Analytic calculation of multi-loop Feynman integrals

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Perturbative calculation of cross sections @ LHC

Fixed order expansion

power-counting: $\alpha_s^2 \sim \alpha$ (i.e. $70^{-1} \sim 137^{-1}$)

$$\sigma = \sigma_0 \qquad \text{LO}$$

$$+ \alpha_s \, \sigma_{\alpha_s} + \alpha_s^2 \, \sigma_{\alpha_s^2} + \alpha_s^3 \, \sigma_{\alpha_s^3} + \dots \qquad \text{QCD}$$

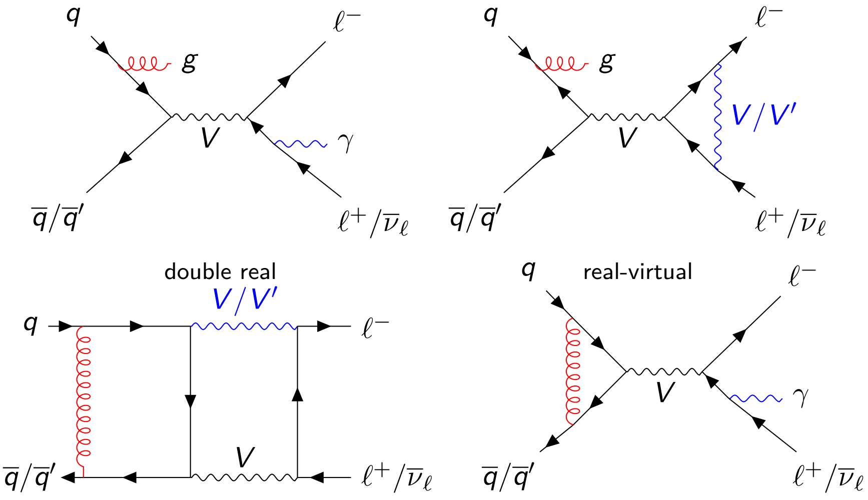
$$+ \alpha \, \sigma_{\alpha} + \alpha^2 \sigma_{\alpha^2} + \alpha^3 \, \sigma_{\alpha^3} + \dots \qquad \text{EW}$$

$$+ \alpha \, \alpha_s \, \sigma_{\alpha\alpha_s} + \alpha \, \alpha_s^2 \, \sigma_{\alpha\alpha_s^2} + \dots \qquad \text{EW} \times \text{QCD}$$

NLO NNLO N3LO

- ✓ QCD NLO — fully differential, matched to PS, automated
- ✓ EW NLO — fully differential, matched to PS, towards automation
- ☺ QCD NNLO — impressive progress (e.g. $t\bar{t}$, VV', Hj, Vj, VBF, single-t)
- ☺ QCD N3LO — $gg \to H$ differential approximated / total σ analytic
- ☹ QCD×EW NNLO — Drell-Yan (“pole approx.”), work towards $gg \to H$

E.g. NNLO QCD×EW corrections to Drell-Yan



double real

real-virtual

double virtual

real-virtual

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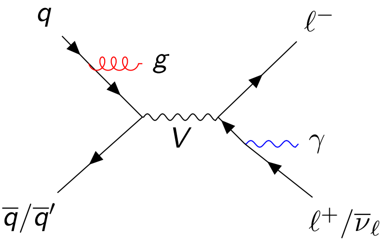
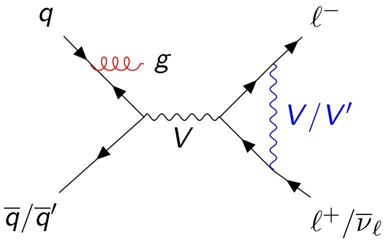
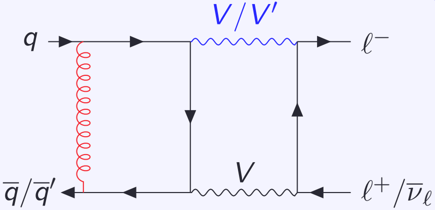


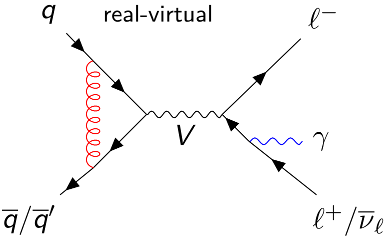
double real



real-virtual



double virtual



real-virtual

Integral families

1. Convert Feynman integrals to scalar integrals already many simplifications
   - Squared summed/averaged amplitude $\sum_{\text{spin,pol}} |\mathcal{M}|^2$
   - Tensor projectors $P_i^{\mu_1\cdots\mu_n}$ e.g. $\mathcal{M} = \mathcal{M}^{\mu_1\cdots\mu_n}\epsilon_{\mu_1}\ldots\epsilon_{\mu_n}$, with $\mathcal{M}^{\mu_1\cdots\mu_n} = \sum_i A_i T_i^{\mu_1\cdots\mu_n}$. Then $A_i = P_i^{\mu_1\cdots\mu_n}\mathcal{M}_{\mu_1\ldots\mu_n}$
   - Helicity amplitudes, $|\mathcal{M}|^2 = \sum_{\vec{h}} |\mathcal{M}_{\vec{h}}|^2$
2. Express scalar products in terms of inverse propagators:

$$\int_k \frac{-2k\cdot p}{(k^2-m^2)[(k-p)^2-m^2][(k-q)^2-m^2]} = \int_k \frac{[(k-p)^2-m^2]-(k^2-m^2)-p^2}{(k^2-m^2)[(k-p)^2-m^2][(k-q)^2-m^2]}$$

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$$- \int_k \frac{(k^2-m^2)}{(k^2-m^2)[(k-p)^2-m^2][(k-q)^2-m^2]}$$

$$- \int_k \frac{p^2}{(k^2-m^2)[(k-p)^2-m^2][(k-q)^2-m^2]}$$

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$$\int_k \frac{-2k\cdot p}{(k^2-m^2)[(k-p)^2-m^2][(k-q)^2-m^2]} = \int_k \frac{1}{(k^2-m^2)[(k-p)^2-m^2]^0[(k-q)^2-m^2]}$$
$$-\int_k \frac{1}{(k^2-m^2)^0[(k-p)^2-m^2][(k-q)^2-m^2]}$$
$$-\int_k \frac{p^2}{(k^2-m^2)[(k-p)^2-m^2][(k-q)^2-m^2]}$$

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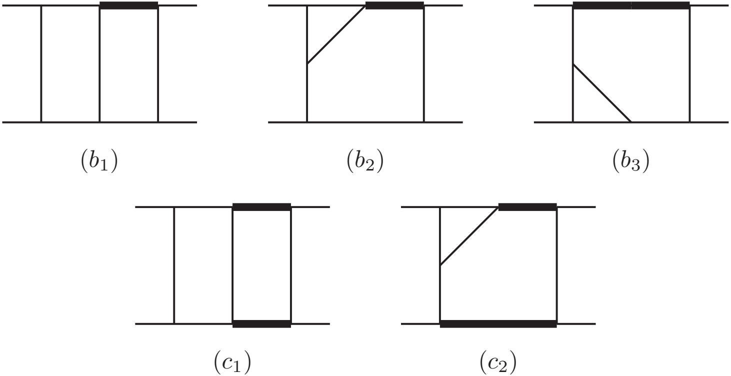
Solving scalar integrals ⇔ solving integral family

$$\int \frac{d^d k}{(k^2 - m^2)^a[(k-p)^2 - m^2]^b[(k-q)^2 - m^2]^c}$$

Possibly add suitable “auxiliary denominators” to close the basis

E.g. Drell-Yan four-point @ NNLO QCD×EW

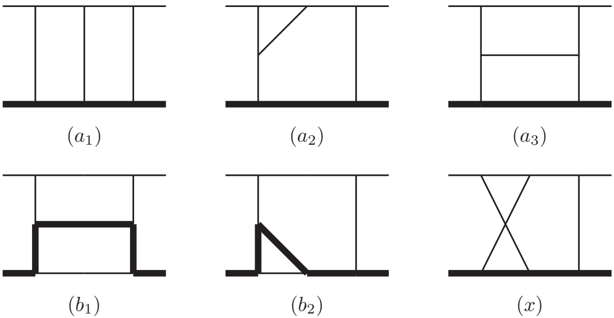


[Bonciani, Mastrolia, Schubert, **DV** 16]

see also [von Manteuffel, Schabinger 17] for the 1-mass families b_i

thick line = W/Z boson line

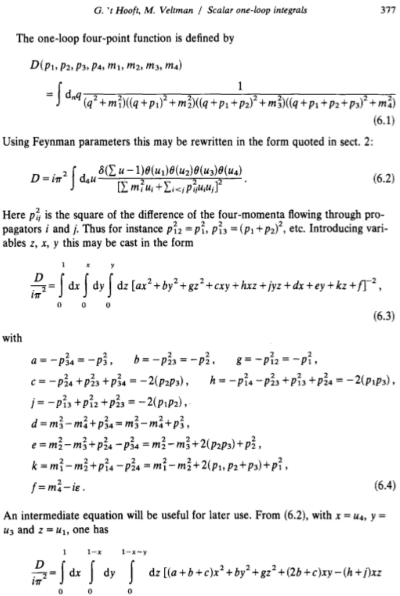
E.g. $\mu e \to \mu e$ four-point @ NNLO QED ($\leftrightarrow t\bar{t}$)



[Mastrolia, Passera, Primo, Schubert 17; Mastrolia, Laporta, Primo, Schubert, **DV** 18]

$m_e = 0,\ m_\mu \neq 0$ (thick) See Uli Schubert's talk

The art of computing Feynman integrals



G. 't Hooft, M. Veltman / Scalar one-loop integrals 377

The one-loop four-point function is defined by

$$D(p_1, p_2, p_3, p_4, m_1, m_2, m_3, m_4) = \int d_n q \frac{1}{(q^2+m_1^2)((q+p_1)^2+m_2^2)((q+p_1+p_2)^2+m_3^2)((q+p_1+p_2+p_3)^2+m_4^2)} \quad (6.1)$$

Using Feynman parameters this may be rewritten in the form quoted in sect. 2:

$$D = i\pi^2 \int d_4 u \frac{\delta(\sum u - 1)\theta(u_1)\theta(u_2)\theta(u_3)\theta(u_4)}{[\sum m_i^2 u_i + \sum_{i<j} p_{ij}^2 u_i u_j]^2}. \quad (6.2)$$

Here p_{ij}^2 is the square of the difference of the four-momenta flowing through propagators i and j. Thus for instance $p_{12}^2 = p_1^2$, $p_{13}^2 = (p_1+p_2)^2$, etc. Introducing variables z, x, y this may be cast in the form

$$\frac{D}{i\pi^2} = \int_0^1 dx \int_0^x dy \int_0^y dz \, [ax^2+by^2+gz^2+cxy+hxz+jyz+dx+ey+kz+f]^{-2}, \quad (6.3)$$

with

$$\begin{aligned}
&a = -p_{34}^2 = -p_3^2, \quad b = -p_{23}^2 = -p_2^2, \quad g = -p_{12}^2 = -p_1^2,\\
&c = -p_{24}^2 + p_{23}^2 + p_{34}^2 = -2(p_2p_3), \quad h = -p_{14}^2 - p_{23}^2 + p_{13}^2 + p_{24}^2 = -2(p_1p_3),\\
&j = -p_{13}^2 + p_{12}^2 + p_{23}^2 = -2(p_1p_2),\\
&d = m_3^2 - m_4^2 + p_{34}^2 = m_3^2 - m_4^2 + p_3^2,\\
&e = m_2^2 - m_3^2 + p_{24}^2 - p_{34}^2 = m_2^2 - m_3^2 + 2(p_2p_3) + p_2^2,\\
&k = m_1^2 - m_2^2 + p_{14}^2 - p_{24}^2 = m_1^2 - m_2^2 + 2(p_1, p_2+p_3) + p_1^2,\\
&f = m_4^2 - i\varepsilon.
\end{aligned} \quad (6.4)$$

An intermediate equation will be useful for later use. From (6.2), with $x = u_4$, $y = u_3$ and $z = u_1$, one has

$$\frac{D}{i\pi^2} = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \, [(a+b+c)x^2+by^2+gz^2+(2b+c)xy-(h+j)xz$$

←1-loop box integral from classic 't Hooft and Veltman paper: much has changed since the old days

- Automation (1-loop “solved”)
- New methods (Mellin Barnes, unitarity-based, differential equations, difference equations, sector decomposition)
- *Divide and conquer* approach: exploit “algebraic redundancies” and reduce the number of integrals to be computed
- New functions ($\log, \mathrm{Li}_2 \to \log, \mathrm{Li}_n$, elliptic...)

Differential equations for Master Integrals [Kotikov 91; Remiddi 97]

Crucial ingredient: Integration by parts identities

Loop integrals in d dimensions satisfy linear identities (IBPs + other). E.g.

$$\int \frac{d^d k}{(k^2-m^2)^2[(k-p)^2-m^2]} \equiv \int \frac{d^d k}{D_1^2 \, D_2}$$

$$= \frac{d-3}{(p^2-4m^2)} \int \frac{d^d k}{D_1 \, D_2} - \frac{d-2}{2m^2(p^2-4m^2)} \int \frac{d^d k}{D_1}$$

Only a finite number of them are independent (hence MIs)! ☺

- Public IBP codes: AIR [Anastasiou, Lazopoulos 04], FIRE [Smirnov 08], REDUZE [Studerus 10; + von Manteuffel 12], LiteRed [Lee 12], Kira [Maierhöfer et al 17]
- Take derivatives wrt external p_{ij}^2's and m_i^2's → use IBPs → obtain system of linear differential equations for the MIs (ODEs or PDEs)

$\mathbf{F} \equiv$ vector of MIs

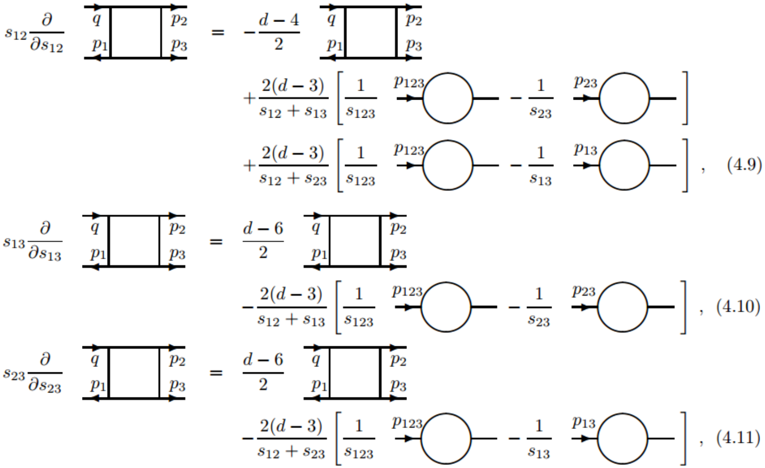
$\mathbb{K} \equiv$ coeff. matrix

$$d\mathbf{F}(\vec{x},\epsilon) = \mathbb{K}(\vec{x},\epsilon)\,\mathbf{F}(\vec{x},\epsilon)$$

$\epsilon = (4-d)/2$

Example: PDEs for $\gamma^* \to 3j$ [Gehrmann, Remiddi 99]

$s_{23} \equiv s,\ s_{13} \equiv t,\ s_{12} \equiv u,$
$s_{123} \equiv s + t + u$



$+$ other equations for the bubbles, *not* involving the boxes

$\Rightarrow$ hierarchical structure: solve simplest, plug into next-to-simplest, ...

Canonical DEs systems and Chen's iterated integrals

A smart change of the MIs basis can bring to big simplifications [Henn 13]

old basis $\leftarrow$ $\mathbf{F}(\vec{x},\epsilon) = \mathbb{B}(\vec{x},\epsilon)\,\mathbf{I}(\vec{x},\epsilon)$ $\rightarrow$ new basis

bad basis ☹

$$d\mathbf{F}(\vec{x},\epsilon) = \mathbb{K}(\vec{x},\epsilon)\,\mathbf{F}(\vec{x},\epsilon)$$

with $d\mathbb{K} - \mathbb{K}\wedge\mathbb{K} = 0$

good basis ☺

$$d\mathbf{I}(\vec{x},\epsilon) = \epsilon\, d\mathbb{A}(\vec{x})\,\mathbf{I}(\vec{x},\epsilon)$$

with $d\mathbb{A}\wedge d\mathbb{A} = 0$

Solution order by order in ϵ — remember Dyson's series for $i\, dU(t,t_0) = \epsilon V(t) U(t,t_0) dt$?

$$\mathbf{I}(\epsilon,\vec{x}) = \mathcal{P}\exp\left\{\epsilon\int_\gamma d\mathbb{A}\right\}\mathbf{I}(\epsilon,\vec{x}_0)$$

$\mathbf{I}(\epsilon,\vec{x}_0) \equiv$ boundary constants, e.g. value at $\vec{x}_0 = 0$ etc

γ is *any* path from $\vec{x}_0$ to $\vec{x}$ (that does not cross branch cuts and singularities of the integrand). $\mathcal{P}$ is like $\mathcal{T}$-ordering, but in more dimensions!

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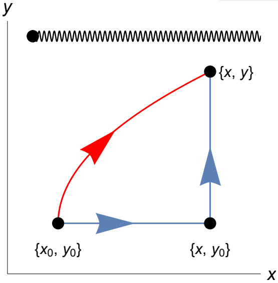

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$$\mathcal{P}\exp\left\{\epsilon\int_\gamma d\mathbb{A}\right\} = \mathbb{1} + \epsilon\int_\gamma d\mathbb{A} + \epsilon^2\int_\gamma d\mathbb{A}\,d\mathbb{A} + \epsilon^3\int_\gamma d\mathbb{A}\,d\mathbb{A}\,d\mathbb{A} + \ldots$$

$$\mathbf{I}(\epsilon,\vec{x}_0) = \mathbf{I}_0(\vec{x}_0) + \epsilon\mathbf{I}_1(\vec{x}_0) + \epsilon^2\mathbf{I}_2(\vec{x}_0) + \epsilon^3\mathbf{I}_3(\vec{x}_0) + \ldots$$

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From Chen's theorem

$$d\mathbb{A}\wedge d\mathbb{A} = 0 \quad \Rightarrow \quad \int_\gamma \underbrace{d\mathbb{A}\ldots d\mathbb{A}}_{\text{k times}} \quad \text{is a homotopy functional}$$

Therefore, with suitable branch cuts and avoiding integrand singularities, it does not depend on the path γ but only on endpoints

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Achieving a "canonical" basis

- constant leading singularity [Henn 13; Bern et al. 14]
- DEs linear in $\epsilon \Rightarrow$ Magnus exp. [Argeri, Mastrolia, Mirabella, Schlenk, Schubert, Tancredi, **DV** 14]
- Investigating parametric representations [Höschele, Hoff, Ueda 14]
- Variations on Moser's algorithm (Fuchsian systems) [Lee 14, Meyer 16]
- decoupling DEs for $\epsilon \to 0$ [Gehrmann, von Manteuffel, Tancredi, Weihs 14; Tancredi 15]
- Factorisation properties of Picard-Fuchs operators [Adams, Chaubey, Weinzierl 17]

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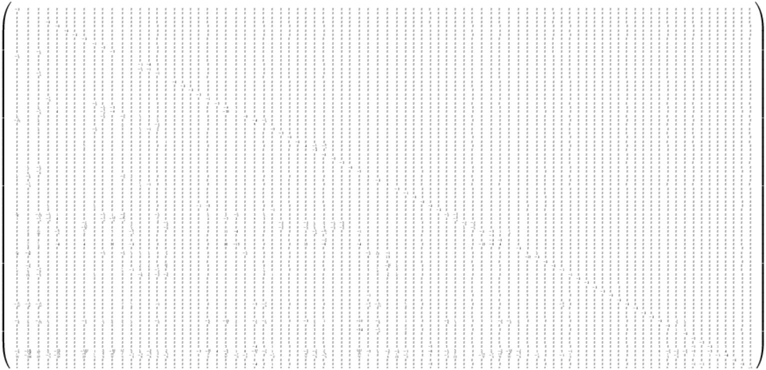
Achieving a “canonical” basis: some tools

- Lee's algorithm (1-scale): `epsilon` [Prausa 17], `Fuchsia` [Gituliar, Magerya 17]
- Meyer's alogithm (generalization to multi-scale) `Canonica` [Meyer 17]

[Higgs + 1Jet 3-loop ladder [Mastrolia, Schubert, Yundin, **DV** 14]]

$1/x$



Canonical coefficients

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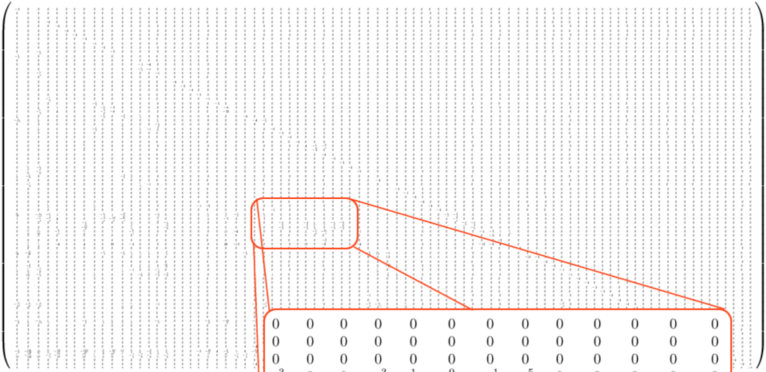

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
$-\frac{3}{2}$	0	0	$-\frac{3}{2}$	$\frac{1}{2}$	$\frac{9}{4}$	$-\frac{1}{2}$	$-\frac{5}{2}$	0	0	0	0	0
$-\frac{3}{2}$	0	0	$-\frac{7}{2}$	$\frac{1}{3}$	2	$-\frac{1}{2}$	$-\frac{5}{2}$	0	0	0	0	0
0	0	0	5	$-\frac{1}{6}$	$-\frac{11}{4}$	0	0	0	-1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	$-\frac{3}{2}$	0	0	0	0	0	0	0
0	0	0	12	0	-6	0	0	0	-4	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	$-\frac{1}{2}$	$\frac{1}{12}$	-1

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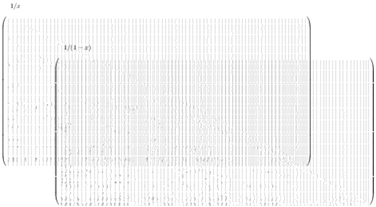

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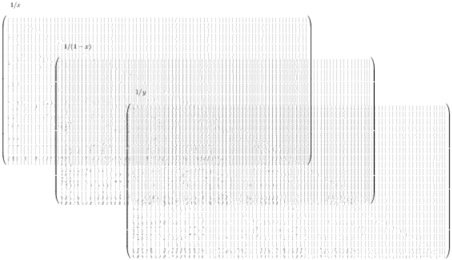

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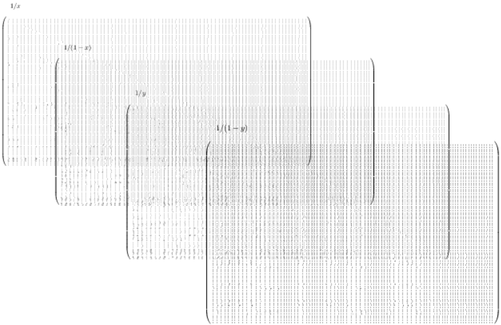

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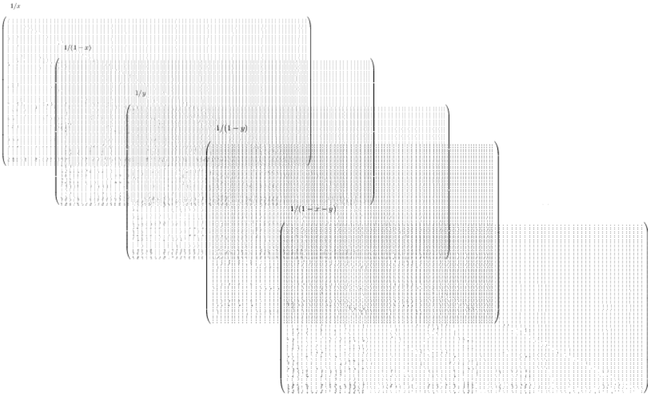

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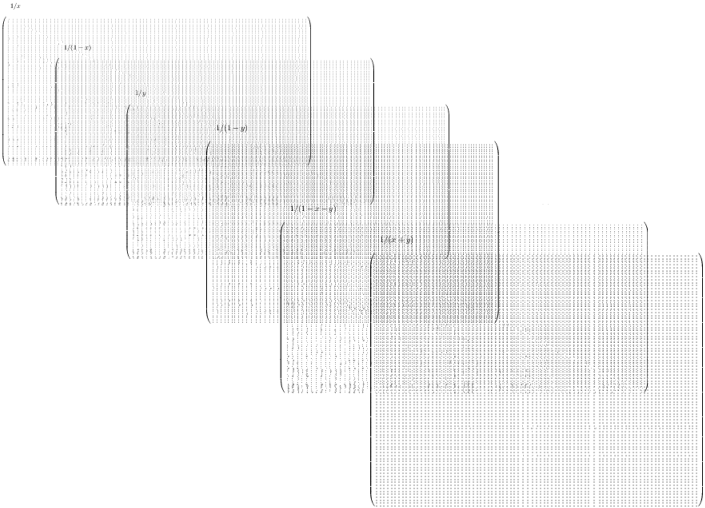

Chen's iterated integrals

[Chen 77]

"Canonical" system $d\mathbf{I}(\vec{x},\epsilon) = \epsilon\, d\mathbb{A}(\vec{x})\, \mathbf{I}(\vec{x},\epsilon)$, $d\mathbb{A}$ is a $d\log$ 1-form

$$d\mathbb{A} = \sum_{i=1}^{n} \mathbb{M}_i\, d\log\eta_i(\vec{x}) \qquad \text{where} \begin{cases} \text{the } \mathbb{M}_i \text{ are matrices of integers} \\ \text{the ``letters'' } \eta_i \text{ are functions of } \vec{x} \end{cases}$$

Therefore the entries of

$$\int_\gamma \underbrace{d\mathbb{A} \ldots d\mathbb{A}}_{\text{k times}}$$

are linear combinations of Chen's iterated integrals of the form

$$\underbrace{\int_\gamma d\log\eta_{i_k} \ldots d\log\eta_{i_1}}_{\equiv \mathcal{C}^{[\gamma]}_{i_k,\ldots,i_1}} \equiv \int_{0\le t_1\le\ldots\le t_k\le 1} g^\gamma_{i_k}(t_k)\ldots g^\gamma_{i_1}(t_1)\, dt_1 \ldots dt_k$$

where, given a parametrization $\gamma(t)$, $t\in[0,1]$, $g^\gamma_i(t) = \frac{d}{dt}\log\eta_i(\gamma(t))$

Chen's iterated integrals

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A particular case: Goncharov Polylogs (GPLs)

$$G_{i_k,\ldots,i_1}(1) \equiv \int_{0\le t_1 \le \ldots \le t_k \le 1} \frac{1}{t_k - i_k} \ldots \frac{1}{t_1 - i_1} \, dt_1 \ldots dt_k$$

Possibly more familiar ...

Line integral representation for complex functions

$$\log(z) - \log(z_0) \equiv \int_\gamma \frac{d\zeta}{\zeta}$$

$$\mathrm{Li}_2(z) - \mathrm{Li}_2(z_0) \equiv -\int_\gamma \frac{\log(1-\zeta)}{\zeta} d\zeta$$

where γ is *any* path in the complex plane that starts at some z_0 and ends at z and does not cross

- the point $\zeta = 0$ for the first integral
- the branch cut, e.g. for $\zeta > 1$, for the second integral

Chen integrals generalize GPLs, which in turn generalize the classical polylogarithms. Public codes are available for GPL evaluation, including their analytic continuation, e.g. GiNaC.

Chen's integral are more general, automation and optimization is harder.

Chen's iterated integrals: properties

[e.g. Brown 09]

- Invariance under path reparametrization
- Reverse path formula: $\mathcal{C}^{[\gamma^{-1}]}_{i_k,\dots,i_1} = (-1)^k \mathcal{C}^{[\gamma]}_{i_k,\dots,i_1}$
- Recursive structure: ($\gamma^s(t) \equiv \gamma(s\,t)$, with $s \in [0,1]$)

$$\mathcal{C}^{[\gamma]}_{i_k,\dots,i_1} = \int_0^1 g^{\gamma}_{i_k}(s)\, \mathcal{C}^{[\gamma_s]}_{i_{k-1},\dots,i_1}\, ds \qquad \frac{d}{ds}\mathcal{C}^{[\gamma_s]}_{i_k,\dots,i_1} = g^{\gamma}_{i_k}(s)\, \mathcal{C}^{[\gamma_s]}_{i_{k-1},\dots,i_1}$$

- Shuffle algebra:

$$\mathcal{C}^{[\gamma]}_{\vec{m}}\, \mathcal{C}^{[\gamma]}_{\vec{n}} = \sum_{\text{shuffles } \sigma} \mathcal{C}^{[\gamma]}_{\sigma(m_M),\dots,\sigma(m_1),\sigma(n_N),\dots,\sigma(n_1)}$$

- Path composition formula: if $\gamma \equiv \alpha\beta$, i.e. first α, then β

$$\mathcal{C}^{[\alpha\beta]}_{i_k,\dots,i_1} = \sum_{p=0}^{k} \mathcal{C}^{[\beta]}_{i_k,\dots,i_{p+1}}\, \mathcal{C}^{[\alpha]}_{i_p,\dots,i_1}$$

- Integration-by-parts formula: get rid of outermost integration

$$\mathcal{C}^{[\gamma]}_{i_k,\dots,i_1} = \log \eta_{i_k}(\vec{x})\, \mathcal{C}^{[\gamma]}_{i_{k-1},\dots,i_1} - \int_0^1 \log \eta_{i_k}(\vec{x}(t))\, g_{i_{k-1}}(t)\, \mathcal{C}^{[\gamma_t]}_{i_{k-2},\dots,i_1}\, dt$$

Connection with GPLs in special cases

η_i' polynomial in $\vec{x}$ (with algebraic roots) $\Leftrightarrow$ GPLs representation

e.g. $\eta = 1 + xy$, with $\gamma = \alpha\beta$ and ($t \in [0,1]$)

$$\alpha(t) = (x_0 + t(x_1 - x_0),\, y_0)\,, \qquad \beta(t) = (x_1,\, y_0 + t(y_1 - y_0))\,,$$

$$\int_{\alpha\beta} d\log(1+xy) = \int_{\alpha} d\log(1+xy) + \int_{\beta} d\log(1+xy)$$
$$= G\left(\frac{1+x_0 y_0}{y_0(x_0-x_1)};1\right) + G\left(\frac{1+x_0 y_0}{x_0(y_0-y_1)};1\right)$$

$$\int_{\alpha\beta} d\log(1+xy)\ d\log(1+xy) = \int_{\alpha} d\log(1+xy)\ d\log(1+xy) + \int_{\alpha} d\log(1+xy) \cdot \int_{\beta} d\log(1+xy)$$
$$+ \int_{\beta} d\log(1+xy)\, d\log(1+xy)$$
$$= G\left(\frac{1+x_0 y_0}{y_0(x_0-x_1)}, \frac{1+x_0 y_0}{y_0(x_0-x_1)};1\right) + G\left(\frac{1+x_0 y_0}{x_0(y_0-y_1)}, \frac{1+x_0 y_0}{y_0(x_0-x_1)};1\right)$$
$$+ G\left(\frac{1+x_0 y_0}{x_0(y_0-y_1)}, \frac{1+x_0 y_0}{x_0(y_0-y_1)};1\right)$$

ϵ-linear DEs $\Rightarrow$ canonical

[Argeri, Mastrolia, Mirabella, Schlenk, Schubert, Tancredi, **DV** 14]

1. start with DE linear in ϵ (may need a bit of trial and error + expertise to get there)

$$\partial_x \mathbf{F}(\epsilon, x) = [A_0(x) + \epsilon\, A_1(x)]\mathbf{F}(\epsilon, x)$$

2. find basis change $\mathbf{F}(\epsilon, x) = B_0(x)\ \mathbf{I}(\epsilon, x)$ that absorbs the A_0 piece: use the Magnus exponential

$$B_0(x) \equiv e^{\Omega[A_0](x,x_0)} \quad \Leftrightarrow \quad \partial_x B_0(x) = A_0(x) B_0(x)$$

(formally) always works, but the matrix exponential might be an infinite series $\leftrightarrow$ elliptic?

3. obtain a canonical system for the $\mathbf{I}$'s

$$\partial_x \mathbf{I}(\epsilon, x) = \epsilon\, \hat{A}_1(x) \mathbf{I}(\epsilon, x)\,, \quad \hat{A}_1(x) = B_0^{-1}(x) A_1(x) B_0(x)$$

4. integrate e.g. again with Magnus's exponential (or Dyson's series)

$$\mathbf{I}(\epsilon, x) = B_1(\epsilon, x) g_0(\epsilon)\ , \qquad B_1(\epsilon, x) = e^{\Omega[\epsilon\, \hat{A}_1](x,x_0)}$$

5. ϵ-expansion of g's will have uniform weight ("transcendentality")

(if $\mathbf{I}(0)$'s are chosen to be uniform too)

In two (or more!) dimensions

[Mastrolia, Schubert, Yundin, **DV** 14]

- the **F**'s obey an ϵ-linear DE system ($x = \frac{s}{m^2}$, $y = \frac{t}{m^2}$)

$$\partial_x \mathbf{F}(x,y,\epsilon) = \left(A_{1,0}(x,y) + \epsilon\, A_{1,1}(x,y)\right) \mathbf{F}(x,y,\epsilon)$$
$$\partial_y \mathbf{F}(x,y,\epsilon) = \left(A_{2,0}(x,y) + \epsilon\, A_{2,1}(x,y)\right) \mathbf{F}(x,y,\epsilon)$$

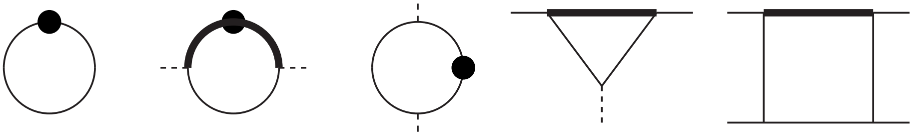
- After getting rid of $A_{i,0}$'s with Magnus (one variable at the time), the g's obey a canonical DE

$$\partial_x \mathbf{I}(x,y,\epsilon) = \epsilon\, \hat{A}_x(x,y) \mathbf{I}(x,y,\epsilon)$$
$$\partial_y \mathbf{I}(x,y,\epsilon) = \epsilon\, \hat{A}_y(x,y) \mathbf{I}(x,y,\epsilon)$$

- which can be cast in *dlog* form

$$d\mathbf{I}(x,y,\epsilon) = \epsilon\, d\mathbb{A}(x,y)\, \mathbf{I}(x,y,\epsilon)$$

- with some *alphabet* $\{\eta_1, \ldots, \eta_n\}$ $\Rightarrow$ Path-ordered exponential



$(\mathcal{T}_1)$ $(\mathcal{T}_2)$ $(\mathcal{T}_3)$ $(\mathcal{T}_4)$ $(\mathcal{T}_5)$

$$\mathsf{F}_1 = \epsilon\,\mathcal{T}_1\,, \qquad \mathsf{F}_2 = \epsilon\,\mathcal{T}_2\,, \qquad \mathsf{F}_3 = \epsilon\,\mathcal{T}_3\,,$$
$$\mathsf{F}_4 = \epsilon^2\,\mathcal{T}_4\,, \qquad \mathsf{F}_5 = \epsilon^2\,\mathcal{T}_5$$

The vector **F** obeys an ϵ-linear DE: we obtain the canonical MIs with the Magnus procedure

$$\mathsf{I}_1 = \mathsf{F}_1\,, \qquad \mathsf{I}_2 = -s\,\mathsf{F}_2\,, \qquad \mathsf{I}_3 = -t\,\mathsf{F}_3\,,$$
$$\mathsf{I}_4 = -t\,\mathsf{F}_4\,, \qquad \mathsf{I}_5 = (s-m^2)\,t\,\mathsf{F}_5$$

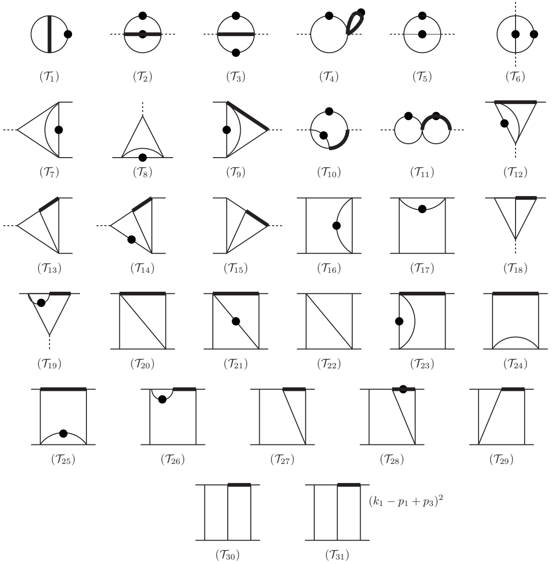
Simple alphabet ($x \equiv -s/m^2$, $y \equiv -s/m^2$) $\Rightarrow$ GPLs

$$\eta_1 = x\,, \qquad \eta_2 = 1+x\,, \qquad \eta_3 = y\,, \qquad \eta_4 = 1-y\,, \qquad \eta_5 = x+y$$

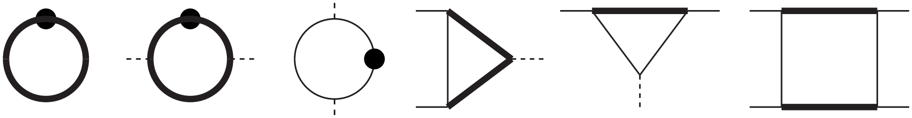
One-mass DY MIs: 2-loop

[Bonciani, Mastrolia, Schubert, DV 16]

- just 1 extra letter
  $\eta_6 = x + y + xy$
- alphabet multilinear in $x, y \Rightarrow$ GPLs
- boundary conditions
  - regularity at pseudo-thresholds
  - zero momentum limits
  - direct integration
- analytic continuation straightforward $\Rightarrow$ complex (s, t, m^2)
- Checked against SecDec (Euclidean and in the physical regions)



[Bonciani, Mastrolia, Schubert, DV 16]



$(\mathcal{T}_1)$ $(\mathcal{T}_2)$ $(\mathcal{T}_3)$ $(\mathcal{T}_4)$ $(\mathcal{T}_5)$ $(\mathcal{T}_6)$

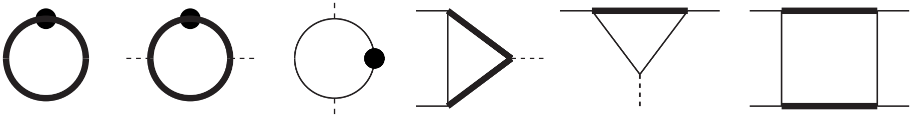
$$\mathrm{F}_1 = \epsilon\,\mathcal{T}_1\,, \qquad \mathrm{F}_2 = \epsilon\,\mathcal{T}_2\,, \qquad \mathrm{F}_3 = \epsilon\,\mathcal{T}_3\,,$$
$$\mathrm{F}_4 = \epsilon^2\,\mathcal{T}_4\,, \qquad \mathrm{F}_5 = \epsilon^2\,\mathcal{T}_5\,, \qquad \mathrm{F}_6 = \epsilon^2\,\mathcal{T}_6$$

Canonical basis

$$\mathrm{I}_1 = \mathrm{F}_1\,, \qquad \mathrm{I}_2 = -s\sqrt{1-\frac{4m^2}{s}}\,\mathrm{F}_2\,, \quad \mathrm{I}_3 = -t\,\mathrm{F}_3\,,$$
$$\mathrm{I}_4 = -s\,\mathrm{F}_4\,, \quad \mathrm{I}_5 = -t\,\mathrm{F}_5\,, \qquad \mathrm{I}_6 = s\,t\sqrt{1-4\frac{m^2}{s}\left(1+\frac{m^2}{t}\right)}\,\mathrm{F}_6$$

Two-mass DY MIs: 1-loop

[Bonciani, Mastrolia, Schubert, DV 16]



$(\mathcal{T}_1)$ $(\mathcal{T}_2)$ $(\mathcal{T}_3)$ $(\mathcal{T}_4)$ $(\mathcal{T}_5)$ $(\mathcal{T}_6)$

Four square roots appear

$$\sqrt{-s},\ \sqrt{4m^2 - s}, \sqrt{-t}, \text{ and } \sqrt{1 - \frac{4\,m^2}{s}\left(1 + \frac{m^2}{t}\right)}$$

A change of variables gets rid of them

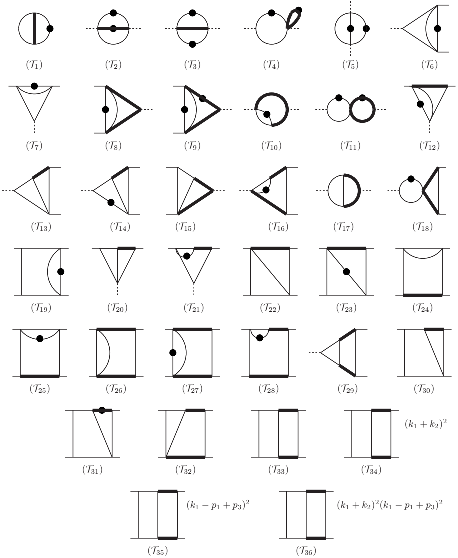
$$-\frac{s}{m^2} = \frac{(1-w)^2}{w}\,, \qquad -\frac{t}{m^2} = \frac{w}{z}\frac{(1+z)^2}{(1+w)^2}\,.$$

$$\eta_1 = z\,, \qquad \eta_2 = 1 + z\,, \qquad \eta_3 = 1 - z\,, \qquad \eta_4 = w\,,$$
$$\eta_5 = 1 + w\,, \qquad \eta_6 = 1 - w\,, \qquad \eta_7 = z - w\,, \qquad \eta_8 = z + w^2\,,$$

Two-mass DY MIs: 2-loop

[Bonciani, Mastrolia, Schubert, DV 16]

- one extra sqrt $\sqrt{1 + \frac{m^4}{t^2} - \frac{2\,m^2}{s}\left(1 - \frac{u}{t}\right)}$
  - in DE for I_{32} at weight 3,4
  - in DEs for $I_{33,\ldots,36}$ at weight 4
  - all the rest $\to$ GPLs
- boundary conditions
  - regularity at pseudo-thresholds
  - zero momentum limits
  - direct integration
- analytic continuation
  - straightforward for $I_{1,\ldots,31}$
  - requires care for $I_{32,\ldots,36}$
- checks against SecDec
  - $I_{1,\ldots,31}$ (Eucl./phys.)
  - $I_{32,\ldots,36}$ (Eucl.)



Mixed Chen-Goncharov representation for DY

[Bonciani et al. 16]

Exploiting the recursive structure, the weight k coefficient of the MIs is

$$\mathbf{I}^{(k)}(\vec{x}) = \mathbf{I}^{(k)}(\vec{x}_0) + \int_0^1 \left[\frac{d\mathbb{A}(t)}{dt} \mathbf{I}^{(k-1)}(\vec{x}_t) \right] dt \,,$$

where $\vec{x}_t$ is the point $(x(t), y(t))$ along the curve identified by γ.

- Need weight-$(k-1)$ coefficient, which is independent of the path
- ☺ Rational alphabet $\to$ factorize over $\mathbb{C}$ $\to$ GPLs GiNaC
- ☹ In our case we have also square roots $\to$ path integration over GPLs
- ▶ Exploit IBP to perform always only 1 numerical path integration

$$\mathcal{C}^{[\gamma]}_{a|\vec{m}|\vec{n}} \equiv \int_0^1 g_a^\gamma(t)\, G^\gamma_{\vec{m}}(x)\, G^\gamma_{\vec{n}}(y)\, dt \,,$$

$$\mathcal{C}^{[\gamma]}_{a|\vec{m}|\varnothing} \equiv \int_0^1 g_a^\gamma(t)\, G^\gamma_{\vec{m}}(x)\, dt \,,$$

$$\mathcal{C}^{[\gamma]}_{a|\varnothing|\vec{n}} \equiv \int_0^1 g_a^\gamma(t)\, G^\gamma_{\vec{n}}(y)\, dt \,,$$

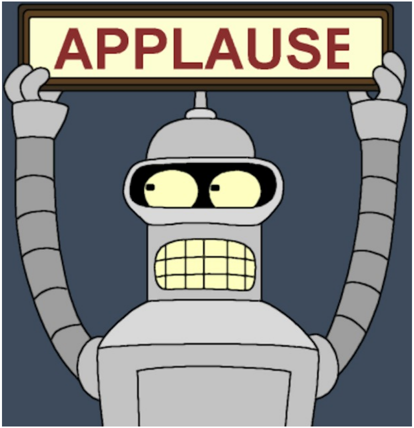
$$\mathcal{C}^{[\gamma]}_{a,\vec{b}|\vec{m}|\vec{n}} \equiv \int_0^1 g_a^\gamma(t)\, \mathcal{C}^{[\gamma t]}_{\vec{b}|\vec{m}|\vec{n}}\, dt \,,$$

where $G^\gamma_{\vec{m}}(x)$ and $G^\gamma_{\vec{n}}(y)$ stand for the GPLs $G_{\vec{m}}(x)$ and $G_{\vec{n}}(y)$ evaluated at $(x, y) = (\gamma^1(t), \gamma^2(t))$.

Summary and perspectives

- We definitely entered a “production era” concerning two-loop calculations for collider physics. Internal masses being tackled too.
- Differential equations approach has been revived and has seen a huge boost in the past few years
- Canonical form exposes analytic properties (and allows for simple integration)
- Magnus exponential to achieve starting from ϵ-linear DEs
- New methods, new classes of functions (e.g. elliptic integrals), deeper understanding of iterated integrals
- Chen’s integrals lend themselves to a semi-analytical approach in extremely difficult cases (e.g. Drell-Yan)
- Future work:
  - Analytic continuation of Chen’s iterated integrals
  - Optimization of numerical evaluation
  - Lots of integrals so far, now need amplitudes and cross-section!

(canonical)



Thanks for your attention!

The Magnus exponential

[Magnus 54]

- a generic matrix linear system of 1st order ODE

$$\partial_x Y(x) = A(x) Y(x) \ , \quad Y(x_0) = Y_0$$

- in the general non-commutative case, the Magnus theorem tells us that

$$Y(x) = e^{\Omega(x,x_0)} \ Y(x_0) \equiv e^{\Omega(x)} \ Y_0$$

- with $\Omega(x) = \sum_{n=1}^{\infty} \Omega_n(x)$ and

$$\begin{aligned}
\Omega_1(x) &= \int_{x_0}^{x} d\tau_1 A(\tau_1) \ , \\
\Omega_2(x) &= \frac{1}{2} \int_{x_0}^{x} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 [A(\tau_1), A(\tau_2)] \\
\Omega_3(x) &= \frac{1}{6} \int_{x_0}^{t} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 \int_{x_0}^{\tau_2} d\tau_3 [A(\tau_1), [A(\tau_2), A(\tau_3)]] + [A(\tau_3), [A(\tau_2), A(\tau_1)]] \\
&\dots
\end{aligned}$$

Relation with Dyson series

[Blanes, Casas, Oteo and Ros 09]

Magnus $\leftrightarrow$ Dyson series. Dyson expansion of the solution Y in terms of the *time-ordered* integrals Y_n

$$Y(x) = Y_0 + \sum_{n=1}^{\infty} Y_n(x)$$
$$Y_n(x) \equiv \int_{x_0}^{x} d\tau_1 \dots \int_{x_0}^{\tau_{n-1}} d\tau_n \; A(\tau_1)A(\tau_2)\cdots A(\tau_n) \ ,$$

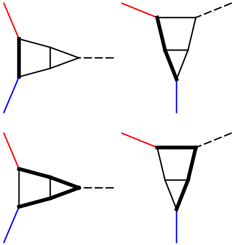
Then

$$Y(x) = e^{\Omega(x)} Y_0 \quad \Rightarrow \quad \sum_{j=1}^{\infty} \Omega_j(x) = \log\left(Y_0 + \sum_{n=1}^{\infty} Y_n(x) \right)$$

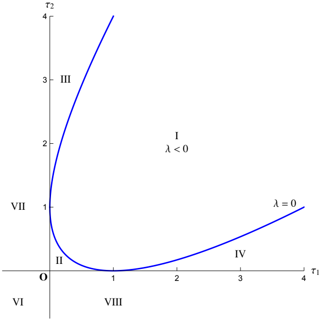
and

$$\begin{aligned} Y_1 &= \Omega_1 \ , \\ Y_2 &= \Omega_2 + \frac{1}{2!}\Omega_1^2 \ , \\ Y_3 &= \Omega_3 + \frac{1}{2!}(\Omega_1\Omega_2 + \Omega_2\Omega_1) + \frac{1}{3!}\Omega_1^3 \end{aligned}$$

E.g. Leading QCD corrections to $X^0 \to W^+W^-$



[Di Vita, Mastrolia, Primo, Schubert 17]



$$-\frac{s}{m^2} = \frac{(1-v)^2}{v}, \qquad \frac{p_1^2}{s} = z\bar{z}, \qquad \frac{p_2^2}{s} = (1-z)(1-\bar{z}).$$

$$\begin{aligned}
&\eta_1 = v\,, && \eta_2 = 1-v\,, && \eta_3 = 1+v\,, && \eta_4 = z\,, \\
&\eta_5 = 1-z\,, && \eta_6 = \bar{z}\,, && \eta_7 = 1-\bar{z}\,, && \eta_8 = z-\bar{z}\,, \\
&\eta_9 = z+v(1-z)\,, && \eta_{10} = 1-z(1-v)\,, && \eta_{11} = \bar{z}+v(1-\bar{z})\,, && \eta_{12} = 1-\bar{z}(1-v)\,, \\
&\eta_{13} = v+z\bar{z}(1-v)^2\,, && \eta_{14} = v+(1-z-\bar{z}+z\bar{z})(1-v)^2\,, && \eta_{15} = v+z(1-v)^2\,, && \eta_{16} = v+(1-z)(1-v)^2 \\
& && \eta_{17} = v+\bar{z}(1-v)^2\,, && \eta_{18} = v+(1-\bar{z})(1-v)^2\,.
\end{aligned}$$