LG Inequalities and neutrino oscillations

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December 15, 2017

Plan of talk

Introduction

Primary focus

To understand the ways in which it may be possible (or NOT) to accommodate quantum theory in a deeper picture of “reality”.

Classical vs Quantum view

Classical view: The physical properties have an existence independent of observation. Measurements merely act to reveal such physical properties.

Quantum view: An unobserved particle does not possess physical properties that exist independent of observation. Rather, such physical properties arise as a consequence of measurements performed upon the system.

This quantum view of Nature was rejected by many physicists, in particular by Albert Einstein.

EPR

- In the famous "EPR paper" in 1935, with Nathan Rosen and Boris Podolsky, Einstein proposed a thought experiment which, according to him, demonstrated that quantum mechanics is not a complete theory of Nature.
- According to EPR an element of reality must be represented in any complete physical theory $\implies$ it must be possible to predict with certainty the value any physical property will have, immediately before measurement.

I recall that during one walk Einstein suddenly stopped, turned to me and asked whether I really believed that the moon exists only when I look at it. The rest of this walk was devoted to a discussion of what a physicist should mean by the term "to exist".

– Abraham Pais

Most physicists did not accept the EPR reasoning as convincing: The attempt to impose on Nature by fiat properties which she must obey seems a most peculiar way of studying her laws. *Indeed, Nature has had the last laugh on EPR.*

In 1964 John Belle formulated a mathematical statement in the form of inequalities which were based on following two assumptions (which the critics of quantum theory wanted to incorporate into the modified version of QM):

Realism & Locality

Realism: A system has well defined values of an observable whether someone measures it or not. Measurement process simply reveals these values to us.
Locality: A measurement made on a system cannot influence other systems instantaneously.

These assumptions are collectively known as the assumptions of *local realism*.

[For details: "*Quantum Computation and Quantum Information*" by M. A. Nielsen & I. L. Chuang]

- A system that can be described by a local realistic theory will satisfy this inequality.
- However, quantum mechanics, and indeed Nature, seems to take delight in violating it ! $\Longrightarrow$ It turns out that Nature experimentally invalidates that point of view, while agreeing with quantum mechanics [A. Aspect et. al. (1981, 1982)].

What can we learn from Bells inequality violation?

The most important lesson is that our deeply held common sense intuitions about how the world works are wrong $\Longrightarrow$ The world is not locally realistic.

Leggett Garg inequality

Leggett Garg inequality

The Leggett-Garg inequality (LGI) was introduced as a means of putting to experimental test a world-view which Leggett and Garg called *Macroscopic Realism*.

Macroscopic Realism is the doctrine that a macroscopic system is always determinately in one or other of the macroscopically distinguishable states available to it, and so is never in a superposition of these states $\Longrightarrow$ No funny-business of quantum superposition is permitted at the macroscopic level.

LGI was derived to allow experimental test of whether or not this doctrine is true.

If the violation of the LGI can be demonstrated on the macroscopic scale, this would challenge the notion of realism even at the macroscopic level.

LGI bears strong formal analogies to Bell-inequalities: In a Bell-inequality one considers measurements occurring on two (or more) systems at spacelike separation, in a LGI, one considers repeated measurements, at different times, of a single observable, on a single system: a timelike, rather than a spacelike separation between measurements.

For this reason, LGIs have often come to be called temporal Bell-inequalities.

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PHYSICAL REVIEW
LETTERS

VOLUME 54 | 4 MARCH 1985 | NUMBER 9

Quantum Mechanics versus Macroscopic Realism: Is the Flux There when Nobody Looks?

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(Received 19 November 1984)

It is shown that, in the context of an idealized "macroscopic quantum coherence" experiment, the predictions of quantum mechanics are incompatible with the conjunction of two general assumptions which are designated "macroscopic realism" and "noninvasive measurability at the macroscopic level." The conditions under which quantum mechanics can be tested against these assumptions in a realistic experiment are discussed.

PACS numbers: 03.65.Bz, 05.30.–d, 74.50.+r, 85.25.+k

Leggett-Garg Inequality

Leggett and Garg begin their discussion by emphasizing that:

"Despite sixty years of schooling in quantum mechanics, most physicists have a very non-quantum-mechanical notion of reality at the macroscopic level, which implicitly makes two assumptions:

Assumptions

- Macroscopic realism (MR): A macroscopic system with two or more macroscopically distinct states available to it will at all times be in one or the other of these states.
- Non-Invasive measurability (NIM): It is possible, in principle, to determine the state of the system with arbitrarily small perturbation on its subsequent dynamics.

A direct extrapolation of quantum mechanics to the macroscopic level denies this."

Goals of LGI tests: To test "realism", the notion that physical systems possess complete sets of definite values for various parameters prior to, and independent of, measurement and *to demonstrate that QM applies on macroscopic scales up to the level at which many-particle systems exhibit decoherence*.

- Dichotomic observable: $Q = \pm 1$

- Two time correlation functions

$$C_{ij} = \frac{1}{N} \sum_{q=1}^{N} \langle Q_i^q Q_j^q \rangle$$

$q \rightarrow$ over an ensemble

  - $-1 \leq C_{ij} \leq 1$
  - $C_{ij} = 1 \implies$ Perfectly correlated
  - $C_{ij} = -1 \implies$ Perfectly anti-correlated
  - $C_{ij} = 0 \implies$ No correlation

- Macrorealism restricts the following combination of two time correlation functions:

$$\begin{aligned} K_3 &= C_{12} + C_{23} - C_{13} \\ &= \langle Q_1 Q_2 \rangle + \langle Q_2 Q_3 \rangle - \langle Q_1 Q_3 \rangle \end{aligned}$$

$$K_3 = \langle Q_1 Q_2 \rangle + \langle \left[Q_2 - Q_1 \right] Q_3 \rangle = \begin{cases} 1 + 0 = 1 \\ -1 + (\pm 2) = 1 \textit{ or } -3 \end{cases}$$

$\Longrightarrow$ $\boxed{-3 \leq K_3 \leq 1}$ $\longleftarrow$ LGI for 3 time measurement scenario

$$-3 \leq C_{12} + C_{23} - C_{13} \leq 1$$

This is the LGI, in one of its standard forms. It will be satisfied if the special conditions of macroscopic realism and noninvasive measurability hold. Leggett and Garg go on to show that it can readily be violated in quantum mechanics.

LGI violations imply that *hidden-variable (or "realistic") alternatives to quantum mechanics cannot adequately describe a system's time evolution.*

Four time measurement:

$$-2 \leq C_{12} + C_{23} + C_{34} - C_{14} \leq 2$$

Quantum correlations in particle physics

Motivation

- The foundations of quantum mechanics are usually studied in optical or electronic systems where the interplay between the various measures of quantum correlations is well known.
- Inspired by the recent technical advances in high energy physics experiments, in particular the neutrino oscillations experiments and B-factories, this quest has now been directed towards mesons and neutrinos.
- The detection efficiency is much higher than that of the corresponding detectors used in optical or electronic systems [Bramon et.al. (2006)].

Quantum correlations in meson systems

[Bramon-Nowakowski (1999)]: Bell-inequalities for Entangled Neutral Kaons were set up.

[Genovese et. al. (2001); Bramonet. al. (2005); Nikitin (2015)]: Several experimental proposals to test Bell-inequalities for entangled mesons were proposed.

[Banerjee-Alok-MacKenzie (2016)]: Interplay between various measures of quantum correlations for the entangled mesons were studied. It was shown that quantum correlations here can be nontrivially different from their stable counterparts.

Quantum correlations in neutrinos

[Alok-Banerjee-Uma Sankar (2016)]: It was shown that various measures of quantum correlations can be expressed in terms of neutrino oscillation probabilities.

[Formaggio et. al. (2016)]: LG-type inequalities were studied in the context of two flavor neutrino oscillations. Using MINOS experimental data, it was shown that neutrino oscillations demonstrate a violation of the classical limits imposed by the LG-type inequality.

[Naikoo-Alok-Banerjee-Uma Sankar-Guarnieri-Hiesmayr (2017)]: LG-type inequalities were studied in the context of three flavor neutrino oscillations. LG-type inequalities were constructed in terms of neutrino transition probabilities. It was shown that these inequalities are sensitive to CP violating phase and sign of Δ_{31}.

Decoherence effects in flavor and neutrino physics

[Farzan-Schwetz-Smirnov (2008)]: It was shown that LSND data could be explained without sterile neutrino if we include decoherence effects.

[Mavromatos et. al. (2008)]: Potential of the CNGS and J-PARC beams in constraining models of quantum-gravity induced decoherence using neutrino oscillations were discussed.

[Alok-Banerjee-Uma Sankar (2015)]: Effect of decoherence on important quantities of the B_d system, such as $\sin 2\beta$ and ΔM_d were studied. It was shown that the values of these two quantities are modulated by the decoherence parameter. An upper bound on this parameter was obtained using Belle data.

Leggett-Garg Inequality for Neutrinos

- Let the initial state of neutrino be prepared in a specific flavor $|\nu_{initial}\rangle = |\nu_\alpha\rangle$ $(\alpha = e/\mu/\tau)$.
- We use the dichotomic observable $\hat{Q} = 2\,|\nu_\alpha\rangle\,\langle\nu_\alpha| - \mathbb{1} \implies Q = \pm 1$ i.e., we ask whether the neutrino is still in the state $|\nu_\alpha\rangle$ (Q = 1) or has undergone a transition to another flavor state $|\nu_\beta\rangle$ (Q = -1).
- We then develop the two time correlation function $C_{ij} = \langle \hat{Q}(t_i)\hat{Q}(t_j)\rangle$,

$$C_{0t} = 4\delta_{\alpha\beta}\langle\nu_\alpha(t)|\nu_\beta\rangle\langle\nu_\beta|\nu_\alpha(t)\rangle - 2\langle\nu_\alpha(t)|\nu_\beta\rangle\langle\nu_\beta|\nu_\alpha(t)\rangle - 2\delta_{\alpha\beta} + 1.$$

- The LG parameter is calculated to be (for initial state $|\nu_\mu\rangle$)

$$K_3 = 1 - 4\mathcal{P}_{\mu\to e}(t) + 4\alpha'(t)\mathcal{P}_{\mu\to e}(2t) + 4\beta'(t)$$

$\alpha'(t)$ *and* $\beta'(t)$ $\leftarrow$ non measurable quantities

LGIs cannot be expressed in terms of the experimentally measurable neutrino oscillation probabilities.

LG-type Inequality: Three flavor neutrino oscillations

In order to bypass this difficulty and obtain experimentally testable inequality, the assumption of NIM could be replaced by a weaker condition of *Stationarity* [Huelga et. al. (1995)].

- With this assumption, $C(t_i, t_j)$ only depends on the time difference $t_j - t_i$, this leading to the following simplification, known as *Leggett-Garg-type Inequalities*:

  $$K_3|_{stat} = 2C(0,t) - C(0,2t) \leq 1.$$

  Here we have assumed that $t_1 = 0$ and $t_2 - t_1 = t_3 - t_2 = t$.

- The LG parameter for an initial ν_μ becomes:

  $$\boxed{K_3 = 2\mathcal{P}_{\mu\to e}(2t) - 4\mathcal{P}_{\mu\to e}(t) + 1}$$

  Using ultrarelativistic approximation $t \approx L$

  $$\boxed{K_3 = 2\mathcal{P}_{\mu\to e}(2L) - 4\mathcal{P}_{\mu\to e}(L) + 1}$$

LG-type inequalities can be expressed in terms of the experimentally measurable neutrino oscillation probabilities.

LG-type Inequality: Three flavor neutrino oscillations

- A problem: Need two detectors placed at L and 2L, respectively. Not possible with the present day facilities.
- Recipe: Eliminate the 2L dependence by searching for new energies such that the following holds [Formaggio et. al. (2016)]:

$$\mathcal{P}_{\mu\to e}(2L, E) = \mathcal{P}_{\mu\to e}(L, \tilde{E})$$

- Finally,

$$\boxed{K_3 = 2\mathcal{P}_{\mu\to e}(\tilde{E}) - 4\mathcal{P}_{\mu\to e}(E) + 1}$$

1. NOνA: $L = 810\,\mathrm{km},\ E \sim 1 - 7$ GeV.
2. T2K: $L = 295$ km, $E \sim 1 - 2$ GeV.

Nova: Maximum flux at 4.7 Gev. T2K: Maximum flux at 1.6 Gev.

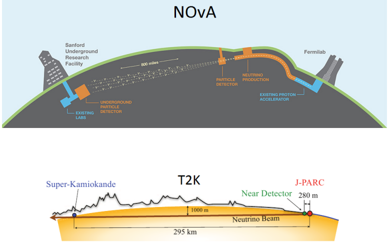


Figure: NOvA and T2K experiments

Both these experiments use ν_μ source

LG-type Inequality: Three flavor neutrino oscillations

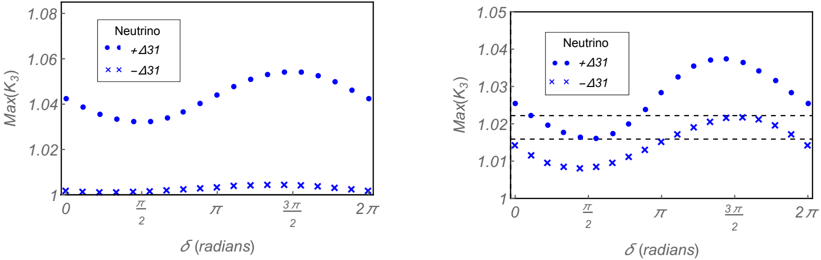


Figure: Maximum of LG function K_3 plotted against CP violating phase δ. (Left panel: NOνA; Right panel: T2K.)

Advantages of LG-type Inequalities

It allows for measurements made on distinct ensemble members to mimic a series of measurements made on a single time-evolving system $\implies$ This bypasses the recent criticism of the LGI whereby measurements on a single system at later times may be influenced by the outcomes of earlier measurements on that same system [Clemente-Kofler (2016)].

Limitations of LG-type Inequalities

LG-type inequalities can test only limited class of realistic models.

It can test class of realistic models that are Markovian, for which the evolution of the system after some time t is independent of the means by which the system arrived in a given state at t [Emary-Lambert-Nori (2014)].

Decoherence effects in B meson systems

Motivation

- The time evolution of neutral mesons are used to measure a number of important parameters in flavor physics.
- In the time evolution of neutral meson systems, a perfect quantum coherence is usually assumed. (i.e. the interaction between the system and the environment is neglected. By decoherence we mean interaction between environment and system.)
- However, any real system interacts with its environment and this interaction can lead to a loss of quantum coherence.
- Hence with the inclusion of decoherence effects, the measured values of some of the parameters can get masked.

We study the effect of decoherence on the important observables in the B_d^0 meson system, such as the CP violating parameter $\sin 2\beta$ and the $B_d^0 - \bar{B}_d^0$ mixing parameter Δm_d.

Decoherence is an unavoidable phenomenon as any physical system is inherently open due to its inescapable interactions with a pervasive environment.

Possible environment

- Environmental effects may arise at a fundamental level, such as the fluctuations in a quantum gravity space-time background [S.W. Hawking (1982); J. R. Ellis et. al. (1984); Huet-Peskin (1995)].
- They may also arise due to the detector environment itself.

- The effect of environment on the neutral meson systems can be taken into account by using the ideas of open quantum systems.
- We use an effective description which is phenomenological in nature. It is independent of the details of the actual dynamics between the system and environment.

- We are interested in the decays of B^0 and $\bar{B}^0$ mesons as well as $B^0 \leftrightarrow \bar{B}^0$ oscillations.
- To describe the time evolution of all these transitions, we need a basis of three states: $|B^0\rangle$, $|\bar{B}^0\rangle$ and $|0\rangle$, where $|0\rangle$ represents a state with no B meson and is required for describing the decays.
- We use the density matrix formalism to represent the time evolution of the B^0 system: $\rho_{B^0(\bar{B}^0)}(0)$ is the initial density matrix for the state which starts out as $B^0(\bar{B}^0)$.
- The time evolution of these matrices is governed by the Kraus operators $K_i(t)$ as $\rho(t) = \sum_i K_i(t)\rho(0)K_i^\dagger(t)$.

The Kraus operators are constructed taking into account the decoherence in the system which occurs due to the evolution under the influence of the environment.

Open time evolution of B mesons

Time dependent density matrices

$$\frac{\rho_{B^0}(t)}{\frac{1}{2}e^{-\Gamma t}} = \begin{pmatrix} a_{ch} + e^{-\lambda t} a_c & -a_{sh} - ie^{-\lambda t} a_s & 0 \\ -a_{sh} + ie^{-\lambda t} a_s & a_{ch} - e^{-\lambda t} a_c & 0 \\ 0 & 0 & 2(e^{\Gamma t} - a_{ch}) \end{pmatrix}$$

$$\frac{\rho_{\bar{B}^0}(t)}{\frac{1}{2}e^{-\Gamma t}} = \begin{pmatrix} a_{ch} - e^{-\lambda t} a_c & -a_{sh} + ie^{-\lambda t} a_s & 0 \\ -a_{sh} - ie^{-\lambda t} a_s & a_{ch} + e^{-\lambda t} a_c & 0 \\ 0 & 0 & 2(e^{\Gamma t} - a_{ch}) \end{pmatrix}$$

- $a_{ch} = \cosh\left(\frac{\Delta\Gamma\, t}{2}\right)$, $a_{sh} = \sinh\left(\frac{\Delta\Gamma\, t}{2}\right)$, $a_c = \cos(\Delta m\, t)$, $a_s = \sin(\Delta m\, t)$.
- $\Gamma = (\Gamma_L + \Gamma_H)/2$, $\Delta\Gamma = \Gamma_L - \Gamma_H$: Γ_L and Γ_H are the respective decay widths of the decay eigenstates B^0_L and B^0_H.
- λ is the decoherence parameter, due to the interaction between one-particle system and its environment.
- To keep expressions simple, CP violation in mixing is neglected.

Hermitian operator describing decay $B^0 \to f$ and $\bar{B}^0 \to f$

$$\mathcal{O}_f = \begin{pmatrix} |A_f|^2 & {A_f}^* \bar{A}_f & 0 \\ A_f \bar{A}_f^* & |\bar{A}_f|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- $A_f \equiv A(B^0 \to f)$ and $\bar{A}_f \equiv A(\bar{B}^0 \to f)$.
- The probability, $P_f(B^0/\bar{B}^0; t)$, of an initial $B^0/\bar{B}^0$ decaying into the state f at time t is given by $\mathrm{Tr}\left[\mathcal{O}_f \, \rho_{B^0(\bar{B}^0)}(t)\right]$.

CP asymmetry of $B_d^0 \to J/\psi K_S$ decay

$$\mathcal{A}_{J/\psi K_S}(t) = \frac{P_{J/\psi K_S}(\bar{B}_d^0; t) - P_{J/\psi K_S}(B_d^0; t)}{P_{J/\psi K_S}(\bar{B}_d^0; t) + P_{J/\psi K_S}(B_d^0; t)}$$

CP asymmetry in $B_d^0 \to J/\psi K_S$

$$\mathcal{A}_{J/\psi K_S}(t) = \frac{\left(|\lambda_f|^2 - 1\right)\cos\left(\Delta m_d t\right) + 2\mathrm{Im}(\lambda_f)\sin\left(\Delta m_d t\right)}{\left(1 + |\lambda_f|^2\right)\cosh\left(\frac{\Delta\Gamma_d t}{2}\right) - 2\mathrm{Re}(\lambda_f)\sinh\left(\frac{\Delta\Gamma_d t}{2}\right)} e^{-\lambda t}$$

- $\lambda_f = A(\bar{B}_d^0 \to J/\psi K_S))/A(B_d^0 \to J/\psi K_S)$.
- By putting $\lambda = 0$, the usual expression for CP asymmetry in the interference of mixing and decay is obtained.

In extracting $\sin 2\beta$ from $\mathcal{A}_{J/\psi K_S}(t)$ it is usually assumed that $\Delta\Gamma_d \approx 0$, $|\lambda_f| \approx 1$ and $\mathrm{Im}(\lambda_f) \approx \sin 2\beta$.

$$\mathcal{A}_{J/\psi K_S}(t) \approx \sin 2\beta\, e^{-\lambda t} \sin\left(\Delta m_d\, t\right)$$

The coefficient of $\sin\left(\Delta m_d\, t\right)$ is $\sin 2\beta\, e^{-\lambda t}$ and not $\sin 2\beta$!

The measurement of $\sin 2\beta$ is masked by the presence of decoherence.

In order to determine $\sin 2\beta$, we need to know Δm_d.

Is the measurement of Δm_d also affected by the presence of decoherence?

LHCb, CDF and D0 experiments determine Δm_d by measuring rates that a state that is pure B_d^0 at time $t = 0$, decays as either as B_d^0 or $\bar{B}_d^0$ as function of proper decay time.

In the presence of decoherence, the survival (oscillation) probability of initial B_d^0 meson to decay as $B_d^0(\bar{B}_d^0)$ at a proper decay time t is:

B_d^0 survival (oscillation) probability

$$P_{\pm}(t,\lambda) = \frac{e^{-\Gamma t}}{2}\left[\cosh(\Delta\Gamma_d t/2) \pm e^{-\lambda t}\cos(\Delta m_d t)\right]$$

The positive (negative) sign *implies* B_d^0 meson decaying with the same (opposite) flavor as its production.

Determination of Δm_d

Time dependent mixing asymmetry: Used to determine Δm_d

$$A_{\rm mix}(t,\lambda) = \frac{P_+(t,\lambda) - P_-(t,\lambda)}{P_+(t,\lambda) + P_-(t,\lambda)} = e^{-\lambda t}\frac{\cos(\Delta m_d t)}{\cosh(\Delta\Gamma_d t/2)}$$

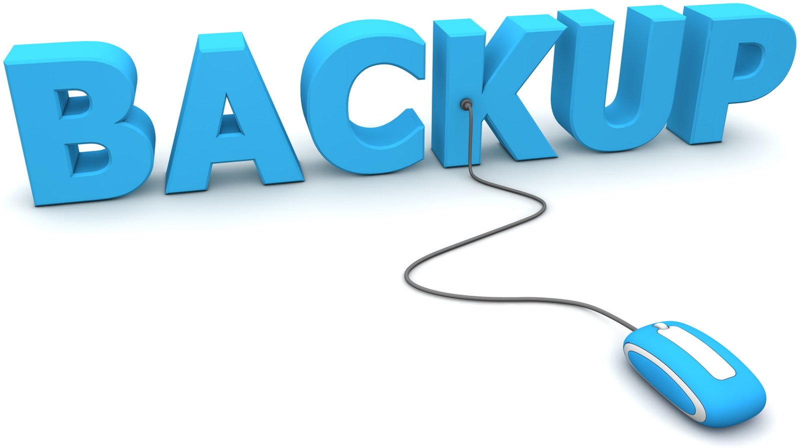
Neglecting $\Delta\Gamma_d$, the otherwise pure cosine dependence is modulated by $e^{-\lambda t}$.

Determination of of Δm_d at Belle and BaBar

- Δm_d is determined by measuring time dependent mixing probability for entangled B_d^0 mesons produced at $\Upsilon(4S)$ resonance.
- The expressions for $P_\pm(t)$ are the same except that the proper time t is replaced by proper decay-time difference Δt between the decays of the two neutral B_d mesons.

The true value of Δm_d, along with $\Delta\Gamma_d$, can be determined by a three parameter $(\Delta m_d, \Delta\Gamma_d, \lambda)$ fit to the time dependent mixing asymmetry $A_{\rm mix}(t,\lambda)$. This in turn will enable a determination of true value of $\sin 2\beta$.


Thank You


BACKUP

Quantum correlations in neutral meson systems

- B factories, electron-positron colliders tailor-made to study the production and decay of B mesons, and ϕ factories, which perform the same function for K mesons, provide an ideal testing ground.
- For the B system, the decay $\Upsilon \to b\bar{b}$ is followed by hadronization into a $B\bar{B}$ pair.
- In the Υ rest frame, the mesons fly off in opposite directions (left and right, say).
- The same considerations apply to the K system, with the Υ replaced by a ϕ meson.
- An important feature of these systems for the study of correlations is the oscillations of the bottom and strangeness flavors, giving rise to $M\bar{M}$ oscillations.
- Another feature about these systems is that they are decaying. Thus one needs to study quantum correlations in unstable, decaying $B\bar{B}$ and $K\bar{K}$ systems.

- A number of well-established measures of quantum correlations, such as Bell's inequality violations teleportation fidelity, concurrence and geometric discord, in the correlated $B\bar{B}$ and $K\bar{K}$ systems were studied in [Banerjee-Alok-MacKenzie (2015)].
- The flavor-space wave function of the correlated $M\bar{M}$ meson systems ($M = K,\ B_d,\ B_s$) at the initial time $t = 0$ is

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left[|M\bar{M}\rangle - |\bar{M}M\rangle\right],$$

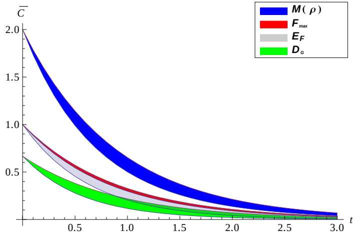
  where the first (second) particle in each ket is the one flying off in the left (right) direction and $|M\rangle$ and $|\bar{M}\rangle$ are flavor eigenstates.
- Thus, the initial state of the neutral meson system is a singlet (maximally entangled) state.

- The state of the two-particle decaying system at time t is given by

$$\rho(t) = \frac{1}{4}\begin{pmatrix} a_- & 0 & 0 & -a_- \\ 0 & a_+ & -a_+ & 0 \\ 0 & -a_+ & a_+ & 0 \\ -a_- & 0 & 0 & a_- \end{pmatrix},$$

  where $a_\pm = 1 \pm e^{-2\lambda t}$.

- The density matrix depends on only one parameter (in addition to time), the decoherence parameter λ, which describes the interaction of the mesons with the environment.
- To take into account the effect of decay in the systems under study, the various correlations are modified by the probability of survival of the pair of particles up to that time, which can be shown to be $e^{-2\Gamma t}$, where Γ is the meson decay width.



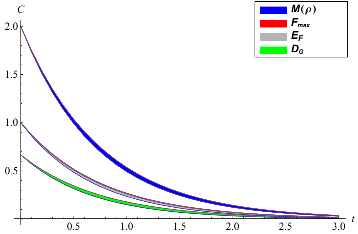


Figure: Average correlation measures, as a function of time t. The left and right panels correspond to the correlations of a $K\bar{K}$ and $B_d\bar{B}_d$ pair created at $t = 0$, respectively. For $K\bar{K}$ pairs, left panel, time is in units of 10^{-10} seconds whereas for the $B_d\bar{B}_d$ pair, time is in units of 10^{-12} seconds (in all cases, the approximate lifetime of the particles). The bands represent the effect of decoherence corresponding to a 3σ upper bound on the decoherence parameter λ.

- On average, Bell's inequality in these correlated-meson systems is violated for about half of the meson lifetime.
- We find that the quantum correlations here can be *nontrivially different* from their stable counterparts. This is made explicit by the interplay between Bell's inequality violation and teleportation fidelity.
- One *particularly surprising result* is that teleportation fidelity does not exceed the classical threshold of 2/3 for all Bell's inequality violations.
- This behavior, *not seen in stable systems*, is interesting since one of the cornerstones in the field of quantum information is the interplay between Bell's inequality violation and teleportation fidelity.

More about Quantum Correlations

Bell's inequality

Given a pair of qubits in the state ρ, the elements of correlation matrix T are $T_{mn} = Tr\left[\rho(\sigma_m \otimes \sigma_n)\right]$. If u_i $(i = 1, 2, 3)$ are the eigenvalues of the matrix $T^\dagger T$ then the Bell-CHSH inequality can be written $M(\rho) < 1$ [Horodecki (1995,1996)], where $M(\rho) = \max(u_i + u_j)$ $(i \neq j)$.

Teleportation Fidelity

- Teleportation provides an operational meaning to entanglement, whenever $F_{\max} > 2/3$, teleportation is possible.
- $F_{\max}$ is computed in terms of the eigenvalues $\{u_i\}$ of $T^\dagger T$.
- $F_{\max} = \frac{1}{2}\left(1 + \frac{1}{3}N(\rho)\right)$ where $N(\rho) = \sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3}$ [Horodecki et. al. (1996)].

An inequality involving $M(\rho)$ and $F_{\max}$

$$F_{max} \geq \frac{1}{2}\left(1 + \frac{1}{3}M(\rho)\right) \geq \frac{2}{3} \text{ if } M(\rho) > 1.$$

Concurrence

For a mixed state ρ of two qubits, the concurrence, which is a measure of entanglement, is $C = \max(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0)$. λ_i are the square root of the eigenvalues, in decreasing order, of the matrix $\rho^{\frac{1}{2}}(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)\rho^{\frac{1}{2}}$ where ρ is computed in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.
For a two-qubit system, concurrence is equivalent to the entanglement of formation which can then be expressed as a monotonic function of concurrence C as $E_F = -\frac{1+\sqrt{1-C^2}}{2}\log_2(\frac{1+\sqrt{1-C^2}}{2}) - \frac{1-\sqrt{1-C^2}}{2}\log_2(\frac{1-\sqrt{1-C^2}}{2})$.

Geometric discord

For the case of two qubits, geometric discord is
$D_G(\rho) = \frac{1}{3}[\|\vec{x}\|^2 + \|T\|^2 - \lambda_{max}(\vec{x}\vec{x}^\dagger + TT^\dagger)]$ where T is the correlation matrix, $\vec{x}$ is the vector whose components are $x_m = \mathrm{Tr}(\rho(\sigma_m \otimes \mathbb{I}_2))$, and $\lambda_{max}(K)$ is the maximum eigenvalue of the matrix K.

Quantum correlations in neutral meson systems: More details

$M\bar{M}$ as an open quantum system

The flavor-space wave function of the correlated $M\bar{M}$ meson systems ($M = K,\ B_d,\ B_s$) at the initial time $t = 0$ is

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left[|M\bar{M}\rangle - |\bar{M}M\rangle\right],$$

where the first (second) particle in each ket is the one flying off in the left (right) direction and $|M\rangle$ and $|\bar{M}\rangle$ are flavor eigenstates. This initial state of the neutral meson system is a singlet (maximally entangled) state.

The usual analysis of such systems is done using a trace-decreasing density matrix. However, such an approach may not be very useful for calculating quantum correlations as the usual methods for computing quantum correlations require a trace-preserving, completely positive description of the system.

The semigroup formalism enables the calculation of a trace-preserving density matrix.

The Hilbert space of a system of two correlated neutral mesons is

$$\mathcal{H} = (\mathcal{H}_L \oplus \mathcal{H}_0) \otimes (\mathcal{H}_R \oplus \mathcal{H}_0).$$

$\mathcal{H}_{L,R}$ are the Hilbert spaces of the left-moving and right-moving decay products, each of which can be either a meson or an anti-meson, and $\mathcal{H}_0$ is that of the zero-particle (vacuum) state.

Thus the total Hilbert space can be seen to be the tensor sum of a two-particle space, two one-particle spaces, and one zero-particle state.

The initial density matrix of the full system is

$$\rho_{\mathcal{H}}(0) = |\psi(0)\rangle \langle\psi(0)| \,.$$

The system, initially in the two-particle subspace, evolves in time into the full Hilbert space, eventually (after the decay of both particles) finding itself in the vacuum state.

Kraus representation

Kraus representation, describes the time evolution of an *open* quantum system, which is not necessarily unitary unlike the evolution of a *closed* quantum system. Real physical systems are always entangled with their ambient environment, alternatingly addressed as the reservoir. Kraus representations are very convenient for handling a number of practical problem of open system dynamics.

Consider a large system S comprising of two subsystems S_a and S_b. At a given time t, let the quantum states corresponding to S, S_a and S_b be represented by $\rho(t)$, $\rho_a(t)$ and $\rho_b(t)$, respectively. Then $\rho_a(t) = Tr_b\{\rho(t)\}$ and $\rho_b(t) = Tr_a\{\rho(t)\}$. Since the total system is unitary, its evolution is given by

$$\rho(t) = U(t)\rho(0)U^\dagger(t),$$

where $U(t)$ is a unitary operator.The evolution of system S_a will look like

$$\rho_a(t) = Tr_b\{U(t)\rho(0)U^\dagger(t)\}.$$

If it is possible to recast above equation in the following form

$$\rho_a(t) = \sum_i E_i(t)\rho_a(0)E_i^\dagger(t),$$

such that $\sum_i E_i(t)E_i^\dagger(t) = \mathbb{1}$, then the evolution of $\rho_a(t)$ has a *Kraus* representation and is completely positive.

The Kraus operators $K_i(t)$ encode the information about the ambient environment of the system of interest and include the decoherence parameter λ, which describes the interaction of the mesons with the environment. These operators are determined using the condition of complete positivity along with trace preservation.

$M\bar{M}$ as an open quantum system

The time evolution of the initial state is described by the following density matrix:

$$\rho_{\mathcal{H}}(t) = \sum_{i,j} K_{ij}(t)\rho_{\mathcal{H}}(0)K_{ij}^{\dagger}(t),$$

where $K_{ij}(t) = K_i(t) \otimes K_j(t)$.

From basic notions of quantum correlations such as entanglement, one needs to have two parties to correlate. For this we need to project from the full Hilbert space $\mathcal{H}$ down to the two-particle sector $\mathcal{H}_L \otimes \mathcal{H}_R$. This can be achieved by using the projector $\mathcal{P}_2$, the projector on to the two-particle sector $\mathcal{H}_L \otimes \mathcal{H}_R$.

The result is

$$\rho(t) = \frac{\mathcal{P}_2\, \rho_{\mathcal{H}}(t)\, \mathcal{P}_2}{\mathrm{Tr}(\mathcal{P}_2\, \rho_{\mathcal{H}}(t))} = \frac{1}{4}\begin{pmatrix} a_- & 0 & 0 & -a_- \\ 0 & a_+ & -a_+ & 0 \\ 0 & -a_+ & a_+ & 0 \\ -a_- & 0 & 0 & a_- \end{pmatrix},$$

where $a_{\pm} = 1 \pm e^{-2\lambda t}$.

$\implies$ $\rho(t)$ is trace-preserving.

Non-locality

$$M(\rho) = (1 + e^{-4\lambda t}).$$

Concurrence

$$C = e^{-2\lambda t}.$$

- Entanglement of formation is

$$E_F = -\frac{1+\sqrt{1-C^2}}{2}\log_2(\frac{1+\sqrt{1-C^2}}{2}) - \frac{1-\sqrt{1-C^2}}{2}\log_2(\frac{1-\sqrt{1-C^2}}{2}).$$

Geometric discord

$$D_G(\rho) = M(\rho)/3.$$

Teleportation fidelity

$$F_{\max} = \frac{1}{12}\left[6 + 2e^{-2\lambda t} + \sqrt{2}\sqrt{\alpha - \sqrt{\beta}} + \sqrt{2}\sqrt{\alpha + \sqrt{\beta}}\right],$$

where

$$\alpha = 1 + \cosh(4\lambda t) - \sinh(4\lambda t), \quad \beta = 3 - 2\alpha + \cosh(8\lambda t) - \sinh(8\lambda t).$$

- For the K meson, $\Gamma = \frac{1}{2}(\Gamma_S + \Gamma_L)$ (where Γ_S and Γ_L are the decay widths of short and long neutral kaon states, respectively); its value is $5.59 \times 10^9\,\mathrm{s}^{-1}$.
- The decay widths for B_d and B_s mesons are $6.58 \times 10^{11}\,\mathrm{s}^{-1}$ and $6.61 \times 10^{11}\,\mathrm{s}^{-1}$, respectively.
- In the case of the K meson system, the value of λ has been obtained by the KLOE collaboration by studying the interference between the initially entangled kaons and the decay product in the channel $\phi \to K_S K_L \to \pi^+\pi^-\pi^+\pi^-$ [F. Ambrosino et al. (KLOE Collaboration 2006)]. The value of λ is at most $1.58 \times 10^9\,\mathrm{s}^{-1}$ at 3σ.
- In the case of B_d meson systems, 3σ upper limit for λ is $0.45 \times 10^{11}\,\mathrm{s}^{-1}$ [Alok-Banerjee-Uma Sankar (2015)].
- For B_s mesons, to the best of our knowledge, there is no experimental information about λ so we will take it to be zero.

Quantum correlations in neutrinos

- In nature, neutrinos are available in three flavors.
- Owing to their non-zero mass, they oscillate from one flavor to another which has been confirmed by a plethora of experiments, using both natural and "man-made" neutrinos.
- Neutrino oscillations are fundamentally three flavor oscillations. However, in some cases, it can be reduced to effective two flavor oscillations.

Motivation

- Neutrino system is particularly interesting as the effect of decoherence as compared to other particles widely used in quantum information processing, is minimal.
- Also, the detection efficiency is much higher than that of the corresponding detectors used in optical or electronic systems [Bramon et.al. (2006)].
- Thus neutrino system has the potential to provide an alternative platform for testing foundations of quantum mechanics.

Quantum correlations in Neutrinos

- The coherent time evolution of neutrino flavor eigenstates implies that there is a linear superposition between the mass eigenstates which make up a flavour state.
- Thus neutrino oscillations are related to the multi-mode entanglement of single-particle states which can be expressed in terms of flavor transition probabilities.
- Hence neutrino is an interesting candidate for study of quantum correlations.

Definition of the problem

We are interested in studying various facets of quantum correlations in neutrinos. In particular, we intend to study:

- The interplay between various aspects of quantum correlations such as non-locality, entanglement and weaker measures such as discord.
- To explore relation between neutrino mixing and coherences in the system.

The three flavour states (eigenstates of weak interaction, which are detectable in lab) of neutrinos, ν_e, ν_μ and ν_τ mix via a 3×3 unitary matrix to form the three mass eigenstates (which are the propagation eigenstates) ν_1, ν_2 and ν_3.

Neutrino oscillations occur only if the three corresponding masses, m_1, m_2 and m_3, are non-degenerate.

Of the three mass-squared differences $\Delta_{ij} = m_i^2 - m_j^2$ (where $i, j = 1, 2, 3$ with $i > j$), only two are independent. Oscillation data tells us that $\Delta_{21} \approx 0.03 \times \Delta_{32}$, hence $\Delta_{31} \approx \Delta_{32}$.

In considering neutrino oscillations, in general, one should use the full three flavour oscillation formulae.

However, in a number of cases, the three flavour formula reduces to an effective two flavour formula, if one or both of the small parameters, Δ_{21}/Δ_{32} and θ_{13}, are set equal to zero.

For example, in long baseline accelerator experiments, both the above parameters can be neglected in doing leading order calculations. Then the problem reduces to that of two flavour mixing of ν_μ and ν_τ to form two mass eigenstates ν_2 and ν_3. The corresponding oscillations are described by one mixing angle θ ($\equiv \theta_{23}$ in three flavour mixing) and one mass-squared difference Δ ($\equiv \Delta_{32}$ in three flavour mixing).

In the case of two flavour mixing, the relation between the flavour and the mass eigenstates is described by a 2×2 rotation matrix, $U(\theta)$,

$$\begin{pmatrix} \nu_\alpha \\ \nu_\beta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_j \\ \nu_k \end{pmatrix} .$$

Each flavour state is given by a superposition of mass eigenstates,

$$|\nu_\alpha\rangle = \sum_j U_{\alpha j} |\nu_j\rangle \ ,$$

where $\alpha = \mu$ or τ and $j = 2, 3$.
The time evolution of the mass eigenstates $|\nu_j\rangle$ is given by

$$|\nu_j(t)\rangle = e^{-iE_j t} |\nu_j\rangle \ ,$$

where $|\nu_j\rangle$ are the mass states at time $t = 0$.
Thus, we can write

$$|\nu_\alpha(t)\rangle = \sum_j e^{-iE_j t} U_{\alpha j} |\nu_j\rangle \ .$$

The evolving flavour neutrino state $|\nu_\alpha\rangle$ can also be projected on to the flavour basis in the form

$$|\nu_\alpha(t)\rangle = \tilde{U}_{\alpha\alpha}(t) |\nu_\alpha\rangle + \tilde{U}_{\alpha\beta}(t) |\nu_\beta\rangle \ ,$$

where $|\nu_\alpha\rangle$ is the flavour state at time $t = 0$ and $|\tilde{U}_{\alpha\alpha}(t)|^2 + |\tilde{U}_{\alpha\beta}(t)|^2 = 1$.

We can thus establish the following correspondence, using the occupation number of neutrinos, with two-qubit states [Blasone et al. (2008, 2013, 2015)]

$$|\nu_\alpha\rangle \equiv |1\rangle_\alpha \otimes |0\rangle_\beta \equiv |10\rangle\,, \quad |\nu_\beta\rangle \equiv |0\rangle_\alpha \otimes |1\rangle_\beta \equiv |01\rangle\;.$$

The time evolution of flavor eigenstate can then be written as

$$|\nu_\alpha(t)\rangle = \tilde{U}_{\alpha\alpha}(t)\,|1\rangle_\alpha \otimes |0\rangle_\beta + \tilde{U}_{\alpha\beta}(t)\,|0\rangle_\alpha \otimes |1\rangle_\beta\;,$$

where,

$$\begin{aligned}\tilde{U}_{\alpha\alpha}(t) &= \cos^2\theta e^{-iE_2 t} + \sin^2\theta e^{-iE_3 t}\;,\\ \tilde{U}_{\alpha\beta}(t) &= \sin\theta\cos\theta(e^{-iE_3 t} - e^{-iE_2 t})\;.\end{aligned}$$

$\Longrightarrow$ The state $|\nu_\alpha(t)\rangle$ has the form of a mode entangled single particle state.

Various measures of quantum correlations can now be determined using the density matrix $\rho_\alpha(t) = |\nu_\alpha(t)\rangle\langle\nu_\alpha(t)|$ as the parameters of the density matrix, mixing angle and mass squared difference, are known [Alok-Banerjee-Uma Sankar (2014); Banerjee-Alok-Srikanth-Heismeyr (2015)].

Non-locality

$$M(\rho) = 1 + \left[3 + \cos 4\theta + 2\cos\phi \sin^2 2\theta\right] \sin^2 2\theta \sin^2 (\phi/2)\,.$$

- $\phi = \frac{\Delta t}{2E}$
- $M(\rho)$ is tied up with neutrino mixing.
- In case of no mixing ($\theta = 0$), $M(\rho) = 1$.

Concurrence

$$C = 2\sqrt{\sin^4\theta + \cos^4\theta + 2\cos^2\theta \sin^2\theta \cos\phi)}\, \sin 2\theta \sin(\phi/2)\,.$$

- In case of no mixing, there is no entanglement.

Geometric discord

$$D_G(\rho) = \frac{2}{3}\sin^2 2\theta \sin^2(\phi/2)\left[3 + \cos 4\theta + 2\cos\phi \sin^2 2\theta\right].$$

- $D_G(\rho)$ for $\theta = 0$ is 0, a classically allowed value of geometric discord.

Teleportation fidelity

$$F_{\max} = \frac{2}{3} + \frac{1}{3}\sin 2\theta \sin(\phi/2)\sqrt{3 + \cos 4\theta + 2\sin^2 2\theta \cos\phi}.$$

- In the absence of mixing, $F_{\max} = 2/3$, the classical value of teleportation fidelity.

Quantum correlations in neutrinos

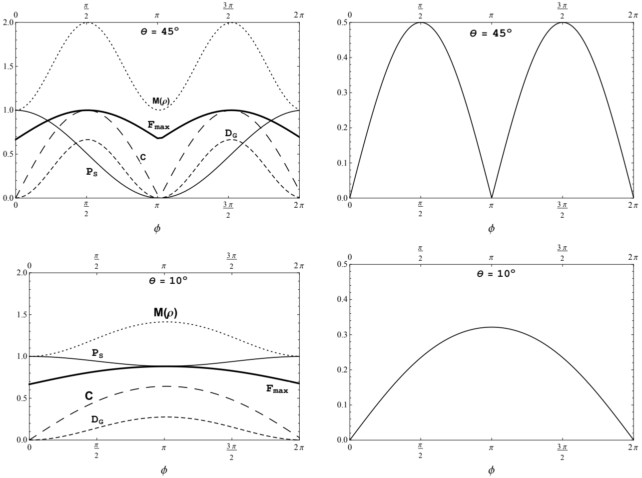


Figure: The left panel of the top and bottom figure depicts various measures of quantum correlations with respect to phase ϕ ($\equiv \Delta t/2E$) for the mixing angle $\theta = 45^\circ$ and $\theta = 10^\circ$, respectively. The thin solid line in the left panel of the top and bottom figure, P_S, represents the neutrino survival probability. The right panel of the figure depicts the magnitude of the off-diagonal elements of the density matrix.

Quantum correlations in neutrinos

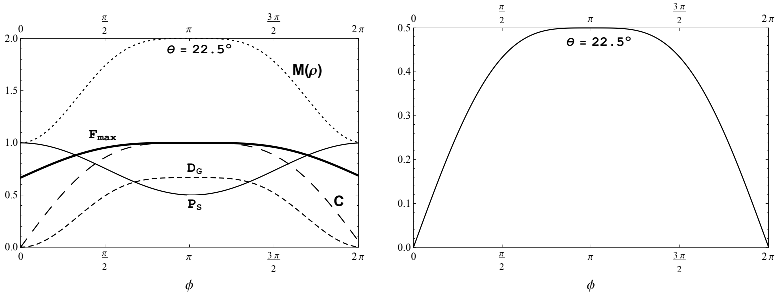
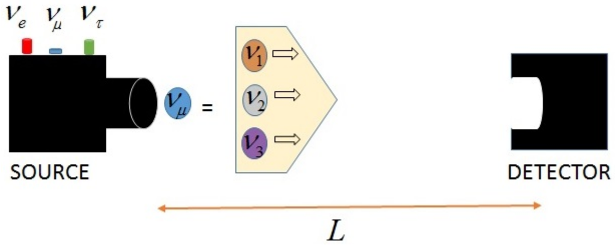


Figure: The left panel of the figure depicts various quantum correlations with respect to phase ϕ for the critical mixing angle $\theta = 22.5^\circ$. The thin solid line in the left panel, P_S, represents the neutrino survival probability. The right panel of the figure depicts the magnitude of the off-diagonal elements of the density matrix.

- Bell's inequality is always violated and hence the evolution of neutrinos is highly non local in nature.
- Teleportation fidelity is always greater than $2/3$ thus obeying the usual relation between Bell's inequality violation and teleportation fidelity, as seen in electronic and photonic systems.
- It is quite remarkable that the measurement of neutrino oscillations due to a non zero value of the mixing angle implies quantum correlations.
- The quantum correlations are seen to be very closely tied to the neutrino mixing angle.
- There exists a critical value of the mixing angle $\pi/8$, for which the Bell's inequality violation is maximal over a broad range of the kinematic variable ϕ.
- Also, it is interesting to note that the off diagonal order, introduced here, is gaining prominence in a number of recent studies related to quantum coherence [Girolami (2014), Bromely et.al. (2015), U. Singh et. al. (2015)].

Neutrino Physics



$\boxed{|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle}$ $\Longrightarrow$ Massive states. In plane wave approximation, these states move with the same momentum. These are the eigenstates of the Hamiltonian

$$\mathcal{H}|\nu_k\rangle = E_k|\nu_k\rangle \qquad E_k = \sqrt{\vec{p}^2 + m_k^2}$$

$$i\frac{d|\nu_k\rangle}{dt} = \mathcal{H}|\nu_k\rangle \qquad \text{Schrodinger equation}$$

$$\Longrightarrow \qquad |\nu_k(t)\rangle = e^{iE_k t}|\nu_k(0)\rangle$$

- Massive states $\rightarrow \{|\nu_1\rangle\,,\;\; |\nu_2\rangle\,,\;\; |\nu_3\rangle\}$;
  Flavor states $\rightarrow \{|\nu_e\rangle\,,\;\; |\nu_\mu\rangle\,,\;\; |\nu_\tau\rangle\}$
- The general state of a neutrino can be expressed in flavor basis as:

$$\Psi(t) = \nu_e(t)\,|\nu_e\rangle + \nu_\mu(t)\,|\nu_\mu\rangle + \nu_\tau(t)\,|\nu_\tau\rangle$$

- Same state in propagation basis looks like:

$$\Psi(t) = \nu_1(t)\,|\nu_1\rangle + \nu_2(t)\,|\nu_2\rangle + \nu_3(t)\,|\nu_3\rangle$$

- The coefficients in two representations are connected by a *unitary* matrix called Pontecorvo Maki Nakagawa Sakata matrix (PMNS matrix)

$$\begin{pmatrix} \nu_e(t) \\ \nu_\mu(t) \\ \nu_\tau(t) \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1(t) \\ \nu_2(t) \\ \nu_3(t) \end{pmatrix}.$$

or,

$$\boldsymbol{\nu}_\alpha(t) = \mathbf{U}\boldsymbol{\nu}_i(t). \tag{1}$$

- A convenient parametrization for $U(\theta_{12}, \theta_{23}, \theta_{13}, \delta)$ is given by

$$U(\theta_{12}, \theta_{23}, \theta_{13}, \delta) = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{23}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{13}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

- where $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$, θ_{ij} being the mixing angles and δ the CP violating phase.
- The mass eigenstates evolve as

$$\begin{pmatrix} \nu_1(t) \\ \nu_2(t) \\ \nu_3(t) \end{pmatrix} = \begin{pmatrix} e^{-iE_1 t} & 0 & 0 \\ 0 & e^{-iE_2 t} & 0 \\ 0 & 0 & e^{-iE_3 t} \end{pmatrix} \begin{pmatrix} \nu_1(0) \\ \nu_2(0) \\ \nu_3(0) \end{pmatrix},$$

or,

$$\boldsymbol{\nu}_i(t) = \mathbf{E}\ \boldsymbol{\nu}_i(0)$$

- $\boldsymbol{\nu}_\alpha(t) = \mathbf{U}\ \mathbf{E}\mathbf{U}^{-1}\ \boldsymbol{\nu}_\alpha(0) = \mathbf{U}_f\ \boldsymbol{\nu}_\alpha(0)$.

- So the flavor state at time $t = 0$ is connected to the flavor state at time t by

$$\begin{pmatrix} \nu_e(t) \\ \nu_\mu(t) \\ \nu_\tau(t) \end{pmatrix} = \begin{pmatrix} a(t) & d(t) & g(t) \\ b(t) & e(t) & h(t) \\ c(t) & f(t) & k(t) \end{pmatrix} \begin{pmatrix} \nu_e(0) \\ \nu_\mu(0) \\ \nu_\tau(0) \end{pmatrix}.$$

- Some elements

$$\begin{aligned} a(t) &= (c_{12}c_{13})^2 \; e^{-iE_1 t} + (s_{12}c_{13})^2 \; e^{-iE_2 t} + s_{13}^2 \; e^{-iE_3 t}, \\ b(t) &= (-s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta})c_{12}c_{13} \; e^{-iE_1 t} \\ &\quad + (c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta})s_{12}c_{13} \; e^{-iE_2 t} + c_{13}s_{23}s_{13}e^{i\delta} \; e^{-iE_3 t}, \\ c(t) &= (s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta})(c_{12}c_{13}) \; e^{-iE_1 t} \\ &\quad + (-c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta})(s_{12}c_{13}) \; e^{-iE_2 t} + (c_{13}c_{23})(s_{13}) \; e^{-iE_3 t}. \end{aligned}$$

Therefore, a neutrino starting in state ν_e at time $t = 0$, evolves to

$$|\nu_e(t)\rangle = a(t)\,|\nu_e\rangle + b(t)\,|\nu_\mu\rangle + c(t)\,|\nu_\tau\rangle$$

Survival probability of being in flvor e: $P_{e\to e}(t) = |\langle\nu_e|\nu_e(t)\rangle|^2 = \boxed{|a(t)|^2}$ $\leftarrow$ *no δ dependence*

Transition probability from flavor e to μ: $P_{e\to\mu}(t) = |\langle\nu_\mu|\nu_e(t)\rangle|^2 = |b(t)|^2$

Transition probability from flavor e to τ: $P_{e\to\tau}(t) = |\langle\nu_\tau|\nu_e(t)\rangle|^2 = |c(t)|^2$

Two flavor scenario

$$P_{e\to\mu} = \sin^2\theta \ \sin^2\left[1.27\Delta m^2 \frac{L(Km)}{E(GeV)}\right] \qquad VACUUM$$

$$P_{e\to\mu} = \sin^2\theta \ \sin^2\left[1.27\frac{\Delta m^2 L(Km)}{E(GeV)}\sqrt{\sin^2(2\theta) + \left(\cos(2\theta) - \frac{2EV}{\Delta m^2}\right)^2}\right] \quad MATTER$$

Three flavor scenario (COMPLICATED FUNCTIONS)

$$P_{e\to\mu} = f(\overbrace{\theta_{12}, \theta_{23}, \theta_{13}, \delta}^{PMNS \ \ parameters}, \Delta m_{21}, \Delta m_{32}, \Delta m_{31}, E, L) \qquad VACUUM$$

$$P_{e\to\mu} = h(\theta_{12}, \theta_{23}, \theta_{13}, \delta, \Delta m_{21}, \Delta m_{32}, \Delta m_{31}, E, L, V) \quad MATTER$$

$$\boxed{\Delta m_{jk} = m_j^2 - m_k^2}$$

Survival and Transitional Probabilities

Initial state $|\nu_e\rangle$

- $\Psi(t) = a(t)\,|\nu_e\rangle + b(t)\,|\nu_\mu\rangle + c(t)\,|\nu_\tau\rangle$
- Survival probability: $|\langle\nu_e|\Psi(t)\rangle|^2 = |a(t)|^2$
- Transition Prob. to $\nu_\mu = |\langle\nu_\mu|\Psi(t)\rangle|^2 = |b(t)|^2$
- Transition Prob. to $\nu_\tau = |\langle\nu_\tau|\Psi(t)\rangle|^2 = |c(t)|^2$

Initial state $|\nu_\mu\rangle$

- $\Psi(t) = d(t)\,|\nu_e\rangle + e(t)\,|\nu_\mu\rangle + f(t)\,|\nu_\tau\rangle$
- Survival probability: $|\langle\nu_\mu|\Psi(t)\rangle|^2 = |e(t)|^2$
- Transition Prob. to $\nu_e = |\langle\nu_e|\Psi(t)\rangle|^2 = |d(t)|^2$
- Transition Prob. to $\nu_\tau = |\langle\nu_\tau|\Psi(t)\rangle|^2 = |f(t)|^2$

Initial state $|\nu_\tau\rangle$

- $\Psi(t) = g(t)\,|\nu_e\rangle + h(t)\,|\nu_\mu\rangle + k(t)\,|\nu_\tau\rangle$
- Survival probability: $|\langle\nu_\tau|\Psi(t)\rangle|^2 = |k(t)|^2$
- Transition Prob. to $\nu_e = |\langle\nu_e|\Psi(t)\rangle|^2 = |g(t)|^2$
- Transition Prob. to $\nu_\mu = |\langle\nu_\mu|\Psi(t)\rangle|^2 = |h(t)|^2$

Matter Effects

[Ohlsson-Snellman (2000)]

1. $U_f(L) = \phi \sum_{n=1}^{3} e^{-i\mathcal{H}_m L} \frac{1}{3\lambda_n^2 + c_1} \left[(\lambda_n^2 + +c_1)\mathbf{I} + \lambda_n \tilde{T} + \tilde{T}^2 \right]$, where $\phi = e^{-i\frac{tr\mathcal{H}_m}{3}L}$, and the Hamiltonian in mass basis is $\mathcal{H}_m = H_m + U^{-1} V_f U$.
2. The matrix $\tilde{T} = UTU^{-1}$, where T is a hermitian matrix given by

$$T = \begin{pmatrix} AU_{e1}^2 - \frac{1}{3}A + \frac{1}{3}(E_{12} + E_{13}) & AU_{e1}U_{e2} & AU_{e1}U_{e3} \\ AU_{e1}U_{e2} & AU_{e2}^2 - \frac{1}{3}A + \frac{1}{3}(E_{21} + E_{23}) & \end{pmatrix},$$

- When neutrinos travel through a series of matter densities with matter density parameters $A_1, A_2, \ldots, A_n$ with thicknesses $L_1, L_2, \ldots, L_n$, the total evolution operator is simply given by

$$U_f^{\rm tot}(L) = U_f(L_1)\ U_f(L_2),\ \ldots, U_f(L_n),$$

where $L = \sum_{i=1}^{n} L_i$ and $U_f(L_i)$ is calculated for density parameter, A_i.

- Mantle Core Mantle approximation for earth
$\rho_{core} = 11.5\ gm/cm^3 \implies A_{core} = 4.35 \times 10^{-13} eV$
$\rho_{mantle} = 4.5\ gm/cm^3 \implies A_{mantle} = 1.70 \times 10^{-13} eV$