Automated calculations of two-loop soft functions in Soft-Collinear Effective Theory

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based on work with Guido Bell, Bahman Dehnadi, Tobias Mohrmann (Siegen) and Jim Talbert (DESY)
Outline

1. *Resummation in SCET*

2. *Universal soft functions - dijet*
   (a) Generic statements
   (b) NLO application
   (c) NNLO application
   (d) SoftSERVE

3. *N-jet soft functions*
Resummation and SCET

- Sudakov logarithms arise from soft and collinear dynamics and can be resummed
- This involves diagrams and soft-collinear expansions
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Diagrams first, or expansion first?
Resummation and SCET

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Diagrams first, or expansion first?

- “direct QCD”
- Coherent branching
- Automation available

- “SCET”
- Factorisation based
- No Automation

[Catani, Trentadue, Turnock, Webber, ‘93]
[Collins, Soper, Sterman, ‘85]
[Bauer, Fleming, Pirjol, Stewart, ’00]

CAESAR/ARES: [Banfi, Salam, Zanderighi, ‘04], [Banfi, McAslan, Monni, Zanderighi, ’14]
If the observable is compatible, SCET matrix elements factorise:

\[ |C_V|^2 \sum_X |\langle 0| \mathcal{O}_{n\bar{n}} |X \rangle|^2 \]

\[ = |C_V|^2 \langle 0 | [\zeta_n^0 W_n^{0,\dagger}] [\zeta_n^0 W_n^{0,\dagger}]^\dagger |0 \rangle \langle 0 | [W_n^0 \zeta_n^0] [W_n^0 \zeta_n^0]^\dagger |0 \rangle \langle 0 | [S_n^\dagger S_n] [S_n^\dagger S_n]^\dagger |0 \rangle \]

\[ \frac{1}{\sigma_0} \frac{d\sigma}{d\tau} = H(Q^2, \mu) \int dp_L^2 \int dp_R^2 \ J(p_L^2, \mu) \ J(p_R^2, \mu) \ S(\tau Q - \frac{p_L^2 + p_R^2}{Q}, \mu) \]

(for dijet thrust)
SCET: Resummation

- In this picture, large logarithms arise from a mismatch of individual factors’ natural scales

- Solution: RG evolve

- We need anomalous dimensions and renormalised functions

- Added difficulty: SCET$_2$ type observables show rapidity divergences
  - Additional analytic regulator required
  - Resummation via collinear anomaly
  - or rapidity renormalisation group

[Becher, Bell, ‘11]
[Becher, Neubert, ‘10]
[Chiu, Jain, Neill, Rothstein, ‘11]
Motivation for Automation

- For NNLL resummation, some two-loop ingredients are required, we look at soft functions

- So far we proceed observable by observable individually, e.g.:
  - Thrust
    - [Becher, Schwartz, '08]
  - C-Parameter
    - [Hoang, Kolodrubetz, Mateu, Stewart, '14]
  - Angularities
    - [Bell, Hornig, Lee, Talbert, WIP]
  - …
  - Threshold Drell-Yan
    - [Becher, Neubert, Xu, ‘07]
  - W/Z/H @ large $p_T$
    - [Becher, Bell, Lorentzen, Marti, ’13,’14]
  - Jet veto
    - [Becher et al. ’13, Stewart et al., ’13]
  - …

- Can this be made more systematic? It’s possible in direct QCD…
Universal dijet soft functions

- The generic form of the dijet soft function we get from the factorisation:

\[ S(\tau, \mu) = \frac{1}{N_c} \sum_X M(\tau, k_i) \text{Tr} |\langle 0 | S_n^+(0) S_n(0) | X \rangle|^2 \]

\[ S_n(x) = P \exp(\alpha_s \int_{-\infty}^{0} n \cdot A_s(x + s n) ds) \]

- The matrix element is independent of the observable and is the source of divergences.

- The measurement function \((M)\) is observable dependent and harmless, e.g.

\[ M_{\text{thrust}}(\tau, \{k_i\}) = \exp \left( -\tau \sum_i \min(k_i^+, k_i^-) \right) \quad \text{(in Laplace space)} \]

- Idea: isolate singularities at each order and calculate the associated coefficient numerically:

\[ S(\tau) \sim 1 + \alpha_s \left\{ \frac{c_2}{e^2} + \frac{c_1}{e^1} + c_0 \right\} + \mathcal{O}(\alpha_s^2) \]
Universal soft functions: NLO

- The virtual corrections are scaleless in dim reg, so the NLO soft function is:

\[
S^{(1)}(\tau, \mu) = \frac{\mu^{2\epsilon}}{(2\pi)^{d-1}} \int \delta(k^2) \theta(k^0) \frac{16\pi\alpha_s C_F}{k_+ k_-} \mathcal{M}(\tau, k) \, d^d k
\]

- To disentangle the soft and collinear divergences we parametrise suitably:

\[
k_- \to \frac{k_T}{\sqrt{y}} \quad k_+ \to k_T \sqrt{y}
\]

- We also must specify the measurement function \(\mathcal{M}\), and assume its form:

\[
\mathcal{M}^{(1)}(\tau, k) = \exp \left( -\tau k_T y^{\frac{n}{2}} f(y, \vartheta) \right)
\]

- The \(k_T\) integration can then be performed analytically, and yields the master formula:

\[
S^{(1)}(\tau, \mu) \sim \Gamma(-2\epsilon) \int_0^\pi d\vartheta \int_0^1 dy \, y^{-1+n\epsilon} f(y, \vartheta)^{2\epsilon}
\]
### Measurement functions: NLO examples

\[ \mathcal{M}^{(1)}(\tau, k) = \exp \left( -\tau k_T y^{\frac{n}{2}} f(y, \vartheta) \right) \]

<table>
<thead>
<tr>
<th>Observable</th>
<th>( n )</th>
<th>( f(y, \vartheta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thrust</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Angularities</td>
<td>( 1 - A )</td>
<td>1</td>
</tr>
<tr>
<td>Recoil-free broadening</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>C-Parameter</td>
<td>1</td>
<td>( \frac{1}{2+y} )</td>
</tr>
<tr>
<td>Threshold Drell-Yan</td>
<td>-1</td>
<td>( 1 + y )</td>
</tr>
<tr>
<td>( W @ \text{large } p_T )</td>
<td>-1</td>
<td>( 1 + y - 2\sqrt{y} \cos \theta )</td>
</tr>
<tr>
<td>( e^+e^- \text{ transverse thrust} )</td>
<td>1 [ \frac{1}{s\sqrt{y}} \left( \sqrt{c \cos \theta + \left( \frac{1}{\sqrt{y}} - \sqrt{y} \right) \frac{s}{2}} \right)^2 + 1 - \cos^2 \theta - \left</td>
<td>c \cos \theta + \left( \frac{1}{\sqrt{y}} - \sqrt{y} \right) \frac{s}{2} \right</td>
</tr>
</tbody>
</table>

- For transverse thrust, \( s = \sin\theta_B, c = \cos\theta_B \), with \( \theta_B = \angle \text{beam axis, thrust axis} \)
Assumptions and classification: NLO

- **Assume**: Exponential function, motivated by Laplace space
  \[ \exp(-\tau \omega(\{k_i\})) = \int_0^\infty d\omega \exp(-\tau \omega) \delta(\omega - \omega(\{k_i\})) \]

- **Assume**: \( \omega \) is linear in mass dimension
  \[ M = \exp(-\tau k_T \hat{f}(y, \vartheta)) \]

- **Classify**: How does the observable behave as \( y \) vanishes?
  \[ M = \exp(-\tau k_T y^{\frac{n}{2}} f(y, \vartheta)) \]

- **Assume**: \( f \) positive and non-vanishing over almost all of phase space

- This is enough to ensure the behaviour of the observable is under control in the critical limits:
  
  **Soft** \( (k_T \to 0) \) ⇒ vanishes, fixed by mass dimension
  **Collinear** \( (y \to 0) \) ⇒ \( f \) finite
Universality: NLO vs. NNLO

NLO:

NNLO:

(a)  (b)  (c)  (d)

(e)  (f)  (g)  (h)
Universality: NLO vs. NNLO

- Consider the double real emission:

\[
S_{RR}^2(\tau) = \frac{\mu^{4\epsilon}}{(2\pi)^{d-2}} \int d^d k \, \delta(k^2) \, \theta(k^0) \int d^d l \, \delta(l^2) \, \theta(l^0) \, |A(k, l)|^2 \, \mathcal{M}(\tau, k, l)
\]

- The matrix elements are no longer nice and easy, see e.g., the $C_F T_F n_f$ color structure:

\[
|A(k, l)|^2 = 128\pi^2 \alpha_s^2 C_F T_F n_f \frac{2k \cdot l (k_- + l_-)(k_+ + l_+)}{(k_- + l_-)^2(k_+ + l_+)^2(2k \cdot l)^2}
\]

- The singularities are partially overlapping, not as easy to extract, but it’s possible
- We then again assume exponential and linearity in mass dimension:

\[
\mathcal{M}^{(2)}(\tau, k, l) = \exp \left(-\tau \, p_T \, y^{n/2} \, F(y, a, b, \vartheta, \vartheta_k, \vartheta_l)\right)
\]

- Why is this enough?
2-loop - Correlated emissions: $C_F C_A, C_F T_f n_f$

- Matrix element divergent in four critical limits:
  - (Global soft) fixed by mass dimension
  - (Individual soft) fixed by IR safety
  - (emissions collinear) fixed by collinear safety
  - (Global “jet-collinear”) unconstrained ➔ Classify!

- Only one unconstrained variable
- Variable definition ensures commuting limits

\[
\mathcal{M}^{(2)}(\tau, k, l) = \exp \left(-\tau \, p_T \, y^{\frac{n}{2}} \, F(y, a, b, \vartheta, \vartheta_k, \vartheta_l)\right)
\]
The completed framework

- Semi-Analytic expressions are available for anomalous dimensions [1805.12414]

- For finite parts we have an implementation using *pySecDec* and a dedicated C++ based program:
  [Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk, Zirke, 1703.09692]

![SoftSERVE](image)

*SoftSERVE* uses the *Cuba* library’s *Divonne* integrator, implements numerical improvements, and has multiprecision variable support [Hahn, hep-ph/0404043]

- Soon™ to be found on *HEPForge*, paper in preparation
## A few results

<table>
<thead>
<tr>
<th>Soft function</th>
<th>(\gamma_1^{CA})</th>
<th>(\gamma_1^{nf})</th>
<th>(c_2^{CA})</th>
<th>(c_2^{nf})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thrust [Kelley et al, ’11]</td>
<td>15.7945 ± 0.00097</td>
<td>3.90981 ± 0.00097</td>
<td>-56.4992 ± 0.00097</td>
<td>43.3902 ± 0.00097</td>
</tr>
<tr>
<td>[Monni et al, ’11]</td>
<td>(15.7945)</td>
<td>(3.90981)</td>
<td>(-56.4990)</td>
<td>(43.3905)</td>
</tr>
<tr>
<td>C-Parameter [Hoang et al, ’14]</td>
<td>15.7947 ± 0.00097</td>
<td>3.90980 ± 0.00097</td>
<td>-57.9754 ± 0.00097</td>
<td>43.8179 ± 0.00097</td>
</tr>
<tr>
<td>[Hoang et al, ’14]</td>
<td>(15.7945)</td>
<td>(3.90981)</td>
<td>[-58.16 ± 0.26]</td>
<td>[43.74 ± 0.06]</td>
</tr>
<tr>
<td>Threshold Drell-Yan [Belitsky, ’98]</td>
<td>15.7946 ± 0.00097</td>
<td>3.90982 ± 0.00097</td>
<td>6.81281 ± 0.00097</td>
<td>-10.6857 ± 0.00097</td>
</tr>
<tr>
<td>W @ large (p_T) [Becher et al, ’12]</td>
<td>15.7947 ± 0.00097</td>
<td>3.90981 ± 0.00097</td>
<td>-2.65034 ± 0.00097</td>
<td>-25.3073 ± 0.00097</td>
</tr>
<tr>
<td>[Becher et al, ’12]</td>
<td>(15.7945)</td>
<td>(3.90981)</td>
<td>(-2.65010)</td>
<td>(-25.3073)</td>
</tr>
<tr>
<td>Transverse Thrust [Becher, Garcia, Piclum, ’15]</td>
<td>-158.278 ± 0.00097</td>
<td>19.3955 ± 0.00097</td>
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</tr>
</tbody>
</table>

\[
\gamma_1 = \gamma_1^{CA} C_F C_A + \gamma_1^{nf} C_F T_F n_f
\]

\[
c_2 = c_2^{CA} C_F C_A + c_2^{nf} C_F T_F n_f + \frac{1}{2} (c_1)^2
\]

- Derived in few minutes to hours on an 8 core desktop machine
- Deviations from analytic results compatible with 1σ error estimate
Results: Soft drop jet mass

- Jet grooming procedures remove radiation from jets to reveal substructure.
- For the soft drop groomer multiple observables have been proposed and factorised in [Frye et al, 1603.09338].
- For Cambridge/Aachen clustering and jet mass as the observable, EVENT2 fits were presented for the anomalous dimension.
- Breaks non-abelian exponentiation.
Extension to N jet directions

- There are now more jet/beam directions -> more Wilson lines:

\[ S(\tau, \mu) = \sum_X M(\tau, k_i) \langle 0 | (S_{n_1} S_{n_2} S_{n_3} \ldots) | X \rangle \langle X | S_{n_1} S_{n_2} S_{n_3} \ldots | 0 \rangle \]

- Tripole and Quadrupole diagrams
  - Assume non-abelian exponentiation: only one tripole (RV)
- Dipole directions are no longer back-to-back
  - Use boost-invariant parametrisation
  - Consequence: transverse space can acquire temporal direction
- more complicated angular integrations
  - 5 angles instead of 3 at NNLO
- external geometry must be sampled
1-Jettiness

Kinematics and Sampling

\[ n_a \cdot n_b = 2 \]
\[ n_a \cdot n_1 = 1 - \cos \theta = n_{a1} \]
\[ n_b \cdot n_1 = 1 + \cos \theta \]

Finite terms for different channels (preliminary)

Dots: [Bell, Dehnadi, Mohrmann, RR, in preparation]
Lines: [Campbell, Ellis, Mondini, Williams, '17]
2-Jettiness

- Kinematics and Sampling

\[ n_a \cdot n_b = n_1 \cdot n_2 = 2 \]
\[ n_a \cdot n_1 = n_b \cdot n_2 = 1 - \cos \theta = n_{a1} \]
\[ n_a \cdot n_2 = n_b \cdot n_1 = 1 + \cos \theta \]

- Some preliminary results - dipoles
2-Jettiness

- **Kinematics and Sampling**

\[
\begin{align*}
\vec{n}_a \cdot \vec{n}_b &= n_1 \cdot n_2 = 2 \\
n_a \cdot n_1 = n_b \cdot n_2 &= 1 - \cos \theta = n_{a1} \\
n_a \cdot n_2 = n_b \cdot n_1 &= 1 + \cos \theta
\end{align*}
\]

- **Some preliminary results - tripole**

Conclusion

- SCET provides an efficient, analytic approach to high-order resummations necessary for precision collider physics.

- We have developed a framework to systematically compute generic NNLO dijet soft functions for wide ranges of observables at lepton and hadron colliders.

- The program(s) based on this framework will soon™ be released into the wild.

- An extension to N-jet observables seems possible, and we have already re-derived a few known results and are working on new ones.
That’s all folks!

Thank you!
2-loop - Uncorrelated emissions: $C_F^2$

- 4 critical limits:
  - (Global soft) fixed by mass dimension
  - (individual soft) fixed by IR safety
  - (one emission “jet-collinear”) unconstrained

- Two “unconstrained” limits
- Worse: Overlapping zeroes:
  \[
  \omega(k, l) = k_T y_k^n f(y_k) + l_T y_l^n f(y_l)
  \]

- Solution: adapt parametrisation for $k_T, l_T$:
  \[
  k_T = q_T \frac{b}{1 + b} \left( \frac{\sqrt{y_l}}{1 + y_l} \right)^n, \quad l_T = q_T \frac{1}{1 + b} \left( \frac{\sqrt{y_k}}{1 + y_k} \right)^n
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- 4 critical limits:
  - (Global soft) fixed by mass dimension
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  - (one emission “jet-collinear”) unconstrained

- Suitable parametrisation solves overlapping limit problem

\[
M^{(2)}(\tau; k, l) = \exp \left( -\tau q_T y_k^{\frac{n}{2}} y_l^{\frac{n}{2}} G(y_k, y_l, b, \vartheta, \vartheta_k, \vartheta_l) \right)
\]
Results: Angularities

- Generalisation of thrust
- Obeys non-abelian observation
- One ingredient for the NNLL’ resummation in [1808.07867]
The parametrisation for the uncorrelated emissions

\[ k_+ = q_T \frac{b}{1 + b} \sqrt{y_k} \left( \frac{\sqrt{y_l}}{1 + y_l} \right)^n \]
\[ l_+ = q_T \frac{1}{1 + b} \sqrt{y_l} \left( \frac{\sqrt{y_k}}{1 + y_k} \right)^n \]
\[ k_- = q_T \frac{b}{1 + b} \frac{1}{\sqrt{y_k}} \left( \frac{\sqrt{y_l}}{1 + y_l} \right)^n \]
\[ l_- = q_T \frac{1}{1 + b} \frac{1}{\sqrt{y_l}} \left( \frac{\sqrt{y_k}}{1 + y_k} \right)^n \]

leads to divergences in \( b, y_k, y_l, q_T \) (analytic)

\[ y_k = \frac{k_+}{k_-} \quad b = \sqrt{\frac{k_+ k_-}{l_+ l_-}} \left( \frac{l_+ + l_-}{k_+ + k_-} \right)^n \]
\[ y_l = \frac{l_+}{l_-} \quad q_T = \sqrt{l_+ l_-} \left( \frac{k_+ + k_-}{\sqrt{k_+ k_-}} \right)^n + \sqrt{k_+ k_-} \left( \frac{l_+ + l_-}{\sqrt{l_+ l_-}} \right)^n \]
Parametrisation correlated

The parametrisation for the correlated emissions

\[
\begin{align*}
  k_+ &= p_T \frac{b}{a + b \sqrt{y}} \\
  l_+ &= p_T \frac{a}{a + b \sqrt{y}} \\
  k_- &= p_T \frac{ab}{1 + ab \sqrt{y}} \\
  l_- &= p_T \frac{1}{1 + ab \sqrt{y}}
\end{align*}
\]

leads to divergences in \( y, b, p_T \) (analytic), and an
overlapping divergence in \( a \rightarrow 1 \) (with transverse angle)

\[
\begin{align*}
  a &= \sqrt{\frac{k_- l_+}{l_- k_+}} = \sqrt{\frac{y_l}{y_k}} \\
  b &= \sqrt{\frac{k_+ k_-}{l_+ l_-}} = \frac{k_T}{l_T} \\
  y &= \frac{k_+ + l_+}{k_- + l_-} \\
  p_T &= \sqrt{(k_+ + l_+)(k_- + l_-)}
\end{align*}
\]