



Three loop QCD corrections to heavy quark form factors

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DESY, Zeuthen

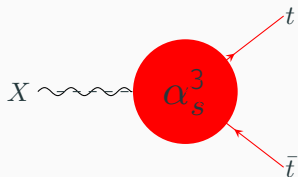
August 30, 2018

DESY-RISC collaboration

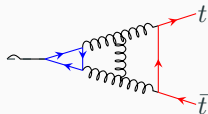
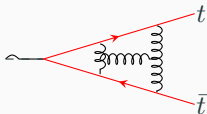
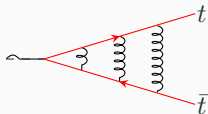
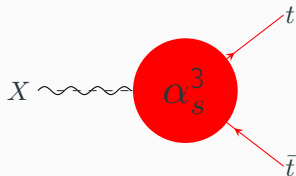
J. Ablinger, J. Blümlein, P. Marquard, C. Schneider

QCD@LHC 2018, Dresden, Germany

Specifically, what we compute



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+ many more



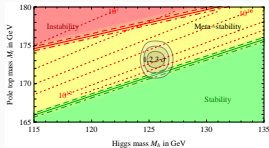
Plan of this talk

1. Preliminary
2. Computational details
3. Renormalization
4. Infrared structure
5. Results
6. Conclusion

The 'top' sector

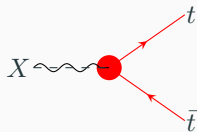
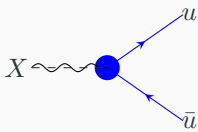
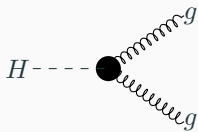
The heaviest SM particle

- ✓ - probes the Higgs sector most
- ✓ - plays unique role in understanding the EW symmetry breaking
- ✓ New physics potential : perfect place to manifest it
- ✓ Does not hadronize - opportunity to study it as a single particle - Spin properties, Interaction vertices, Precise description of mass
- ✓ High precision will be achieved at the future electron-positron colliders
 - In order to match the experimental accuracy, precise predictions are required on the theoretical side as well



Form factors

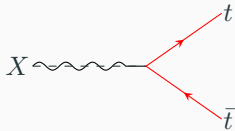
- ✓ The form factors are basic building blocks for many physical quantities
- ✓ They exhibit a universal infrared behavior - leads to information on anomalous dimensions
- ✓ The massive cusp anomalous dimension controls the infrared structure of massive form factors - studying the form factors helps in better understanding of the massive cusp
- ✓ Another important motive is to study high energy behavior of the massive form factors



Preliminary

The process

We consider decay of a color neutral massive particle to a pair of heavy quark of mass m .



Notation

$$X(q) \rightarrow t(q_1) + \bar{t}(q_2)$$

$$X = V, A, S, P$$

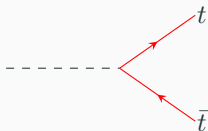
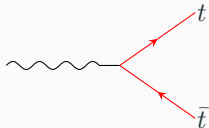
$$s = \frac{q^2}{m^2} = -\frac{(1-x)^2}{x}$$

The general structure

Vector and Axial Vector

$$V: -i\delta_{ij}v_Q\left(\gamma^\mu F_{V,1} + \frac{i}{2m}\sigma^{\mu\nu}q_\nu F_{V,2}\right)$$

$$A: -i\delta_{ij}a_Q\left(\gamma^\mu\gamma_5 F_{A,1} + \frac{1}{2m}q^\mu\gamma_5 F_{A,2}\right)$$



Scalar and Pseudo Scalar

$$-\frac{m}{v}\delta_{ij}\left[s_Q F_S + ip_Q\gamma_5 F_P\right]$$

The form factors are expanded in the strong coupling constant as

$$F_I = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi} \right)^n F_I^{(n)}$$

To obtain $F_I^{(n)} \Rightarrow$ appropriate projector on the amplitudes

$$P_{V,i} = \frac{i}{v_Q} \frac{\not{q}_2 - m}{m} \left(\gamma_\mu g_{V,i}^1 + \frac{1}{2m} (q_{2\mu} - q_{1\mu}) g_{V,i}^2 \right) \frac{\not{q}_1 + m}{m},$$

$$P_{A,i} = \frac{i}{a_Q} \frac{\not{q}_2 - m}{m} \left(\gamma_\mu \gamma_5 g_{A,i}^1 + \frac{1}{2m} (q_{1\mu} + q_{2\mu}) \gamma_5 g_{A,i}^2 \right) \frac{\not{q}_1 + m}{m},$$

$$P_S = \frac{v}{2ms_Q} \frac{\not{q}_2 - m}{m} (g_S) \frac{\not{q}_1 + m}{m}, \quad P_P = \frac{v}{2mp_Q} \frac{\not{q}_2 - m}{m} (i\gamma_5 g_P) \frac{\not{q}_1 + m}{m},$$

$g_I \equiv g_I(s, d)$ and are determined by applying the projectors on the generic Lorentz structure.

NLO and beyond NLO

$F_{V,i}^{(1)}, F_{A,i}^{(1)}$ [Arbuzov, Bardin, Leike '92; Djouadi, Lampe, Zerwas '95]

$F_S^{(1)}, F_P^{(1)}$ [Braaten, Leveille '80; Sakai '80; Drees, Hikasa '90]

$F_{V,i}^{(2)}, F_{A,i}^{(2)}$ [Altarelli, Lampe '93; Ravindran, van Neerven '98; Catani, Seymour '99]

$F_S^{(2)}, F_P^{(2)}$ [Gorishnii et. al. '91; Chetyrkin, Kwiatkowski '95; Harlander, Steinhauser '97]

NNLO

$F_I^{(2)}$ [Bernreuther, Bonciani, Gehrmann, Heinesch, Leineweber, Mastrolia, Remiddi '04,'05]

$F_{V,i}^{(2)}(\mathcal{O}(\epsilon))$ [Gluzo, Mitov, Moch, Riemann '09]

$F_I^{(2)}(\mathcal{O}(\epsilon^2))$ [Ablinger, Behring, Blümlein, Falcioni, Freitas, Marquard, Rana, Schneider '17]

NNNLO

$F_{V,i}^{(3)}|_{\text{large } N}$ [Henn, Smirnov, Smirnov, Steinhauser '16]

$F_{V,i}^{(3)}|_{n_l \text{ contributions}}$ [Lee, Smirnov, Smirnov, Steinhauser '18]

$F_{V,i}^{(4)}|_{\text{partial poles in large } Q^2}$ [Ahmed, Henn, Steinhauser '17]

Goal

In this talk, we present

- *Automatizing the technique to compute the first order factorizable system of differential equations.*
- *Results for the master integrals which contribute to color-planar diagrams and full light quark dependence.*
- *Color-planar (N_C^3) and complete light quark (n_l) contributions for $F_{V,i}^{(3)}$, $F_{A,i}^{(3)}$, $F_S^{(3)}$ and $F_P^{(3)}$.*

J. Ablinger, J. Blümlein, P. Marquard, N. Rana and C. Schneider,
Heavy Quark Form Factors at Three Loops in the Planar Limit,
[Phys.Lett. B782 \(2018\) 528-532](#) (arXiv:1804.07313 [hep-ph]).

J. Ablinger, J. Blümlein, P. Marquard, N. Rana and C. Schneider,
DESY 18-053.

* A parallel computation in [JHEP 1805 \(2018\) 187](#) (arXiv:1804.07310 [hep-ph])

Computational details

The generic procedure

$$d = 4 - 2\epsilon$$

- Diagrammatic approach -> QGRAF [Nogueira] to generate diagrams
- FORM [Vermaseren] for algebraic manipulation :
Lorentz, Dirac and Color [Ritbergen, Schellekens, Vermaseren] algebra
- Decomposition of the dot products to obtain scalar integrals

$$\frac{2l \cdot p}{l^2(l-p)^2} = \frac{l^2 - (l-p)^2 + p^2}{l^2(l-p)^2} = \frac{1}{(l-p)^2} - \frac{1}{l^2} + \frac{p^2}{l^2(l-p)^2}$$

- Identity relations among scalar integrals : IBPs
- Algebraic linear system of equations relating the integrals



Master integrals (MIs)

- **Crusher** [Marquard, Seidel] for reduction to master integrals
- Computation of MIs : *Differential eqns.*

Integration by parts identities (IBPs)

- IBPs : reduce the number of integrals to compute [Tkachov, Chetyrkin]

Generalization of Gauss's theorem in d dimension

Within dimensional regularization, all integrals in d dimension are well-defined and convergent \Rightarrow integrand must be zero at boundary

$$\int \prod_{i=1}^l \mathcal{D}^{d_i} \frac{\partial}{\partial l_j^\mu} \left(\frac{v^\mu}{D_1^{n_1} \dots D_m^{n_m}} \right) = 0 \quad \Big|_{v \equiv l, p}$$

Example : Consider

$$\mathcal{I}(n) = \int \frac{d^d l}{(2\pi)^{d/2}} \frac{1}{l^2 (l-p_1)^2 ((l-p_1-p_2)^2)^n}$$

The identity for $v \equiv l$ gives a recursion relation

$$\mathcal{I}(n+1) = (-1)^n \frac{(d-(n+3)) \dots (d-4)(d-3)}{n! s^n} \mathcal{I}(1)$$

- $\sim 10^6$ identities for our process
- So the problem boils down to solve an algebraic linear system of equations relating the integrals - several programs (AIR, FIRE, Kira, LiteRed, Reduze ...) available - we use CRUSHER [Marquard, Seidel]

Computing the master integrals

A scalar integral can be expressed as

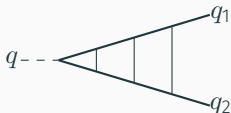
$$J(\nu_1, \dots, \nu_n) = \left((4\pi)^{2-\epsilon} e^{\epsilon\gamma_E} \right)^3 \int \frac{d^d l_1}{(2\pi)^d} \frac{d^d l_2}{(2\pi)^d} \frac{d^d l_3}{(2\pi)^d} \frac{1}{D_1^{\nu_1} \dots D_n^{\nu_n}}$$

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For example



$$\begin{aligned} & l_1^2 - m^2, (l_1 - q)^2 - m^2, (l_1 - l_2)^2, \\ & l_2^2 - m^2, (l_2 - q)^2 - m^2, (l_2 - l_3)^2, \\ & l_3^2 - m^2, (l_3 - q)^2 - m^2, (l_1 - q_1)^2 \end{aligned}$$

To evaluate the integral \rightarrow Feynman parametrization, Mellin-Barnes ...

We use the method of differential equations.

Using differential equations

The integral is a function of d , q^2 and m^2 .

$$\frac{q^2}{m^2} = -\frac{(1-x)^2}{x}$$

$$J(1, 1, 1, 1, 1, 1, 1, 1) \equiv f(d, q^2, m^2) \equiv f(d, x)$$

The idea is to obtain a differential eqn. for the integral *w.r.t.* x and solve it.

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$$\frac{d}{dx} J_i = \text{some combinations of integrals}$$

⇓ IBP identities

$$= \sum_j c_{ij} J_j$$

c_{ij} 's are rational function of d and x .

Using differential equations

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$$d_x \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \end{bmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix}$$

$$d_x \mathbb{J} = \mathbb{A}(d, x) \mathbb{J}$$

Canonical Basis

The choice of MIs is not unique!

The idea is to find a basis such that the deqs decouple as $\epsilon \rightarrow 0$

[Kotikov; Henn]

$$d_x \tilde{\mathbb{J}} = \epsilon \tilde{\mathbb{A}}(x) \tilde{\mathbb{J}}$$

Now one can perform Laurent series expansion in ϵ and trivially solve the deqs as at each order the homogeneous solutions are constants.

Algorithm to find such basis by [Lee]
and implementation in `Epsilon`, `Canonica` ...

Drawback : Finding such a basis is not always possible.

Generic Basis

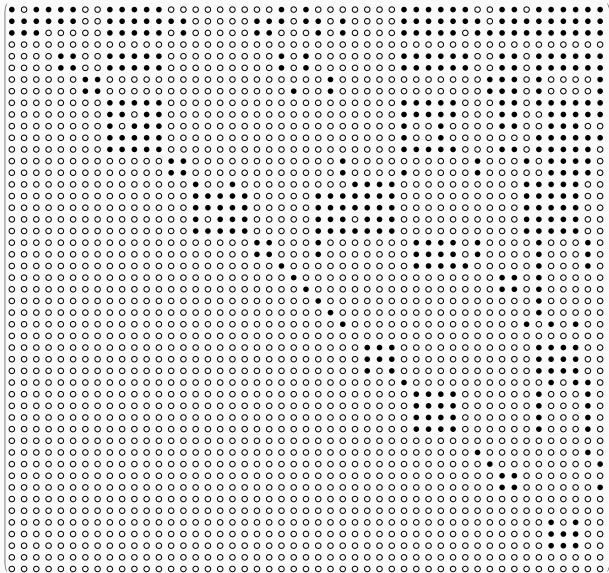
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To solve such a system, it would be best to organize it in such a way that it diagonalizes, or at least it takes a block-triangular form.

$$d_x \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ 0 & \bullet & \bullet & \bullet & \cdots & \bullet \\ 0 & \bullet & \bullet & \bullet & \cdots & \bullet \\ 0 & 0 & 0 & \bullet & \cdots & \bullet \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \bullet \end{bmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix}$$



Let's consider the 12th blob from below

$$\frac{d}{dx} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} + \begin{pmatrix} R_1(\epsilon, x) \\ R_2(\epsilon, x) \\ R_3(\epsilon, x) \end{pmatrix},$$

$$c_{11} = \frac{(7 + 6x + 7x^2 - 2d(1 + x + x^2))}{x(1 + x)^2},$$

$$c_{12} = \frac{(-4 + d)(-10 + 3d)}{2(-3 + d)^2(1 + x)^2},$$

$$c_{13} = \frac{(d^2(15 + 8x + 15x^2) + 8(20 + 9x + 20x^2) - 2d(49 + 24x + 49x^2))}{4(-3 + d)^2x(1 + x)^2}, \dots$$

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For these topologies, the integrals can have, at max, a cubic pole in ϵ .

$$J_i = \frac{1}{\epsilon^3} J_i^{-3} + \frac{1}{\epsilon^2} J_i^{-2} + \frac{1}{\epsilon} J_i^{-1} + J_i^0 + \epsilon J_i^1 + \dots$$

Series expansion and compare each order of ϵ !

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Each order in ϵ -expansion gives a much simpler form

$$\frac{d}{dx} \begin{pmatrix} J_1^{-3} \\ J_2^{-3} \\ J_3^{-3} \end{pmatrix} = \begin{bmatrix} \frac{1}{x} + \frac{2}{1-x} & 0 & \frac{1}{1+x} - \frac{2}{x} - \frac{3}{1-x} \\ -\frac{1}{x} + \frac{2}{1+x} & \frac{1}{1+x} - \frac{1}{x} - \frac{1}{1-x} & \frac{1}{x} - \frac{2}{1+x} \\ \frac{1}{x} + \frac{2}{1-x} & 0 & \frac{1}{1+x} - \frac{2}{x} - \frac{3}{1-x} \end{bmatrix} \begin{pmatrix} J_1^{-3} \\ J_2^{-3} \\ J_3^{-3} \end{pmatrix} + \begin{pmatrix} R_1^{-3}(x) \\ R_2^{-3}(x) \\ R_3^{-3}(x) \end{pmatrix},$$

Algorithm - I : the homogeneous part

- It boils down to solving a system of linear first order diff. eqns.

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An example of a 2×2 coupled system

$$\frac{d}{dx} \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} + \begin{pmatrix} R_1(\varepsilon, x) \\ R_2(\varepsilon, x) \end{pmatrix},$$

$$J_2 = p'(x) \frac{dJ_1}{dx} + q'(x) J_1 + r'(x)$$

$$\frac{d^2 J_1}{dx^2} + p(x) \frac{dJ_1}{dx} + q(x) J_1 = r(x)$$

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- But, it gives a complicated decoupled equation.

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Note that, the inverse operation is trivial!
- The classical/naive method to achieve this uncoupling is the cyclic vector algorithm
- But, it gives a complicated decoupled equation.
- Few smarter uncoupling algorithms

Zürcher
Incomplete Zürcher
Abramov and Zima

Implemented in **OreSys!**

The 12th blob from below

Using one of the algorithms to ϵ^{-3} part

$$\left[\frac{d^2}{dx^2} - \frac{2}{1-x} \frac{d}{dx} + \left(\frac{2}{x} - \frac{2}{1+x} - \frac{2}{(1+x)^2} \right) \right] I_1^{-3}(x) = r_1^{-3}(x)$$
$$\left[\frac{d}{dx} - \left(\frac{1}{1-x} + \frac{1}{x} - \frac{1}{1+x} \right) \right] I_2^{-3}(x) = r_2^{-3}(x)$$

The 12th blob from below

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Solving for homogeneous part of each diff. eqns.

$$y_1(x) = \frac{x}{1-x^2}, \quad y_2(x) = 1 - \frac{2x}{1-x^2} H(0, x); \quad \mu(x) = \frac{1}{x} - x;$$

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$$\left[\frac{d^2}{dx^2} - \frac{2}{1-x} \frac{d}{dx} + \left(\frac{2}{x} - \frac{2}{1+x} - \frac{2}{(1+x)^2} \right) \right] I_1^{-3}(x) = r_1^{-3}(x)$$
$$\left[\frac{d}{dx} - \left(\frac{1}{1-x} + \frac{1}{x} - \frac{1}{1+x} \right) \right] I_2^{-3}(x) = r_2^{-3}(x)$$

Solving for homogeneous part of each diff. eqns.

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Next, use variation of constant to obtain the solution

$$I_1^{-3}(x) = y_1(x) \left[C_1 - \int dx \frac{r_1^{-3}(x) y_2(x)}{W(y_1, y_2)} \right] + y_2(x) \left[C_2 + \int dx \frac{r_1^{-3}(x) y_1(x)}{W(y_1, y_2)} \right]$$
$$I_2^{-3}(x) = \frac{1}{\mu(x)} \left(C_3 + \int dx \mu(x) r_2^{-3}(x) \right)$$

$W(y_1, y_2)$ is the Wronskian.

Finally $J_i^{-3}(x) = f(\{I_1^{-3}(x), I_2^{-3}(x)\})!$

Algorithm - II : the nonhomogeneous part

- Structure of homogeneous part is same at each order in ϵ -expansion
- Hence the homogeneous solutions and uncoupling procedure are similar for each order
- Start with the ϵ^{-3} part
- Find the best uncoupling for the sub-system and solve for the corresponding homogeneous solutions
- Now at each order in ϵ , find the nonhomogeneous parts keeping the uncoupling structure fixed
- Solve order by order in ϵ using variation of constant

Automatization using

Sigma [Schneider], **OreSys** [Gerhold]

and **HarmonicSums** [Ablinger, Blümlein, Schneider]

The results are obtained in terms of HPLs and Cyclotomic HPLs.

Boundary conditions

Boundary conditions are fixed by imposing regularity of the integrals in the limit of vanishing space-like momentum $q^2 \rightarrow 0$ *i.e.* $x \rightarrow 1$.

- In the planar limit, the integrals are regular in $y = 1 - x$, and hence can be expanded as follows

$$J_i(y) = \sum_{n=0}^{\infty} \sum_{j=-3}^r \epsilon^j C_{i,j}(n) y^n$$

- $q^2 \rightarrow 0$ reduces them to known two-point integrals, providing $C_{i,j}(0)$
- The differential equations now can be solved to obtain $C_{i,j}(n)$ for sufficiently high order (n) in y

Iterated integrals and Harmonic polylogarithms (HPLs)

Given a set of integration kernels $K_i(t)$, one can define

$$\mathcal{I}(i_n, \dots, i_1, x) = \int_{x_0}^x K_{i_n}(t) \mathcal{I}(i_{n-1}, \dots, i_1, t) dt$$

Classic example is $\text{Li}_n(x)$. We have five kernels K_m

$$\left\{ -1, 0, 1, \{6, 0\}, \{6, 1\} \right\} \equiv \left\{ \frac{1}{1+x}, \frac{1}{x}, \frac{1}{1-x}, \frac{1}{1-x+x^2}, \frac{x}{1-x+x^2} \right\}$$

and correspondingly we define the HPLs as

$$H(m_n, \dots, m_1, x) = \int_0^x K_{m_n}(t) H(m_{n-1}, \dots, m_1, t) dt$$

Some important properties :

Shuffle algebra, Scaling invariance and integration-by-parts identities

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New!

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Some important properties :

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Renormalization

We consider a hybrid scheme for UV renormalization.

Heavy quark mass and wave function ($Z_{m,\text{OS}}, Z_{2,\text{OS}}$): **On-shell**
QCD strong coupling constant (Z_{a_s}): **\overline{MS}**

The renormalization of F_I for these topologies, is straightforward

$$\begin{aligned} F_{V,i} &= Z_{2,\text{OS}} \hat{F}_{V,i} & F_S &= Z_{m,\text{OS}} Z_{2,\text{OS}} \hat{F}_S \\ F_{A,i} &= Z_{2,\text{OS}} \hat{F}_{A,i} & F_P &= Z_{m,\text{OS}} Z_{2,\text{OS}} \hat{F}_P \end{aligned}$$

where \hat{F}_I contains the counterterms from mass renormalization.

For Axial-Vector and Pseudo-Scalar currents, for these topologies *i.e.* the non-singlet case, both the γ_5 -matrices appear in the same chain of Dirac matrices, which allows us to use anti-commuting γ_5 in D space-time dimensions.

Ward identities

Chiral Ward identity

$$q_\mu \Gamma_A^{\mu, ns} = 2m \Gamma_P^{ns}$$

$$2F_{A,1}^{ns} + \frac{1}{2} \left(-\frac{(1-x)^2}{x} \right) F_{A,2}^{ns} = 2m F_P^{ns}$$

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Anomalous Ward identity

$$q_\mu \Gamma_A^{\mu, s} = 2m \Gamma_P^s - i \frac{\alpha_s}{4\pi} T_F \langle G\tilde{G} \rangle_Q$$

$\langle G\tilde{G} \rangle_Q$ denotes the truncated matrix element of the gluonic operator $G\tilde{G}$ between the vacuum and an on-shell heavy quark pair ($Q\bar{Q}$).

Infrared structure

The infrared singularities factorize as a multiplicative factor

[Becher, Neubert '09]

$$F_I(\epsilon, x) = Z(\epsilon, x, \mu) F_I^{fin}(x, \mu)$$

$Z(\epsilon, x, \mu)$ is universal/independent of current

$F_I^{fin}(x, \mu)$ is finite as $\epsilon \rightarrow 0$

Renormalization group evolution of $Z(\epsilon, x, \mu)$ provides

$$\begin{aligned} Z(\epsilon, x, \mu) = & 1 + \left(\frac{\alpha_s}{4\pi}\right) \left[\frac{\Gamma_0}{2\epsilon}\right] + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[\frac{1}{\epsilon^2} \left(\frac{\Gamma_0^2}{8} - \frac{\beta_0 \Gamma_0}{4}\right) + \frac{1}{\epsilon} \left(\frac{\Gamma_1}{4}\right)\right] \\ & + \left(\frac{\alpha_s}{4\pi}\right)^3 \left[\frac{1}{\epsilon^3} \left(\frac{\Gamma_0^3}{48} - \frac{\beta_0 \Gamma_0^2}{8} + \frac{\beta_0^2 \Gamma_0}{6}\right) + \frac{1}{\epsilon^2} \left(\frac{\Gamma_0 \Gamma_1}{8} - \frac{\beta_1 \Gamma_0}{6}\right) + \frac{1}{\epsilon} \left(\frac{\Gamma_2}{6}\right)\right] \end{aligned}$$

Γ_n is the n^{th} order massive cusp anomalous dimension.

[Korchemsky, Radyushkin '87, '92; Grozin, Henn, Korchemsky, Marquard '14, '15]

Results

Results & Checks

- We have computed the master integrals to obtain color-planar and full light quark contributions of massive form factors.
- We have obtained corresponding results for aforementioned currents

$$\overline{F_{V,1}^{(3)}, F_{V,2}^{(3)}, F_{A,1}^{(3)}, F_{A,2}^{(3)}, F_S^{(3)}, F_P^{(3)}}$$

- ✓ $F_{V,i}^{(3)}$ match with the results from Henn *et al.* (color-planar limit)
- ✓ Complete light quark contributions of $F_{V,i}^{(3)}$ match with the results from Lee *et al.*
- ✓ We agree the results for all other currents with Lee *et al.*
- ✓ The results reproduce the universal infrared structure
- ✓ Chiral Ward identity is satisfied between $F_{A,i}^{(3)}$ and $F_P^{(3)}$

Form factors at various kinematical regions

Low energy region $q^2 \ll m^2$ or $x \rightarrow 1$

We redefine x as $x = e^{i\phi}$ and expand around $\phi = 0$.

Note that, for $\phi = 0$ $F_{V,1} = 1$, $F_{V,2} =$ Anomalous magnetic moment

High energy region $q^2 \gg m^2$ or $x \rightarrow 0$

In the massless limit ($x = 0$),

- the chirality flipping form factors $F_{V,2}$ & $F_{A,2}$ vanishes.
- $F_{V,1}$ is equal to $F_{A,1}$, as expected
- F_S is equal to F_P too

Threshold region $q^2 \sim 4m^2$ or $x \rightarrow -1$

We define $\beta = \sqrt{1 - \frac{4m^2}{q^2}}$ and expand around $\beta = 0$.

Near threshold, the form factors provide the total cross-section.

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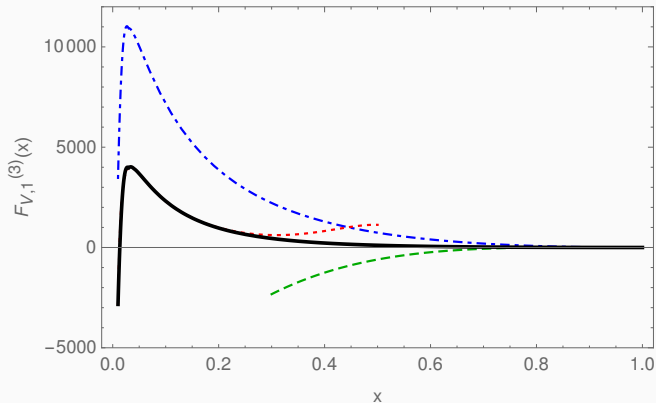
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Dotted : large q , *Dashed* : small q

$$N_C = 3, n_l = 5$$

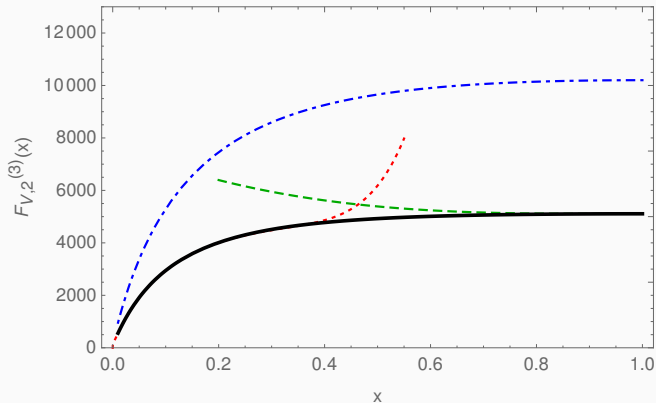
$\mathcal{O}(\epsilon^0)$ part of $F_{V,1}^{(3)}$



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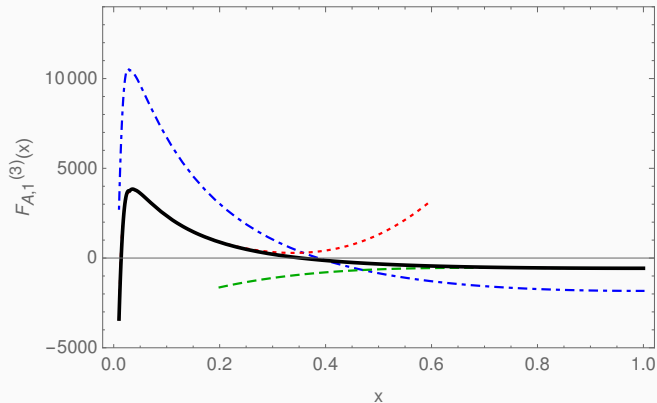
$\mathcal{O}(\epsilon^0)$ part of $F_{V,2}^{(3)}$



Dotted : large q , *Dashed* : small q

$$N_C = 3, n_l = 5$$

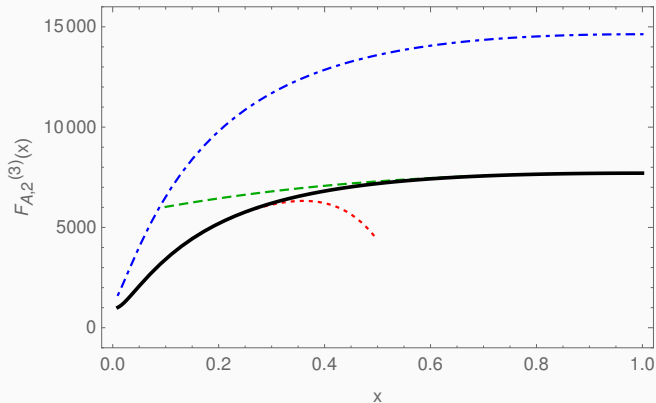
$\mathcal{O}(\epsilon^0)$ part of $F_{A,1}^{(3)}$



Dotted: large q , *Dashed*: small q

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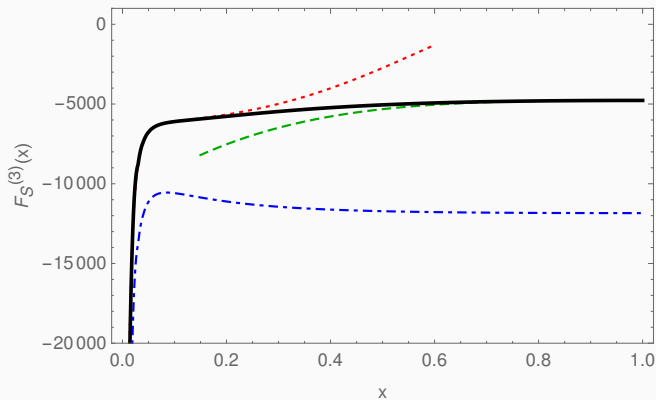
$\mathcal{O}(\epsilon^0)$ part of $F_{A,2}^{(3)}$



Dotted : large q , *Dashed* : small q

$$N_C = 3, n_l = 5$$

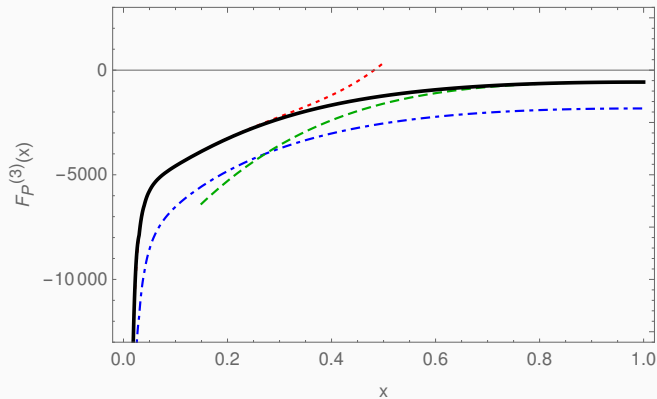
$\mathcal{O}(\epsilon^0)$ part of $F_S^{(3)}$



Dotted: large q , *Dashed*: small q

$$N_C = 3, n_l = 5$$

$\mathcal{O}(\epsilon^0)$ part of $F_P^{(3)}$



Conclusion

Summary

- We have computed the master integrals which appear in the three-loop massive form factors in the color-planar limit and for full light quark (n_l) contributions.
- We have solved the first order factorizable system of differential equations by a new method.
- The use of **OreSys** to uncouple the differential equations has made it possible to automatize the procedure.
- The method applies to all first order factorizable systems in any basis.
- The alphabet contains sixth root of unity letters in the real representation.
- Finally, we have obtained the color-planar and complete light quark three-loop contributions to the heavy quark form factors for vector, axial-vector, scalar and pseudo-scalar currents.

- For non-planar contributions, large differential equations \sim 50 MBs!
- Also first-order non-factorizable differential equations appear!
 \Rightarrow Solutions are **Elliptic integrals**
- A proper generalization for iteration of elliptic integrals is needed!

Thank You!