DIFFERENTIAL EQUATIONS FOR FEYNMAN INTEGRALS

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based on arXiv: 1709.07435, 1806.08241

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LOOP AMPLITUDE

- Amplitude given by Feynman diagrams

$$A = \sum_i a_i I_i$$

- A

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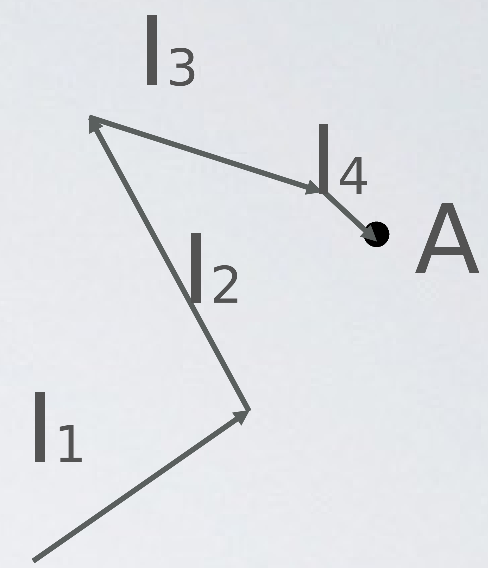

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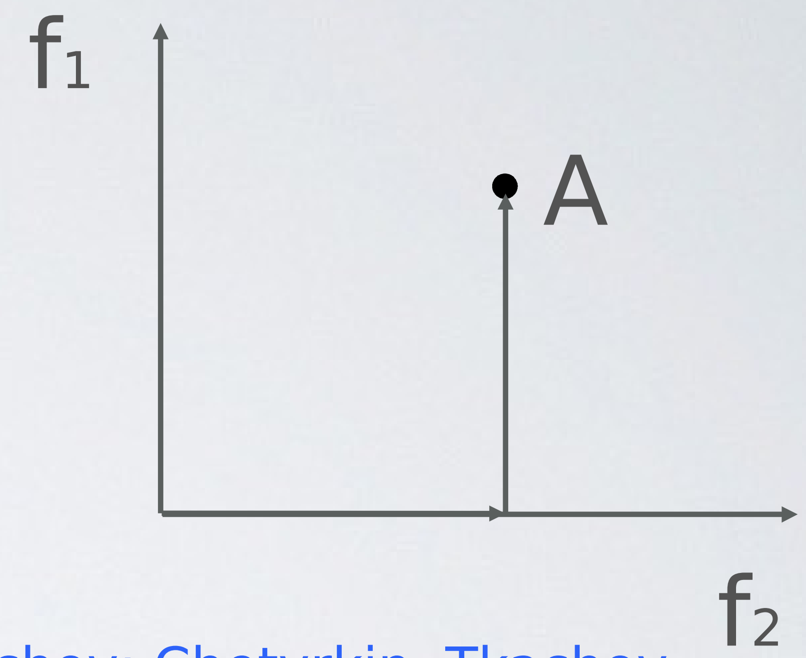

- Project onto basis

$$A = \sum_i c_i f_i$$

  - Integration-by-parts identities — Tkachov; Chetyrkin, Tkachov
  - Integrand Reduction — Ossola, Papadopoulos, Pittau; Ellis, Giele, Kunszt; Mastrolia, Zhang; Mastrolia, Mirabella, Ossola, Peraro
  - General Unitarity — Bern, Dixon, Dunbar, Kosower; Cachazo, Svrcek, Witte Britto, Cachazo, Feng
  - Numerical Unitarity — Ita; Abreu, Febres Cordero, Ita, Jaquier, Page

LOOP AMPLITUDE

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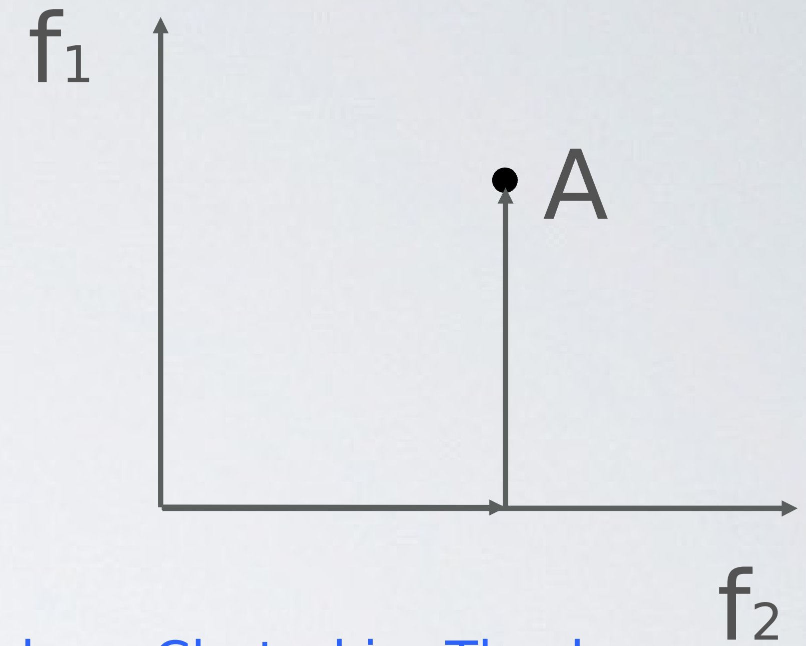

$$A = \sum_i a_i I_i$$

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  - Integration-by-parts identities — Tkachov; Chetyrkin, Tkachov
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- Calculation of master integrals
  - Feynman parameter
  - Mellin-Barnes — Smirnov; Tausk; Czakon; Smirnov, Smirnov
  - Differential equations — Kotikov; Remiddi; Gehrmann, Remiddi
  - Difference equation — Laporta; Lee, Smirnov, Smirnov

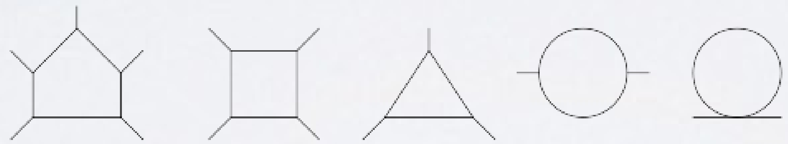
ONE-LOOP AMPLITUDES

- Techniques implemented in public codes
  - BlackHat — Bern, Dixon, Febre-Cordero, Forde, Hoecke, Ita, Kosower Maitre Oze
  - FeynArts/FormCalc/LoopTools — Hahn et. al
  - MadLoop — Hirschi, Frederix, Frixione, Garzelli, Maltoni, Pittau
  - HelacNLO — Bevilacqua, Czakon, van Hameren, Papadopoulos, Pittau, Worek
  - Njets — Badger, Biederman, Uwer, Yundin
  - OpenLoops — Cascioli, Maierhoefer, Pozorini
  - Recola — Actis, Denner, Hofer, Scharf, Uccirati
  - Rocket — Ellis, Giele, Kunszt, Melnikov, Zanderighi
  - GoSam — Cullen, Greiner, Heinrich, Mastrolia, Ossola, Reiter,, Tramontano

ONE-LOOP AMPLITUDES

- Techniques implemented in public codes
  - BlackHat :: On-shell recurrence+ Generalised Unitarity
  - FeynArts/FormCalc/LoopTools :: Feynman Diag. + Tensor Red./Integrand Red.
  - MadLoop :: tree-level recurrence+ Integrand Red.
  - HelacNLO :: tree-level recurrence+ Integrand Red.
  - Njets :: on-shell recurrence+ Generalised Unitarity
  - OpenLoops :: recursive tensors+ Tensor Red./Integrand Red.
  - Recola :: recursive tensors + Tensor Red.
  - Rocket :: tree-level recurrence + Generalised Unitarity
  - GoSam :: Feynman Diag. + Tensor Red./ Integrand Red.

ONE-LOOP VS TWO-LOOP

	One-Loop	Two-Loop
Graphs	only planar	planar and non-planar
Integral Basis	known for any process 	determined case-by-case ?
Integrals	known for general mass configurations	only for certain cases
IR Poles	cancellation between one-loop and tree-level	cancellation between two- and one-loop and tree-level
Functions	logs and di-logs	logs, polylogs, elliptic functions and more ?

Differential

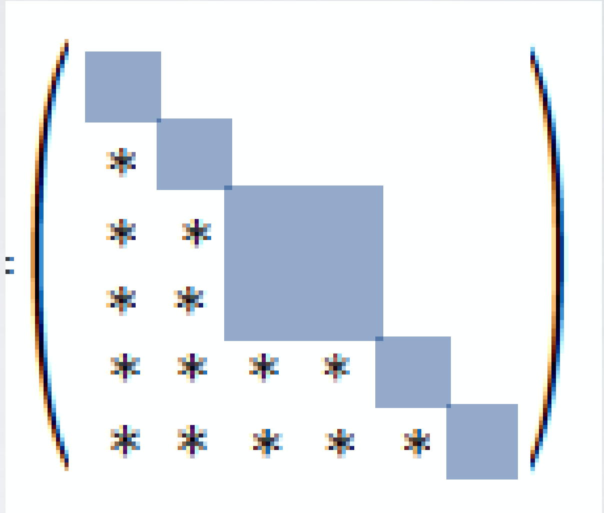
Equations

DIFFERENTIAL EQUATIONS

- **Derivative in space spanned by MI**

$$\partial_x \vec{f} = A_x \vec{f}$$

- A_x **inhabits properties of IBP**
  - Block triangular
  - Rational in x and ε=(4-d)/2



Bottom up Approach	Matrix Approach
• Solve each block separately • Previously solved integrals appear as inhomogeneous part	• Conjecture: There is a basis such that: Henn $\partial_x \vec{g} = \epsilon \tilde{A}_x \vec{g}$ • Makes integration simple • But: Finding basis is difficult

CANONICAL DIFFERENTIAL EQUATIONS

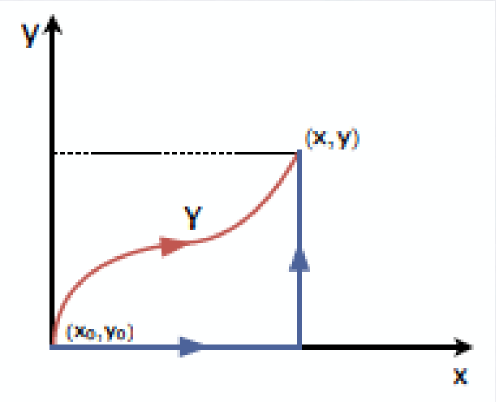
- **Factorization of ε often coincides with dlog-form**

$$d\vec{g}(x,\epsilon) = \epsilon \sum_i M_i dlog(\eta_i)\vec{g}(x,\epsilon)$$

  - Kinematic Dependence encoded in η
  - ηs form the alphabet

- **Solution given by**

$$\vec{g}(x,\epsilon) = \left[1 + \sum_{i=1}^{\infty} \int_\gamma dA \dots dA\right] \vec{g}(x_0,\epsilon)$$



- **Many strategies to find such forms**
  - Unit leading singularity — Henn
  - Magnus Theorem — Ageri, Di Vita, Mastrolia, Mirabella, Schlenk, Tancredi, US
  - Rational Ansatz for basis change — Gehrmann, von Manteuffel, Tancredi, Weihs
  - Reduction to fuchsian form and Eigenvalue normalisation — Lee; Gituliar, Magerya
  - Expand basis change in ε — Meyer
  - Factorisation of Picard-Fuchs operator — Adams, Chaubery, Weinzierl

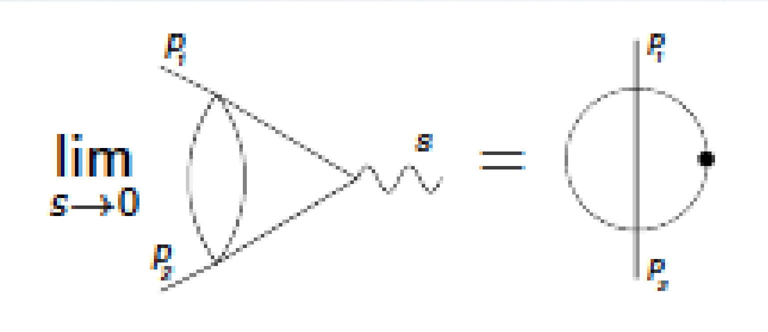
BOUNDARY CONDITIONS

- **Solution given by**

$$\vec{g}(x,\epsilon) = \left[1 + \sum_{i=1}^{\infty} \int_{\gamma} dA \ldots dA\right] \vec{g}(x_0,\epsilon)$$

- **Two general ways to fix the boundary**

Known limits	Pseudo-thresholds
• Taking the limit x to x_0 • Fix boundary constant by matching the solution to known function	• Solution has unphysical divergences • Demanding absence of unphysical divergences gives relations between boundary constant • Leftover constants must be provided



Example 1:
Muon-Electron scattering

MUON G-2

$$g_\mu = 2(1 + a_\mu)$$

- **High precision test of Standard model**
  - E821 experiment at BNL measured

$$a_\mu^{E821} = 116592089(63) \times 10^{-11}$$

Bennett et al. [Muon g-2 Collaboration]

  - Standard model prediction

$$a_\mu^{SM} = 116591802(49) \times 10^{-11}$$

Davier, Hoecker, Malasecu, Zhang

- **g-2 experiment at Fermilab could push difference to 5σ**

- **Biggest theory uncertainty from hadronic contribution**

$$a_\mu^{SM} = a_\mu^{QED} + a_\mu^{Weak} + a_\mu^{Hadr}$$

$$a_\mu^{\mathrm{QED}} = 116584718.95(8) \times 10^{-11}$$

Aoyama, Hayakawa, Kinoshita, Nio

$$a_\mu^{\mathrm{Weak}} = 153(2) \times 10^{-11}$$

Gnendinger, Stoeckinger, Stoeckinger-Kim

$$a_\mu^{\mathrm{Had,LO}} = 6949(58) \times 10^{-11}$$

Hagiwara, Liao, Martin, Nomura, Teubner

$$a_\mu^{\mathrm{Had,NLO}} = -98.4(4) \times 10^{-11}$$

Davier, Hoecker, Malaescu, Zhang

$$a_\mu^{\mathrm{HLbL}} = 105(26) \times 10^{-11}$$

Prades, de Rafael, Vainshtein

LEADING HADRONIC CONTRIBUTION

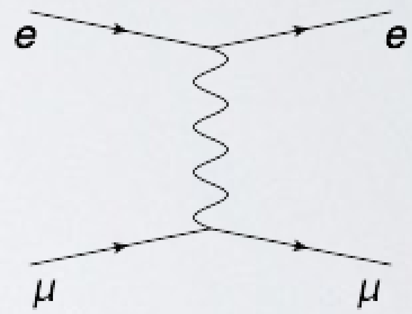
- **Extract a_μ^{HLO} from experimental data** Bouchiat; Michele; Durand,;Gourdin, de Rafael

$$a_\mu^{HLO} = \frac{1}{4\pi^3}\int_{4m_\pi^2}^{\infty} ds \int_0^1 dx \frac{x^2(1-x)}{x^2+(1-x)s/m^2}\sigma_{e^+e^-\to Had}(s)$$

- **Low energy region plagued by production thresholds**

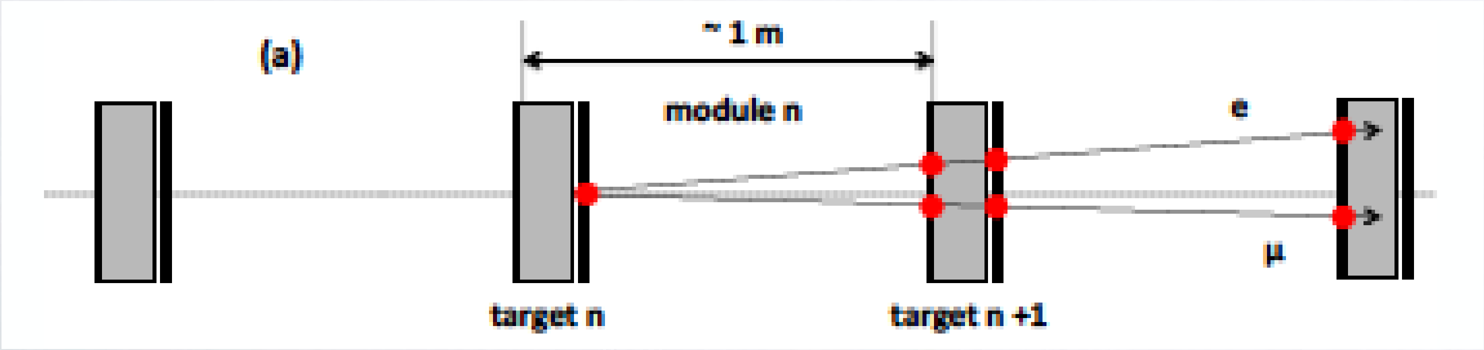
- **Alternatively compute from space-like data**

$$a_\mu^{HLO} = \frac{\alpha}{\pi}\int_0^1 dx(1-x)\Delta\alpha_{Had}[t(x)] \qquad t(x) = \frac{x^2 m_\mu^2}{x-1} < 0$$



- **Extract $\Delta\alpha_{Had}[t(x)]$ from running of α in μe scattering**

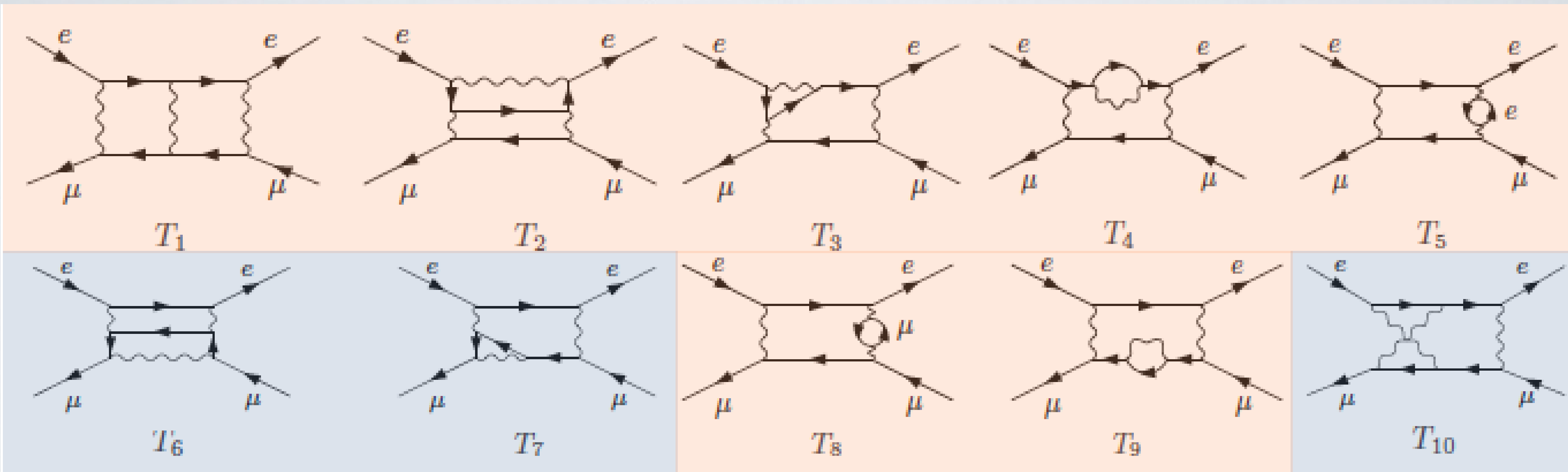
- **Proposed experiment MUonE: 150GeV μ-beam on atomic e**



Carloni Calame, Passera et al; Abbiendi, Carloni Calame, Marconi et al

MUON ELECTRON SCATTERING AT NNLO

- **Four-point topologies at NNLO**



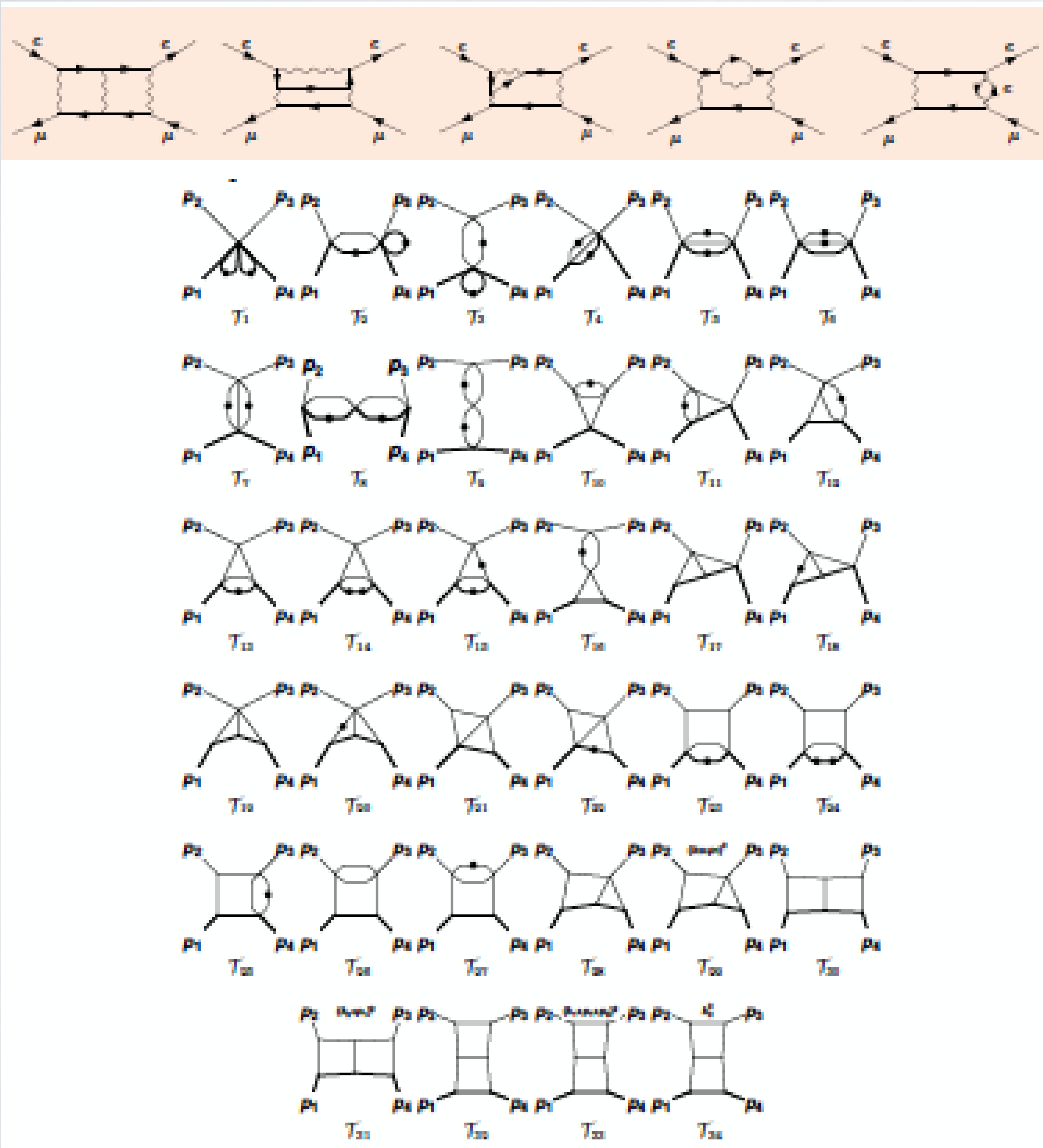
- **Most planar integrals known analytically**
  - $t\bar{t}$ production in QCD — Gehrmann, Remiddi, Bonciani, Mastrolia, Remiddi
  - Bhabha scattering in QED — Bonciani, Ferroglia; Asatrian Greub, Pecjak
  - heavy-to-light quark decay in QCD — Bonciani, Ferroglia, Gehrmann

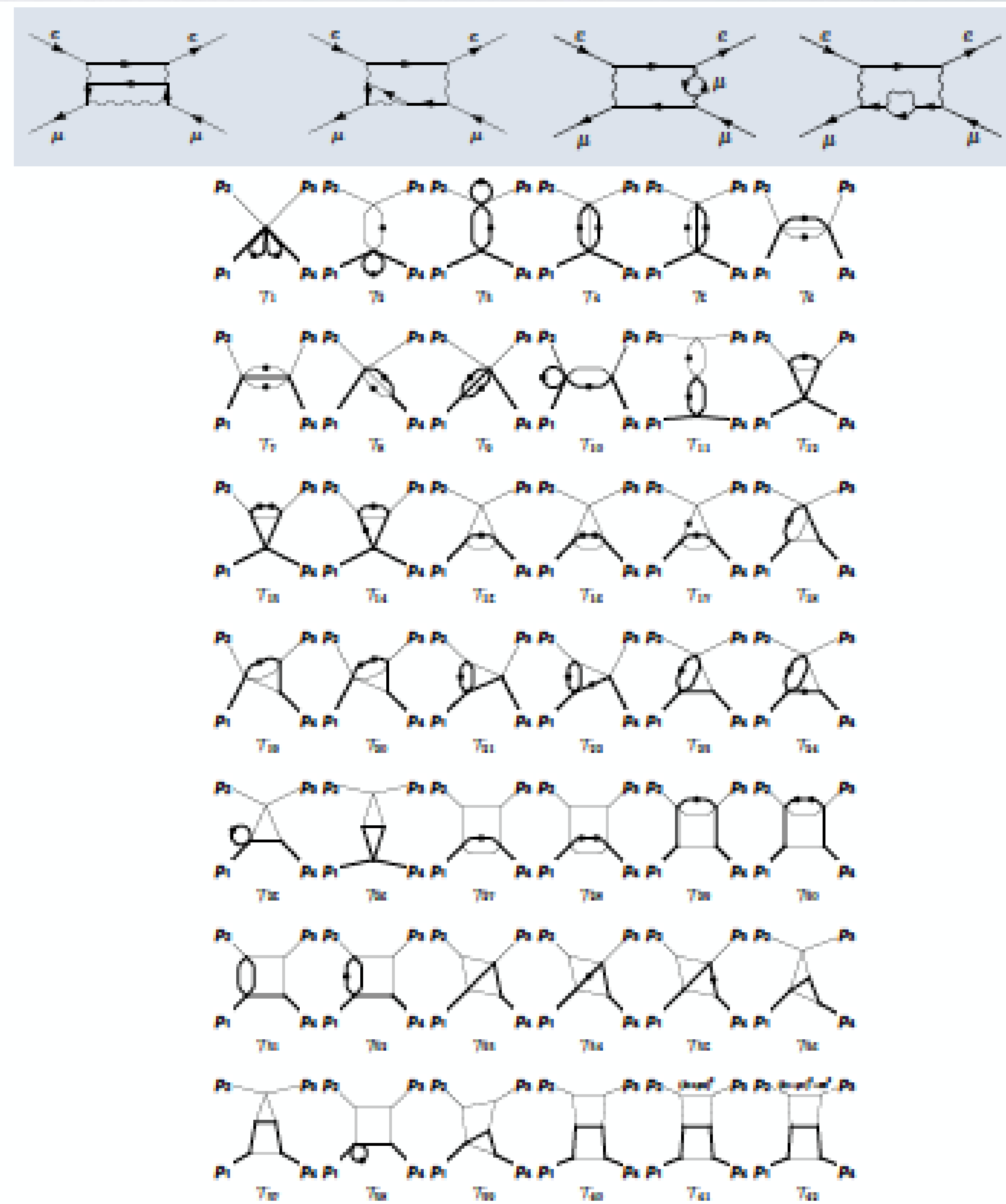
- **Unknown integrals with more massive lines**

PLANAR INTEGRALS

Mastrolia, Passera, Primo, U.S.

- **65 distinct master integrals identified with Reduze**





PLANAR INTEGRALS

Mastrolia, Passera, Primo, U.S.

- **Variables**

$$-\frac{s}{m^2} = x \qquad -\frac{t}{m^2} = \frac{(1-y)^2}{y}$$

- **MIs satisfy pre-canonical form**

$$\partial_x \vec{f} = (A_{0,x} + \epsilon A_{1,x})\vec{f} \qquad \partial_y \vec{f} = (A_{0,y} + \epsilon A_{1,y})\vec{f}$$

- **Use Magnus exponential to obtain canonical form**

$$\partial_x \vec{g} = \epsilon \tilde{A}_x \vec{g} \qquad \partial_y \vec{g} = \epsilon \tilde{A}_y \vec{g}$$

- **Combine to total differential**

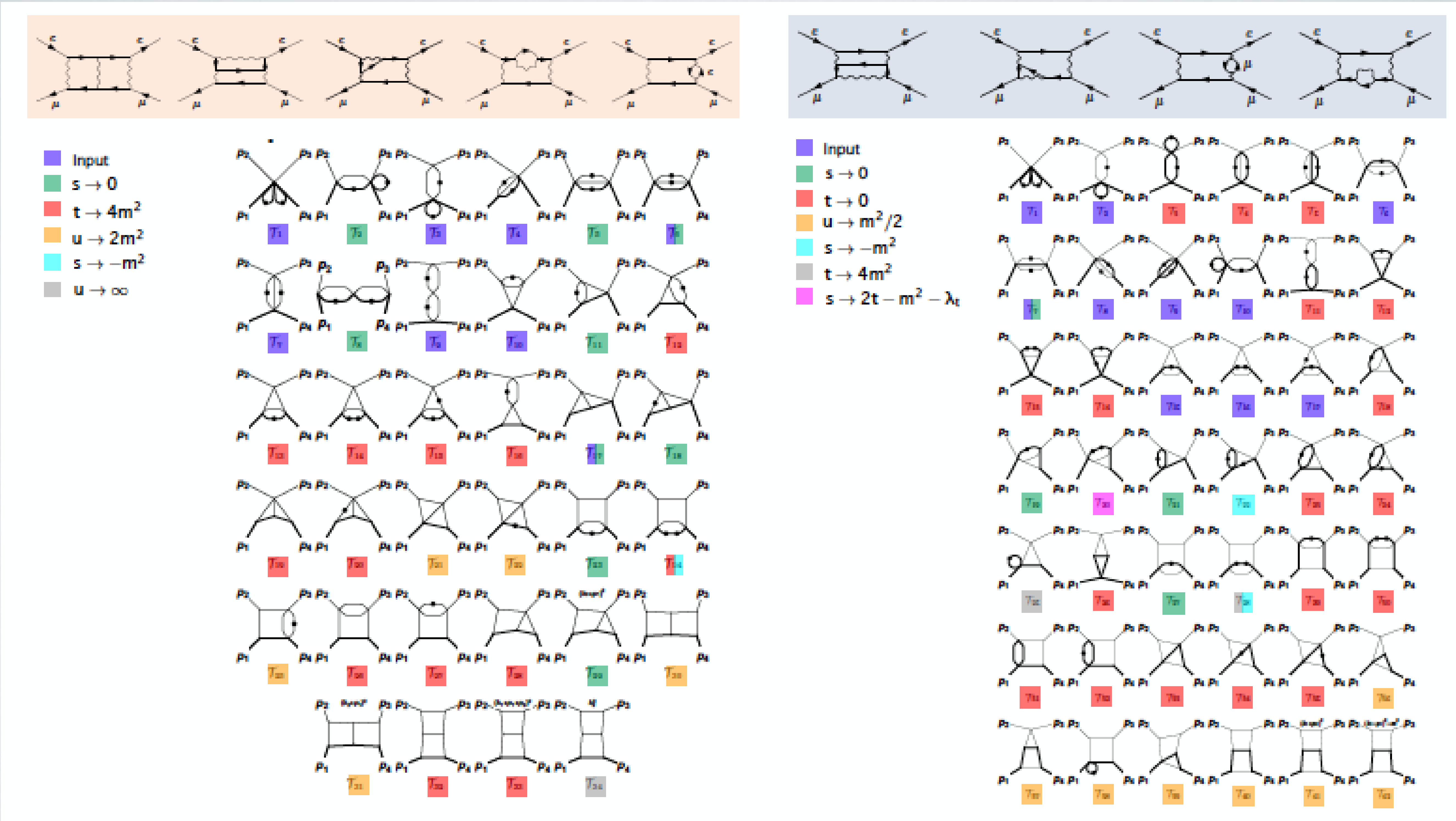
$$d\vec{g} = \epsilon dA \vec{g}$$

$$\begin{aligned} dA = {} & M_1 dlog(x) + M_2 dlog(1+x) + M_3 dlog(1-x) \\ & + M_4 dlog(y) + M_5 dlog(1+y) + M_6 dlog(1-y) \\ & + M_7 dlog(x+y) + M_8 dlog(1+xy) \\ & + M_9 dlog(1-y(1-x-y)) \end{aligned}$$

- **Arguments of dlog form alphabet**

BOUNDARY FIXING

Mastrolia, Passera, Primo, U.S.

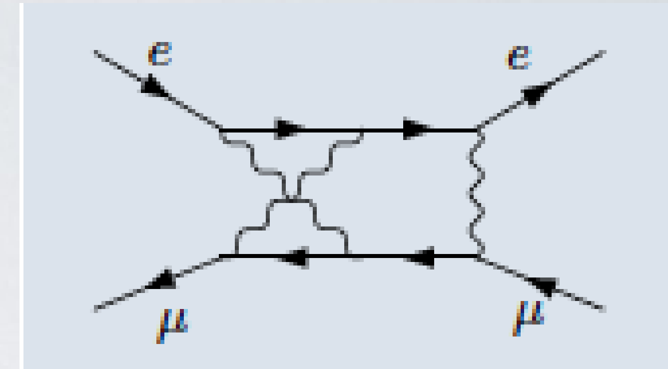


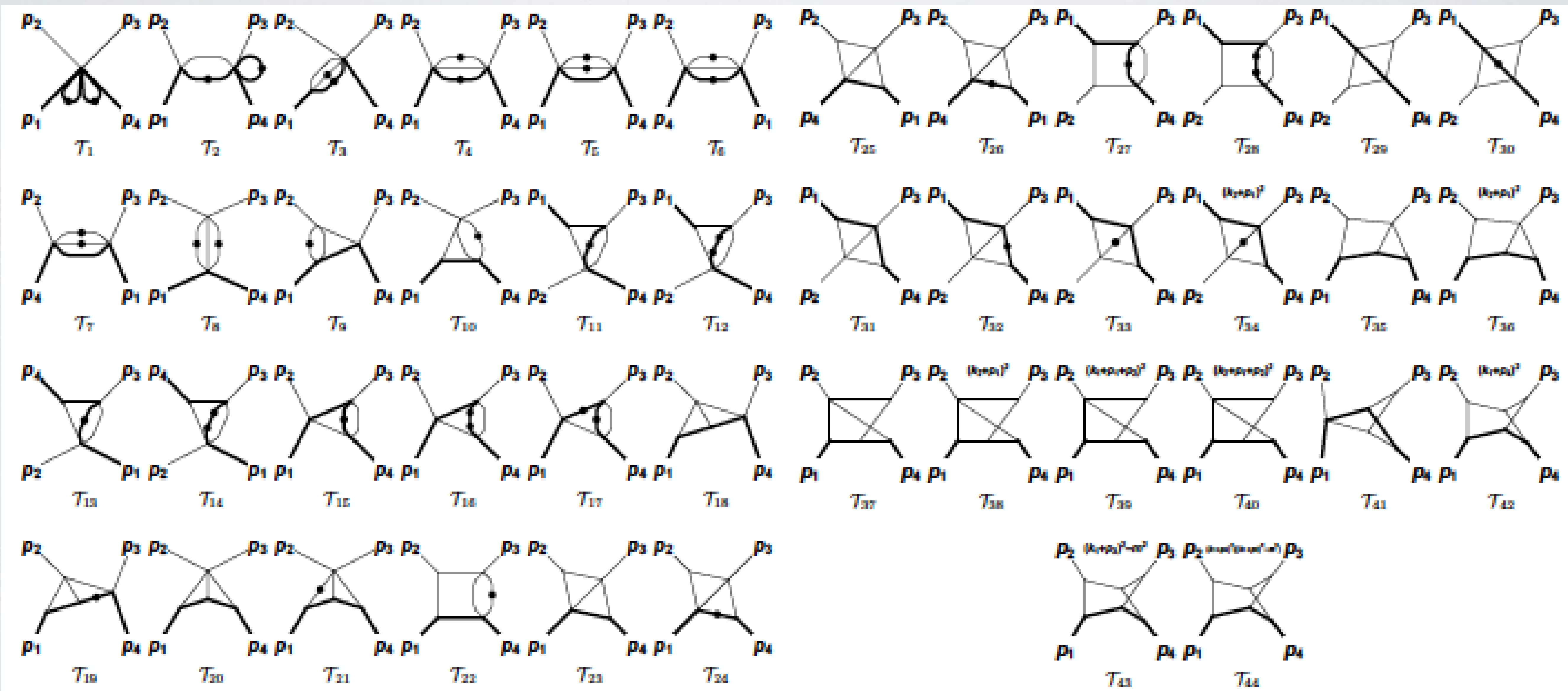
- **All Integrals checked numerically with SecDec**

NON-PLANAR INTEGRALS

Di Vita, Laporta, Mastrolia, Primo, U.S.

- **44 distinct master integrals identified with Reduze**

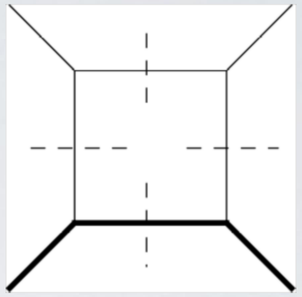




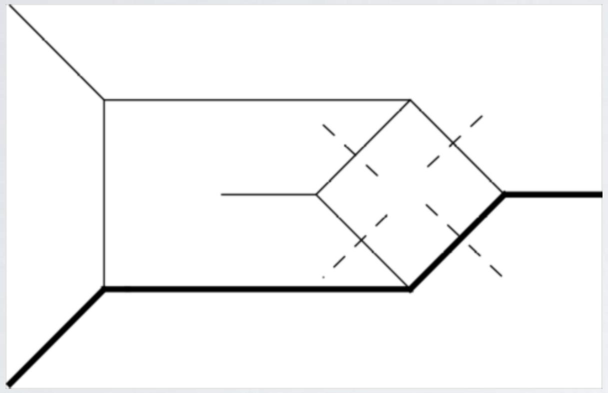
NON-PLANAR INTEGRALS

Di Vita, Laporta, Mastrolia, Primo, U.S.

- **Identify candidates via unitarity cuts** Henn

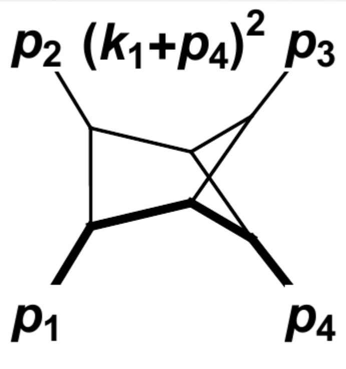


$$= \frac{1}{t(s-m^2)}$$

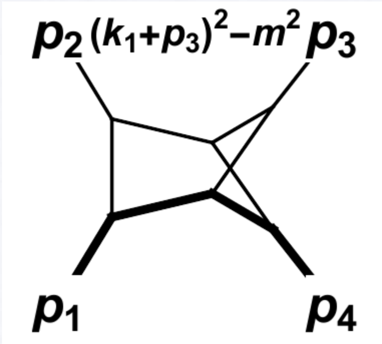


$$= \int d^4k_1 \frac{1}{(k_1^2-m^2)(k_1+p_1)^2(k_1+p_1+p_2)^2(k_1+p_4)^2((k_1+p_3)^2+m^2)}$$

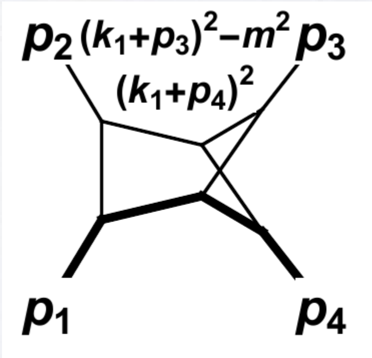
- **Pentagon-type integrals are not good choices**
  → Cancel propagators arising from cut



$\mathcal{T}_{42}$



$\mathcal{T}_{43}$



$\mathcal{T}_{44}$

NON-PLANAR INTEGRALS

Di Vita, Laporta, Mastrolia, Primo, U.S.

- **Variables**

$$\frac{s}{m^2} = 1 + \frac{(1-w)^2}{w - z^2} \qquad -\frac{t}{m^2} = \frac{(1-w)^2}{w}$$

- **MIs satisfy pre-canonical form**

$$\partial_z \vec{f} = (A_{0,z} + \epsilon A_{1,z})\vec{f} \qquad \partial_w \vec{f} = (A_{0,w} + \epsilon A_{1,w})\vec{f}$$

- **Use Magnus exponential to obtain canonical form**

$$\partial_w \vec{g} = \epsilon \tilde{A}_w \vec{g} \qquad \partial_z \vec{g} = \epsilon \tilde{A}_z \vec{g}$$

- **Combine to total differential**

$$d\vec{g} = \epsilon dA \vec{g}$$

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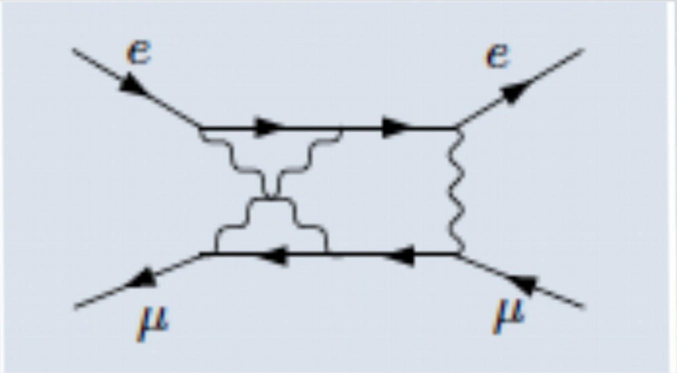
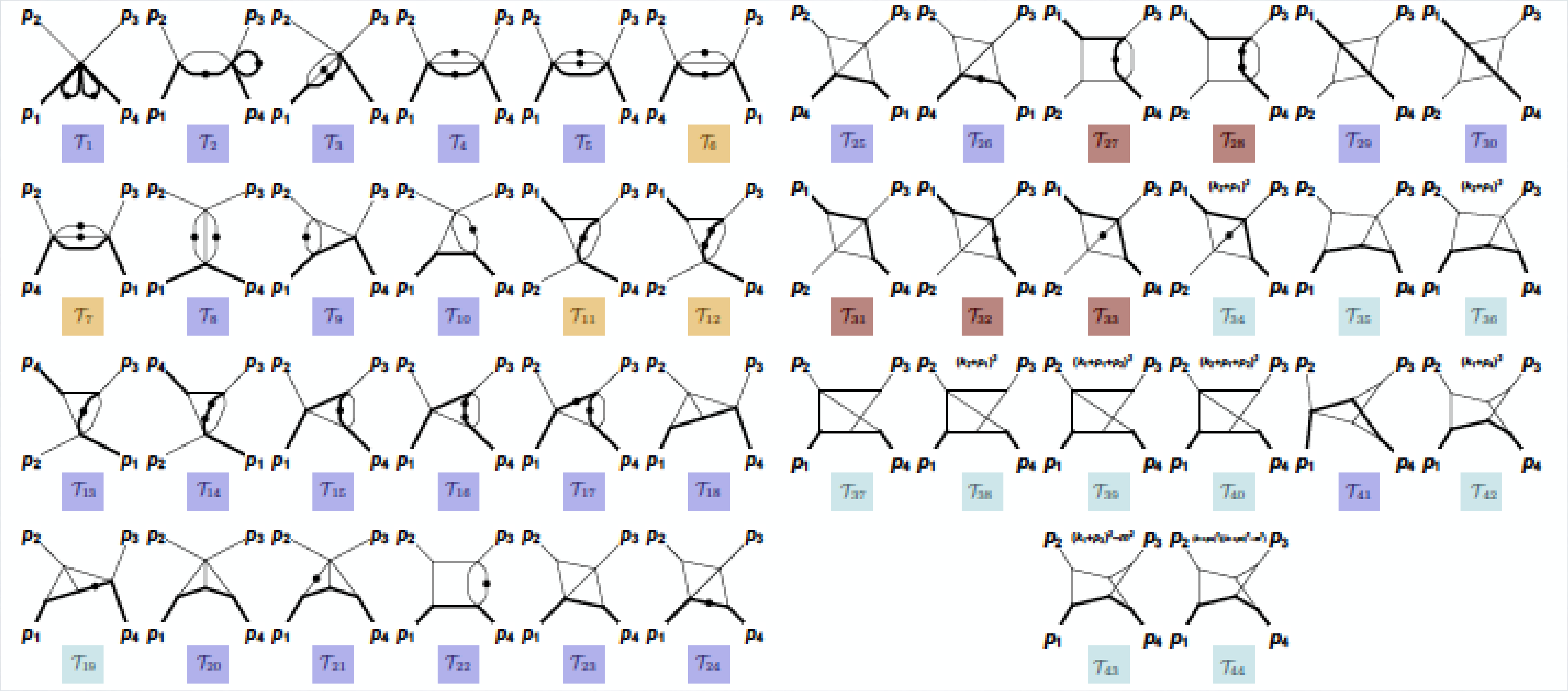
NON-PLANAR INTEGRALS

Di Vita, Laporta, Mastrolia, Primo, U.S.

- Input
- u → 0
- t → 0
- z → 0





➢ **All integrals checked against SecDec or in-house numerical code**

Example 2:
Non-Planar Vertex

VERTEX WITH TWO OFF-SHELL LEGS

Primo, Tancredi; Hidding, Moriello

- **Variables**

$$x = -\frac{s}{m^2} \quad y = -\frac{p_2^2}{m^2}$$

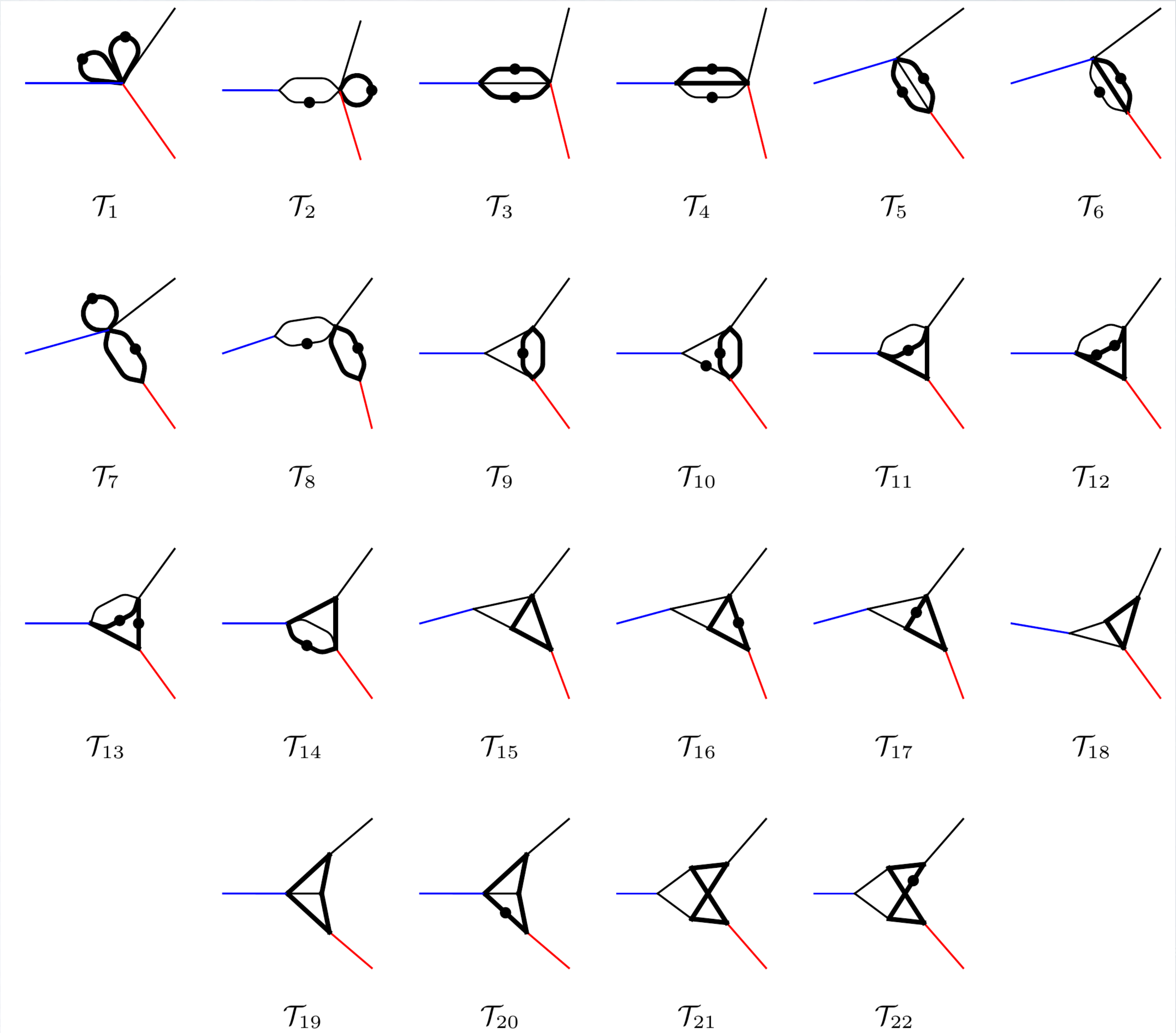
- **DEQ is in pre-canonical form**

$$\partial_x \vec{f} = (A_{0,x} + \epsilon\, A_{1,x})\, \vec{f}$$

$$\partial_y \vec{f} = (A_{0,y} + \epsilon\, A_{1,y})\, \vec{f}$$

- **Magnus finds canonical basis for first 20 integrals**

- **But Magnus series does not converge for last two integrals**



VERTEX WITH TWO OFF-SHELL LEGS

Primo, Tancredi; Hidding, Moriello

- **Knowing ε^0 solution equivalent to finding canonical form**

$$\partial_x B(x) = A_0 B(x)$$

- **Investigate DEQ**

$$\partial_x \begin{pmatrix} \mathrm{F}_{21} \\ \mathrm{F}_{22} \end{pmatrix} = A_{0,x} \begin{pmatrix} \mathrm{F}_{21} \\ \mathrm{F}_{22} \end{pmatrix} + \epsilon A_{1,x} \begin{pmatrix} \mathrm{F}_{21} \\ \mathrm{F}_{22} \end{pmatrix} + C_x \vec{\mathrm{I}}_{sub}$$
$$\partial_y \begin{pmatrix} \mathrm{F}_{21} \\ \mathrm{F}_{22} \end{pmatrix} = A_{0,y} \begin{pmatrix} \mathrm{F}_{21} \\ \mathrm{F}_{22} \end{pmatrix} + \epsilon A_{1,y} \begin{pmatrix} \mathrm{F}_{21} \\ \mathrm{F}_{22} \end{pmatrix} + C_y \vec{\mathrm{I}}_{sub} \,,$$

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Primo, Tancredi;
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- **Knowing ε^0 solution equivalent to finding canonical form**

$$\partial_x B(x) = A_0 B(x)$$

- **Unitarity cut is solution to homogenous DEQ**

Lee, Smirnov; Primo, Tancredi

$$\partial_x \begin{pmatrix} F_{21} \\ F_{22} \end{pmatrix} = A_{0,x} \begin{pmatrix} F_{21} \\ F_{22} \end{pmatrix} + \epsilon A_{1,x} \begin{pmatrix} F_{21} \\ F_{22} \end{pmatrix} + \cancel{C_x \vec{I}_{sub}}$$

$$\partial_y \begin{pmatrix} F_{21} \\ F_{22} \end{pmatrix} = A_{0,y} \begin{pmatrix} F_{21} \\ F_{22} \end{pmatrix} + \epsilon A_{1,y} \begin{pmatrix} F_{21} \\ F_{22} \end{pmatrix} + \cancel{C_y \vec{I}_{sub}}$$

VERTEX WITH TWO OFF-SHELL LEGS

Primo, Tancredi;
Hidding, Moriello

- **Knowing ε^0 solution equivalent to finding canonical form**

$$\partial_x B(x) = A_0 B(x)$$

- **d=4 Unitarity cut is solution to ε^0-part of homogenous DEQ**

Lee, Smirnov; Primo, Tancredi

$$\partial_x \begin{pmatrix} \mathrm{F}_{21} \\ \mathrm{F}_{22} \end{pmatrix} = A_{0,x} \begin{pmatrix} \mathrm{F}_{21} \\ \mathrm{F}_{22} \end{pmatrix} + \epsilon A_{1,x} \begin{pmatrix} \mathrm{F}_{21} \\ \mathrm{F}_{22} \end{pmatrix} + C_x \vec{\mathrm{I}}_{sub}$$

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Lee, Smirnov; Primo, Tancredi

- **Unitarity cut reveals elliptic nature of integral**

$$\mathrm{Cut}_1\left(\mathrm{F}_{21}\right) = \mathrm{F}_{21,1} = \omega K\left(\omega^2\right)$$

$$\omega = \frac{s - p_2^2}{\sqrt{s^2 - 2\,s\,p_2^2 + 16\,s\,m^2 + p_2^4}}\,.$$

- **2^{nd} independent Solution found by properties of elliptic integrals**

$$\mathrm{F}_{21,2} = wK\left(1 - \omega^2\right)$$

- **Find other solutions through DEQ**

$$\mathrm{F}_{22,1} = -\frac{\omega(s + p_2^2)(16m^2 - s)}{16m^2 s} E\left(\omega^2\right)$$
$$\mathrm{F}_{22,2} = -\frac{\omega(s + p_2^2)(16m^2 - s)}{16m^2 s}\left(E\left(1 - \omega^2\right) - K\left(1 - \omega^2\right)\right)$$

VERTEX WITH TWO OFF-SHELL LEGS

Primo, Tancredi; Hidding, Moriello

- **Build new basis from found solutions**

$$\mathrm{I}_{21} = -2\frac{(s-p_2^2)^2}{\omega}\left(E\left(1-\omega^2\right) - K\left(1-\omega^2\right)\right)\,\mathrm{F}_{21} + \frac{16m^2 s(s-p_2^2)^2}{(s+p_2^2)\omega}K\left(1-\omega^2\right)\,\mathrm{F}_{22}\,,$$

$$\mathrm{I}_{22} = -2\frac{(s-p_2^2)^2}{\omega}E\left(\omega^2\right)\,\mathrm{F}_{21} - \frac{16m^2 s(s-p_2^2)^2}{(s+p_2^2)\omega}K\left(\omega^2\right)\,\mathrm{F}_{22}\,,$$

- **ε-factorized DEQ depends on elliptic integrals**

- **Solution given by iterated integrals with elliptic functions in the integrand**

- **Checked against SecDec**

Conclusions

CONCLUSIONS

- Canonical DEQ revived the field
- Magnus Exponential can find canonical basis if the initial DEQ is linear in ε
  - QED vertex at two-loop, 2 to 2 massless box, Higgs+Jet at two-loop, Ladder topology for Higgs+Jet at three-loop, mixed QCD-EW corrections to Drell-Yan, leading QCD corrections for H to WW at two-loop, Muon-Electron scattering at NNLO
- Amplitudes for muon-electron scattering at NNLO are coming
  - Important cross-check for leading hadronic contribution to muon g-2
- Unitarity cuts are important tools to find ε-factorized DEQ
- Extensions to elliptic integrals are being explored
- Is there a “canonical” form for elliptic integrals?
- Do all DEQ have an ε-factorized form ?

Thank you for your attention

INTEGRATION-BY-PARTS IDENTITIES

- **Generated from Stokes Theorem**

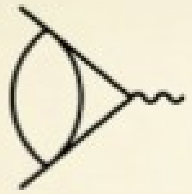
$$\int \prod_{i=1}^{L} d^d k_i \frac{\partial}{\partial k_{\mu,i}} \left(\frac{q_j^\mu}{D_1^{\alpha_1} \ldots D_N^{\alpha_N}} \right) = 0 \qquad \leftrightarrow \qquad A\vec{I} = 0$$

- **Rank of As null space gives number of master integrals**

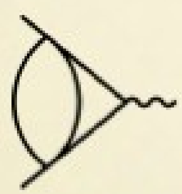
- **Limiting factors**
  - Algebra in Gaussian Elimination
    - Finite Field Method — von Manteuffel, Schabinger; Maierhoefer, Usovitsch, Uwer; Peraro
  - Solving unnecessary Equations
    - Generate IBPs without higher powers — Larsen, Zhang
    - IBPs on the cut — Larsen, Zhang
- **Implemented in Public Codes**
  - Reduze — Studerus, von Manteuffel
  - Fire — Smirnov
  - Air — Anastasiou, Lazopolus
  - Kira — Maierhoefer, Usovitsch, Uwer

QED VERTEX

Bonciani, Remiddi, **P.M.** (2013)

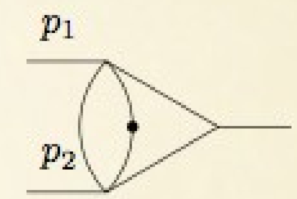


$$
\begin{aligned}
M_{-2} &= \frac{1}{2}\,,\\
M_{-1} &= \frac{5}{2} - \left[1 - \frac{2}{(1-x)}\right] H(0,x)\,,\\
M_0 &= \frac{19}{2} + \zeta(2) + \left[1 - \frac{2}{(1-x)}\right]\left[\zeta(2) - 5H(0,x) + 2H(-1,0,x)\right]\\
&\quad + \frac{2}{(1-x)} H(0,0,x) + \left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right]\left[\zeta(2) H(0,x)\right.\\
&\quad \left. + H(0,0,0,x)\right].
\end{aligned}
$$

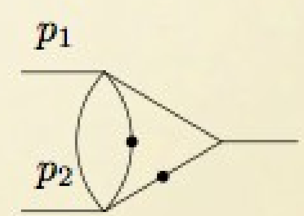
 $(p_2 \cdot k_1)$

$$
\begin{aligned}
\frac{N_{-2}}{a} &= \frac{1}{8} + \frac{1}{16}\left[x + \frac{1}{x}\right],\\
\frac{N_{-1}}{a} &= \frac{9}{32}\left[2 + x + \frac{1}{x}\right] - \frac{1}{8}\left[4 + x - \frac{1}{x}\right] H(0,x) + \frac{1}{(1-x)} H(0,x)\,,\\
\frac{N_0}{a} &= \frac{63}{32} + \frac{\zeta(2)}{2} + \frac{63}{64}\left[\left(1 + \frac{16}{63}\zeta(2)\right) x + \frac{1}{x}\right] - \frac{\zeta(2)}{(1-x)} - \frac{1}{16}\left[32 + 9x\right.\\
&\quad \left. - \frac{9}{x}\right] H(0,x) + \frac{(16 + \zeta(2))}{4(1-x)} H(0,x) - \frac{\zeta(2)}{4(1+x)} H(0,x) - \frac{1}{4}\left[2 - \frac{1}{x}\right.\\
&\quad \left. - \frac{4}{(1-x)}\right] H(0,0,x) + \frac{1}{4}\left[4 + x - \frac{1}{x} - \frac{8}{(1-x)}\right] H(-1,0,x)\\
&\quad + \frac{1}{4}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] H(0,0,0,x)\,.
\end{aligned}
$$

AdVMMSST (2014)



$$
\begin{aligned}
g_{12}^{(0)} &= 0\,,\\
g_{12}^{(1)} &= 0\,,\\
g_{12}^{(2)} &= 0\,,\\
g_{12}^{(3)} &= -\,\mathrm{H}(0,0,0;x) - \zeta_2 \mathrm{H}(0;x)\,,\\
g_{12}^{(4)} &= -\,2\,\mathrm{H}(-1,0,0,0;x) + 2\,\mathrm{H}(0,-1,0,0;x) + 2\,\mathrm{H}(0,0,-1,0;x)\\
&\quad - 3\,\mathrm{H}(0,0,0,0;x) - 4\,\mathrm{H}(0,1,0,0;x) + \zeta_2(-2\,\mathrm{H}(-1,0;x)\\
&\quad + 6\,\mathrm{H}(0,-1;x) - \mathrm{H}(0,0;x)) + 2\,\zeta_3\,\mathrm{H}(0;x) + \frac{\zeta_4}{4}\,,
\end{aligned}
$$



$$
\begin{aligned}
g_{13}^{(0)} &= 0\,,\\
g_{13}^{(1)} &= 0\,,\\
g_{13}^{(2)} &= \mathrm{H}(0,0;x) + \frac{3\,\zeta_2}{2}\,,\\
g_{13}^{(3)} &= -\,2\,\mathrm{H}(-1,0,0;x) - 2\,\mathrm{H}(0,-1,0;x) + 4\,\mathrm{H}(0,0,0;x) + 4\,\mathrm{H}(1,0,0;x)\\
&\quad + \zeta_2(-6\,\mathrm{H}(-1;x) + 2\,\mathrm{H}(0;x) - 3\log 2) - \frac{\zeta_3}{4}\,,\\
g_{13}^{(4)} &= 4\,\mathrm{H}(-1,-1,0,0;x) + 4\,\mathrm{H}(-1,0,-1,0;x) - 8\,\mathrm{H}(-1,0,0,0;x)\\
&\quad - 8\,\mathrm{H}(-1,1,0,0;x) + 4\,\mathrm{H}(0,-1,-1,0;x) - 8\,\mathrm{H}(0,-1,0,0;x)\\
&\quad - 8\,\mathrm{H}(0,0,-1,0;x) + 10\,\mathrm{H}(0,0,0,0;x) + 12\,\mathrm{H}(0,1,0,0;x)\\
&\quad - 8\,\mathrm{H}(1,-1,0,0;x) - 8\,\mathrm{H}(1,0,-1,0;x) + 16\,\mathrm{H}(1,0,0,0;x)\\
&\quad + 16\,\mathrm{H}(1,1,0,0;x) + 12\,\mathrm{Li}_4\frac{1}{2} + \frac{\log^4 2}{2} + 2\,\zeta_2\,(12\log 2\,\mathrm{H}(-1;x)\\
&\quad + 12\log 2\,\mathrm{H}(1;x) + 6\,\mathrm{H}(-1,-1;x) - 2\,\mathrm{H}(-1,0;x) - 8\,\mathrm{H}(0,-1;x)\\
&\quad + \mathrm{H}(0,0;x) - 12\,\mathrm{H}(1,-1;x) + 4\,\mathrm{H}(1,0;x) + 3\log^2 2)\\
&\quad - 2\,\zeta_3(5\,\mathrm{H}(-1;x) + 4\,\mathrm{H}(0;x) + 11\,\mathrm{H}(1;x)) - \frac{47\,\zeta_4}{4}\,,
\end{aligned}
$$