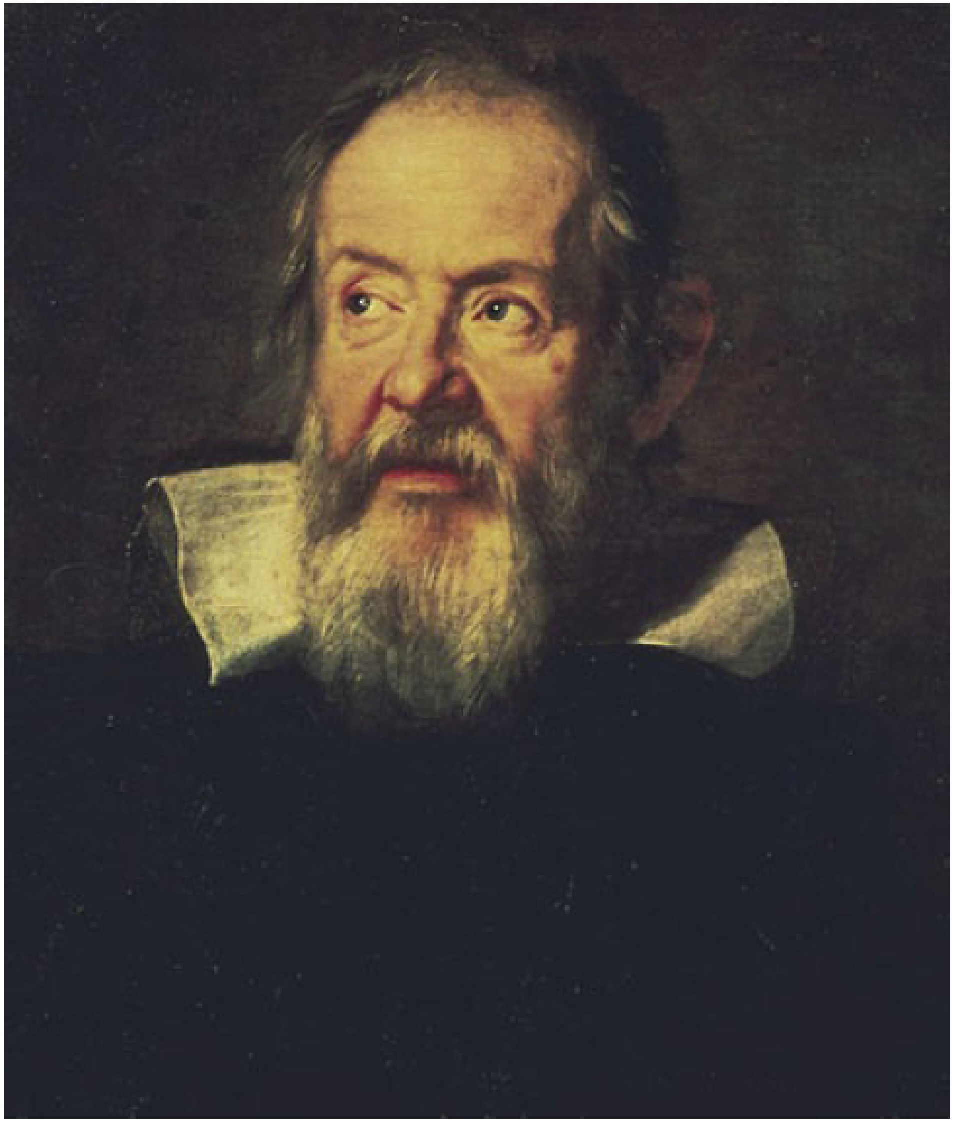
Avoid approximations
Include fluctuations (showers are not all equal !)

These analytic methods are useful to gain
physical understanding, and are in fact also useful
To Speed-up the calculation of very large showers
(describing sub-showers with the average development).

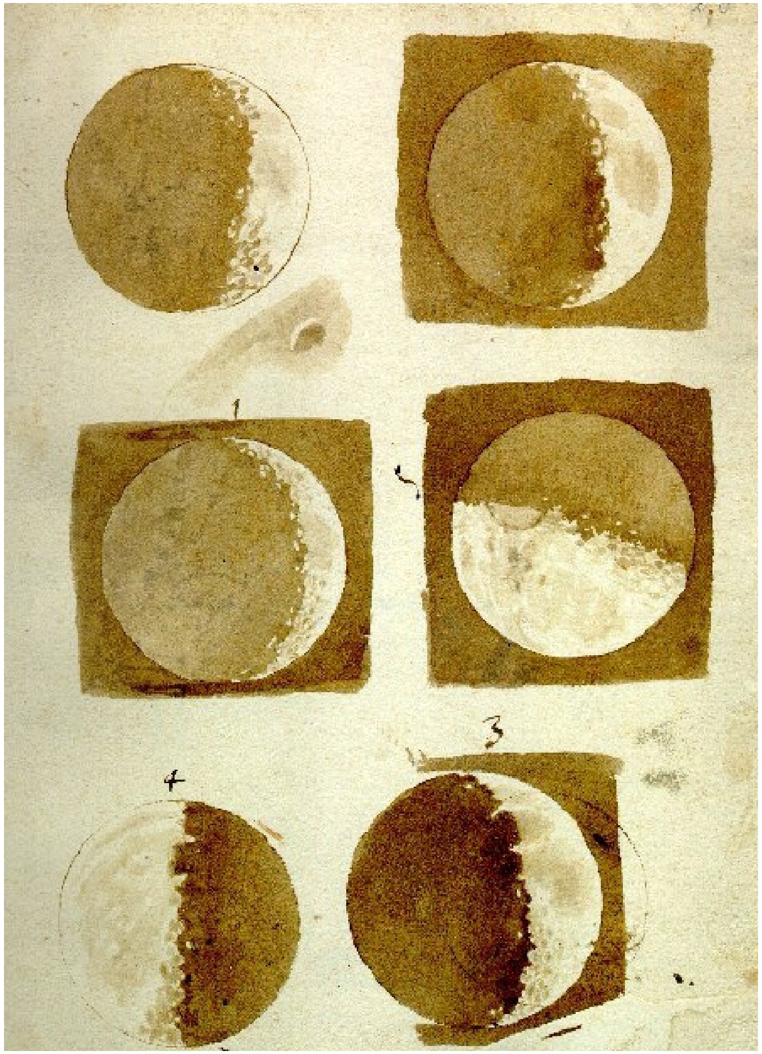
This is the end of my two lectures.....

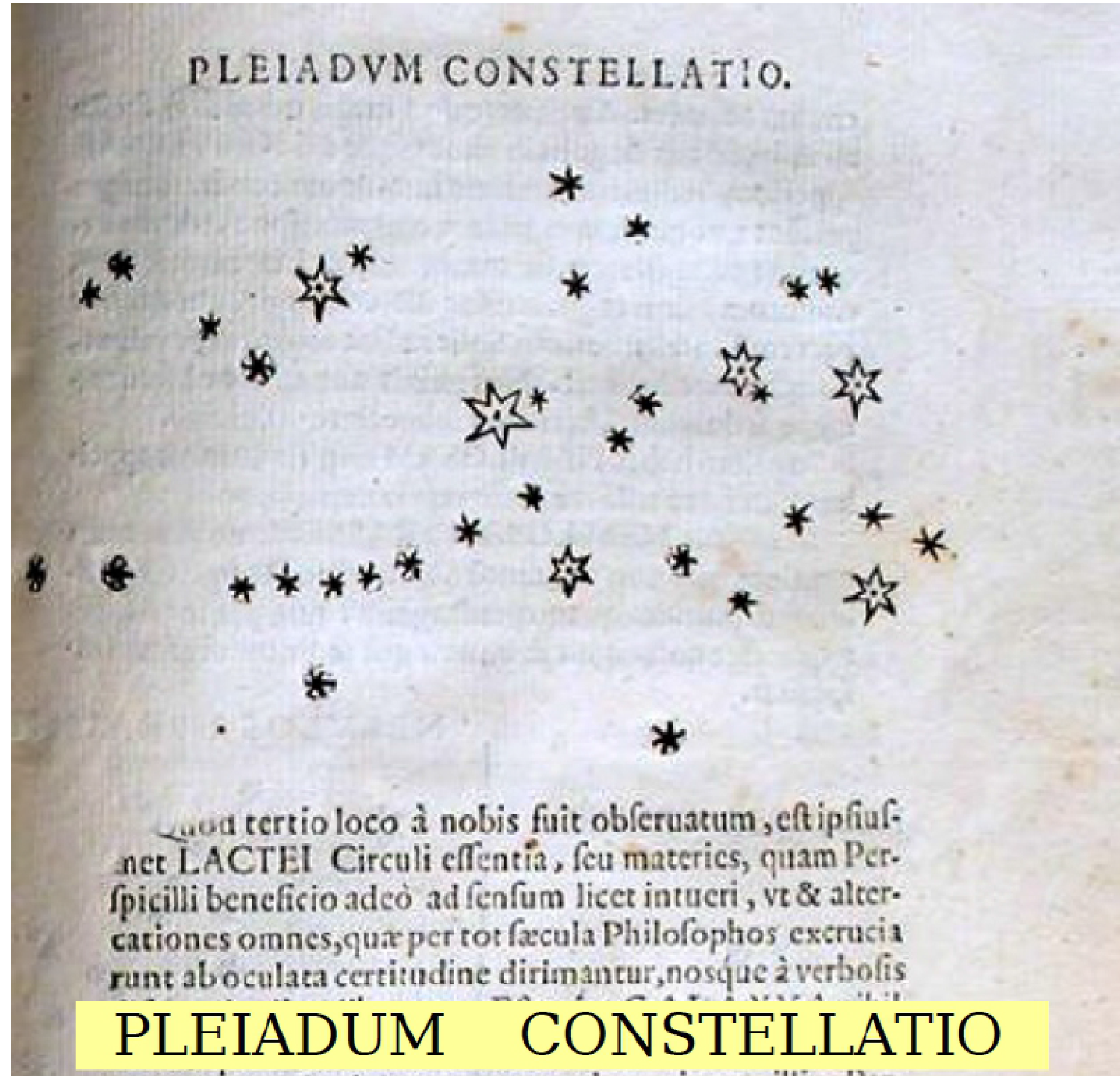
(Many apologies for the imperfection...)

Galileo Galilei



The telescope



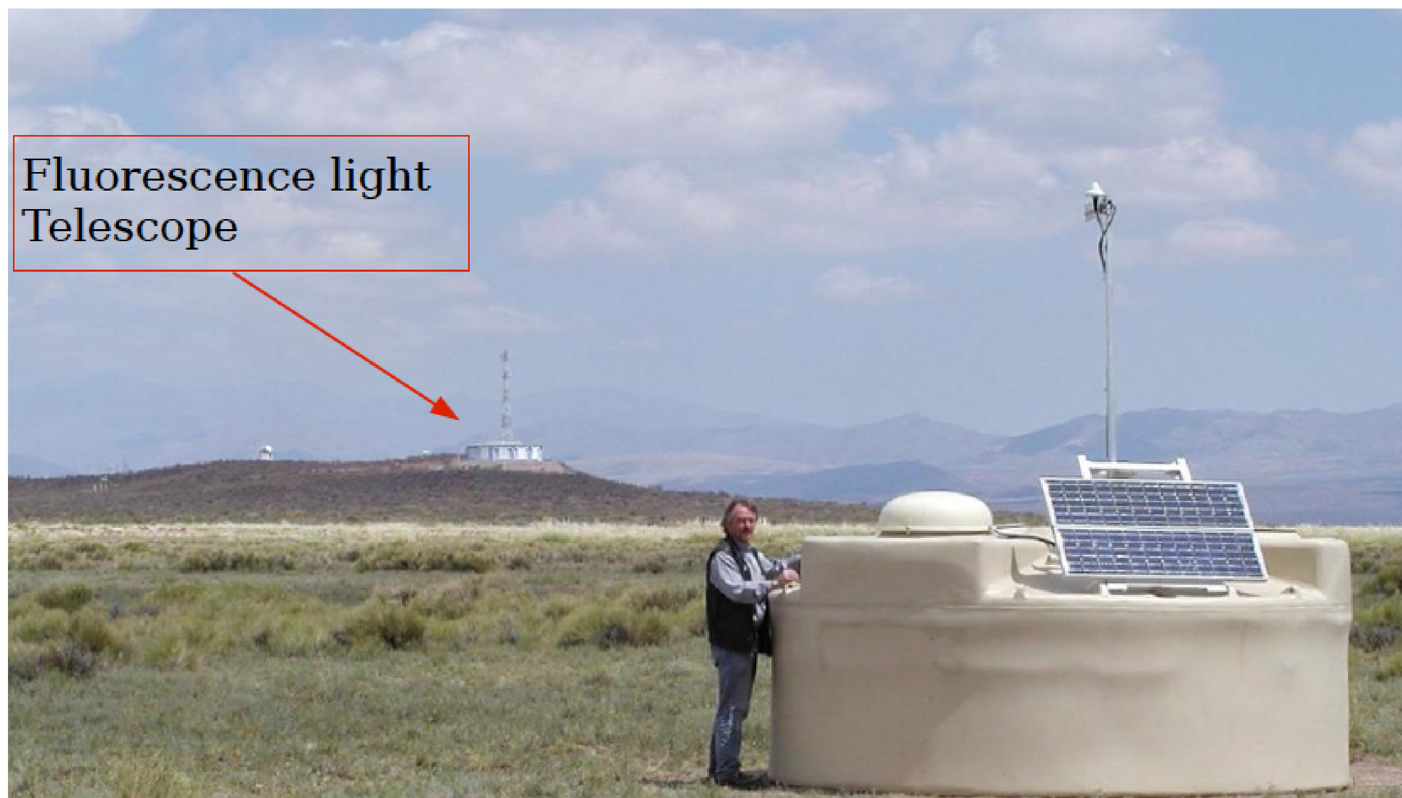


PLEIADVM CONSTELLATIO.

Quod tertio loco à nobis fuit obſeruatum, eſt ipſius-met LACTEI Circuli eſſentia, ſeu materies, quam Perſpicilli beneficio adeò ad ſenſum licet intueri, vt & altercationes omnes, quæ per tot ſæcula Philoſophos excruciarunt ab oculata certitudine dirimantur, noſque à verboſis

Deeper understanding of our universe

The contemporary "telescopes"




My best wishes to all of you !