Soft logarithms beyond Leading Colour

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with René Ángeles Martinez, Jeff Forshaw, Simon Plätzer and Mike Seymour

and work in progress with Jeff Forshaw and Simon Plätzer

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Outline

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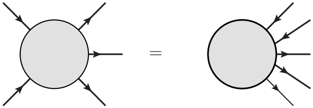
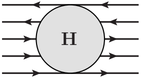
The cross section for emitting n soft gluons:

$$\sigma_0 = \mathrm{Tr}(\mathbf{V}_{\mu,Q}\mathbf{H}(Q)\mathbf{V}^\dagger_{\mu,Q}) \equiv \mathrm{Tr}\ \mathbf{A}_0(\mu)$$

$$\begin{aligned} \mathrm{d}\sigma_1 &= \mathrm{Tr}(\mathbf{V}_{\mu,E_1}\mathbf{D}^\mu_1\mathbf{V}_{E_1,Q}\mathbf{H}(Q)\mathbf{V}^\dagger_{E_1,Q}\mathbf{D}^\dagger_{1\mu}\mathbf{V}^\dagger_{\mu,E_1})\ \mathrm{d}\Pi_1 \\ &\equiv \mathrm{Tr}\ \mathbf{A}_1(\mu)\,\mathrm{d}\Pi_1 \\ &\text{etc.} \end{aligned}$$

$|\mathcal{M}\rangle =$ 

$|\mathcal{M}\rangle\,\langle\mathcal{M}| =$ 

$$\mathbf{D}^\mu_i = \sum_j \mathbf{T}_j E_i \frac{p^\mu_j}{p_j \cdot q_i}; \qquad \mathbf{V}_{a,b} = \exp\left(-\int_a^b \frac{\mathrm{d}E}{E}\mathbf{\Gamma}\right)$$

$$\mathbf{\Gamma} = \frac{\alpha_s}{\pi}\sum_{i<j}(-\mathbf{T}_i \cdot \mathbf{T}_j)\left\{\int \frac{\mathrm{d}\Omega_k}{4\pi}\omega_{ij}(\hat{k}) - i\pi\tilde{\delta}_{ij}\right\}$$

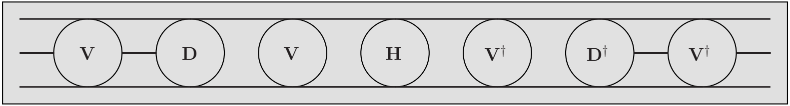
$$\omega_{ij}(\hat{k}) = E_k^2\frac{p_i \cdot p_j}{(p_i \cdot k)(p_j \cdot k)}$$

$$\mathrm{d}\Pi_i = -\frac{\alpha_s}{\pi}\frac{\mathrm{d}E_i}{E_i}\frac{\mathrm{d}\Omega_i}{4\pi}$$

$$\mathrm{d}\sigma_n = \mathrm{Tr}(\mathbf{A}_n(\mu))\mathrm{d}\Pi_n$$

- In principle, μ is equal to 0.
- But μ is equal to Q_0 if the observable is fully inclusive for $E < Q_0$.

Example $\mathbf{A}_1$



$$\mathbf{A}_n(E) = \mathbf{V}_{E,E_n}\mathbf{D}_n^{\mu}\mathbf{A}_{n-1}(E_n)\mathbf{D}_{n\mu}^{\dagger}\mathbf{V}_{E,E_n}^{\dagger}\theta(E \le E_n)$$

- Recurrence relation gives each extra emission.

$$\Sigma(\mu) = \sum_n \int \mathrm{d}\sigma_n u_n(q_1, q_2, \ldots, q_n)$$

where $u_n(q_1, q_2, \ldots, q_n)$ is the measurement function.

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Colour flow basis I

- Some key results concerning the colour flow basis

Some block

The basis tensors are labelled by permuations, σ, of the colour indices

$$|\sigma\rangle = \left| \begin{array}{ccc} 1 & \cdots & n \\ \sigma(1) & \cdots & \sigma(n) \end{array} \right\rangle = \delta^{\alpha_1}_{\bar{\alpha}_{\sigma(1)}} \cdots \delta^{\alpha_n}_{\bar{\alpha}_{\sigma(n)}}$$

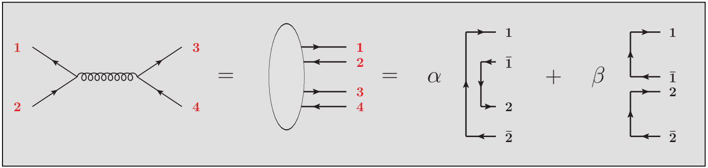
where α ($\bar{\alpha}$) are the fundamental (anti-fundamental) indices assigned to the colour (anti-colour) legs. There are $n!$ colour flows, corresponding to $n!$ basis tensors. The inner products of these basis tensors are given by

$$\langle\sigma|\tau\rangle = \delta^{\alpha_1}_{\bar{\alpha}_{\sigma(1)}} \cdots \delta^{\alpha_n}_{\bar{\alpha}_{\sigma(n)}} \delta^{\bar{\alpha}_{\tau(1)}}_{\alpha_1} \cdots \delta^{\bar{\alpha}_{\tau(n)}}_{\alpha_n} = N_c^{n-\#\text{transpositions}(\sigma,\tau)}$$

where $\#\text{transpositions}(\sigma, \tau)$ is the number of transpositions by which σ and τ differ.

Colour flow basis II

Example



$$|\mathcal{M}\rangle = \alpha \begin{vmatrix} 1 & 2 \\ \bar{2} & \bar{1} \end{matrix}\Big\rangle + \beta \begin{vmatrix} 1 & 2 \\ \bar{1} & \bar{2} \end{matrix}\Big\rangle$$

$$= \alpha \,|2 \quad 1\rangle + \beta \,|1 \quad 2\rangle$$

i	c_i	$\bar{c}_i$	λ_i	$\bar{\lambda}_i$
1	1	0	$\sqrt{T_R}$	0
2	0	$\bar{1}$	0	$\sqrt{T_R}$
3	2	0	$\sqrt{T_R}$	0
4	0	$\bar{2}$	0	$\sqrt{T_R}$

- A colour (anti-colour) index, c_i ($\bar{c}_i$) is assigned to each external leg i of a scattering amplitude.
- Colour index labels are counted from 1 and c_i ($\bar{c}_i$) $= 0$ indicates that i only carries anti-colour (colour).
- All momenta of the amplitude are taken to be outgoing.

The binary variables λ_i and $\bar{\lambda}_i$ can be summarised as $\lambda_i = \sqrt{T_R}$, $\bar{\lambda}_i = 0$ for a quark, $\lambda_i = 0$, $\bar{\lambda}_i = \sqrt{T_R}$ for a antiquark and $\lambda_i = \bar{\lambda}_i = \sqrt{T_R}$ for a gluon, where in QCD $T_R = 1/2$.

Non-orthognal colour bases

We want to compute

$$\mathrm{Tr}(\mathbf{O}) = \mathrm{Tr}([\tau|\,\mathbf{O}\,|\sigma]\,\langle\sigma|\tau\rangle) \qquad (1)$$

where $\mathbf{O} = \sum_{\sigma,\tau} [\tau|\,\mathbf{O}\,|\sigma]\,|\tau\rangle\langle\sigma| = \sum_{\sigma,\tau} \mathcal{O}_{\tau\sigma}\,|\tau\rangle\langle\sigma|$.
Introduce dual basis vectors since basis is non-orthonormal

$$\sum_{\sigma} |\sigma\rangle\,[\sigma| = \mathbb{I}; \qquad [\sigma|\tau\rangle = \delta_{\sigma\tau}$$

As our operator, **O**, can be written as a chain of operators, **R**, which represent evolution operators (be them real or virtual), one can write

$$\mathbf{R}\,|\alpha\rangle = C_R^{\alpha}\,|\beta\rangle\,; \qquad C_R^{\alpha} = [\beta|\mathbf{R}|\alpha\rangle\,; \qquad [\tau|\,\mathbf{O}'\mathbf{R}\,|\sigma_3] = [\tau|\,\mathbf{O}'\,|\sigma_2]\,C_R^{\sigma_2}$$

In this way, we can recursively strip off evolution operators leaving behind reduced matrix elements and c-number factors.

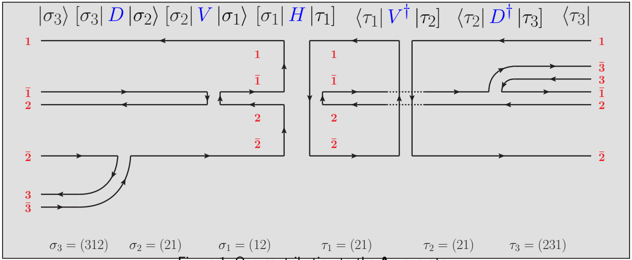


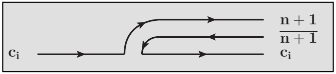
Figure 1: One contribution to the $\mathbf{A}_1$ operator

Real emissions I

$$\mathbf{T}_i = \lambda_i\, \mathbf{t}_{c_i} - \bar{\lambda}_i\, \bar{\mathbf{t}}_{\bar{c}_i} - \frac{1}{N_c}(\lambda_i - \bar{\lambda}_i)\, \mathbf{s}$$

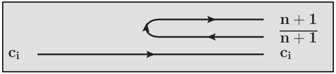
Operator $\mathbf{t}_{c_i}$

$$\mathbf{t}_\alpha \left| \begin{matrix} 1 & \cdots & \alpha & \cdots & n \\ \sigma(1) & \cdots & \sigma(\alpha) & \cdots & \sigma(n) \end{matrix} \right\rangle = \left| \begin{matrix} 1 & \cdots & \alpha & \cdots & n & n+1 \\ \sigma(1) & \cdots & n+1 & \cdots & \sigma(n) & \sigma(\alpha) \end{matrix} \right\rangle$$



Operator s

$$\mathbf{s} \left| \begin{matrix} 1 & \cdots & \cdots & n \\ \sigma(1) & \cdots & \cdots & \sigma(n) \end{matrix} \right\rangle = \left| \begin{matrix} 1 & \cdots & \cdots & n & n+1 \\ \sigma(1) & \cdots & \cdots & \sigma(n) & n+1 \end{matrix} \right\rangle$$



where $\bar{\mathbf{t}}_{\bar{\alpha}}\, |\sigma\rangle = \mathbf{t}_{\sigma^{-1}(\bar{\alpha})}\, |\sigma\rangle$ for the inversion permutation σ^{-1} for which $\alpha = \sigma^{-1}(\sigma(\alpha))$.

Real emissions II

$$\mathbf{T}\,|\sigma_n\rangle \cdots\cdots \langle\tau_n|\,\mathbf{T} = |\sigma_{n+1}\rangle \cdots\cdots\cdots \langle\tau_{n+1}|$$

If σ_n and τ_n differ by n transpositons, then

- $\mathbf{s}\,|\sigma_n\rangle$ and $\mathbf{s}\,|\tau_n\rangle$ will still differ by n transpositions. This is also true for $\mathbf{t}_\alpha\,|\sigma_n\rangle$ and $\mathbf{t}_\beta\,|\tau_n\rangle$, if $\sigma(\alpha) = \tau(\beta)$.
- $\mathbf{t}_\alpha\,|\sigma_n\rangle$ and $\mathbf{s}\,|\tau_n\rangle$ will differ by $n+1$ transpositions
- $\mathbf{t}_\alpha\,|\sigma_n\rangle$ and $\mathbf{t}_\beta\,|\tau_n\rangle$ will differ by $n+2$ transpositions if $\sigma(\alpha) \neq \tau(\beta)$

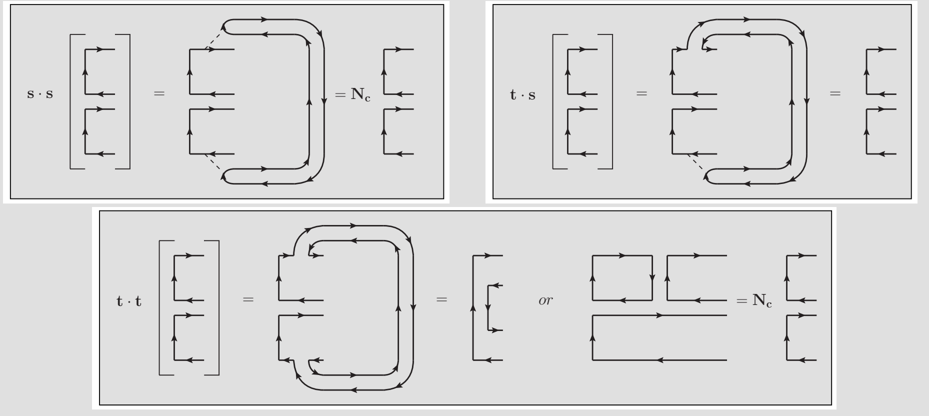
Note that we cannot reduce the number of transpositions between σ_n and τ_n using real emissions and there is a factor of $1/N_c$ associated with each $\mathbf{s}$ operator.

$$\begin{aligned}
[\sigma|\mathbf{T}_i\,\mathbf{A}\,\mathbf{T}_j|\tau] = \Big\{ \Big(& -\lambda_i\bar{\lambda}_j\delta_{c_i\sigma^{-1}(\bar{c}_n)}\delta_{\bar{c}_j\tau(c_n)} - (i,\sigma \leftrightarrow j,\tau) \Big) \\
& + \lambda_i\lambda_j\delta_{c_i\sigma^{-1}(\bar{c}_n)}\delta_{c_j\tau^{-1}(\bar{c}_n)} + \bar{\lambda}_i\bar{\lambda}_j\delta_{\bar{c}_i\sigma(c_n)}\delta_{\bar{c}_j\tau(c_n)} \\
\Big(& -\frac{1}{N_c}\Big(\lambda_i\delta_{c_i\sigma^{-1}(\bar{c}_n)} - \bar{\lambda}_i\delta_{\bar{c}_i\sigma(c_n)}\Big)(\lambda_j - \bar{\lambda}_j)\delta_{c_n\tau^{-1}(\bar{c}_n)} - (i,\sigma \leftrightarrow j,\tau) \Big) \\
& + \frac{1}{N_c^2}(\lambda_i - \bar{\lambda}_i)(\lambda_j - \bar{\lambda}_j)\delta_{c_n\sigma^{-1}(\bar{c}_n)}\delta_{c_n\tau^{-1}(\bar{c}_n)} \Big\} \\
& \times [\tau\backslash n|\mathbf{A}|\sigma\backslash n] \ .
\end{aligned}$$

Virtual corrections I

The products of the colour-line operators, colour reconnectors, are

- $\mathbf{s} \cdot \mathbf{t} = \mathbf{t} \cdot \mathbf{s} = \mathbb{I}$
- $\mathbf{s} \cdot \mathbf{s} = N_c \mathbb{I}$
- $\mathbf{t} \cdot \mathbf{t} = N_c \mathbb{I}$ if $c_i = \sigma^{-1}(\bar{c}_j)$ or a tensor with one transposition relative to σ.



$$[\tau|\,\mathbf{\Gamma}\,|\sigma\rangle = N_c\delta_{\tau\sigma}\Gamma_\sigma + \Sigma_{\sigma\tau} + \frac{1}{N_c}\delta_{\tau\sigma}\rho$$

where $\mathbf{\Gamma}$ is the anomalous dimension matrix which contains all $\mathbf{T}_i \cdot \mathbf{T}_j$ [Plätzer, Eur. Phys. JC (2014) 74, arXiv: 1312.2448] . The off-diagonal elements in the matrix representation of $\mathbf{T}_i \cdot \mathbf{T}_j$ are non-zero only if σ and τ differ by at most one transposition.

$$[\tau|\mathbf{T}_i \cdot \mathbf{T}_j|\sigma\rangle = -N_c\delta_{\tau\sigma}\left(\lambda_i\bar{\lambda}_j\ \delta_{c_i,\sigma^{-1}(\bar{c}_j)} + \lambda_j\bar{\lambda}_i\ \delta_{c_j,\sigma^{-1}(\bar{c}_i)} + \frac{1}{N_c^2}(\lambda_i - \bar{\lambda}_i)(\lambda_j - \bar{\lambda}_j)\right)$$
$$+ \sum_{(ab)}\delta_{\tau_{(ab)},\sigma}\left(\lambda_i\lambda_j\delta_{(ab),(c_ic_j)} + \bar{\lambda}_i\bar{\lambda}_j\delta_{(ab),(\sigma^{-1}(\bar{c}_i)\sigma^{-1}(\bar{c}_j))}\right.$$
$$\left.-\lambda_i\bar{\lambda}_j\delta_{(ab),(c_i,\sigma^{-1}(\bar{c}_j))} - \lambda_j\bar{\lambda}_i\delta_{(ab),(c_j,\sigma^{-1}(\bar{c}_i))}\right)$$

Subleading contributions I

The main challenge is to compute the Sudakov matrix elements as this involves the exponentiation of a possibly large colour matrix [Plätzer, Eur. Phys. JC (2014) 74] :

$$[\tau|\exp(\boldsymbol{\Gamma})|\sigma\rangle = \sum_{l=0}^{\infty} \frac{(-1)^l}{N_c^l} \sum_{\sigma_0,\ldots,\sigma_l} \delta_{\tau\sigma_0}\delta_{\sigma_l\sigma} \times \left(\prod_{\alpha=0}^{l-1} \Sigma_{\sigma_\alpha,\sigma_{\alpha+1}}\right) \times R(\{\sigma_0,\ldots,\sigma_l\}) \qquad (2)$$

- Define succesive summations at (next-to)^d-leading colour (N^d LC) by truncating the sum over l at $l = d$.

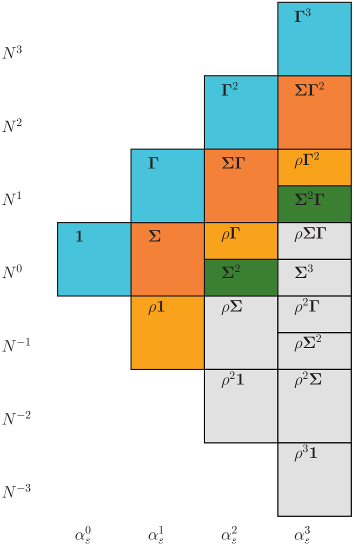
Example of $d = 1$

$$[\tau|\exp(\boldsymbol{\Gamma})|\sigma\rangle|_{\mathrm{NLC}} = \delta_{\tau\sigma} e^{(-N_c\Gamma'_\sigma)} - \frac{1}{N_c}\Sigma_{\tau\sigma}\frac{e^{(-N_c\Gamma'_\tau)} - e^{(-N_c\Gamma'_\sigma)}}{\Gamma'_\tau - \Gamma'_\sigma}; \qquad \Gamma'_\sigma = \Gamma_\sigma(1-\rho/N_c^2)$$

- If $\sigma = \tau$, the R functions revert to their degenerate form:

$$R(\{\sigma,\sigma\}) = -N_c e^{-N_c\Gamma'_\sigma}$$

- Note that as we treat the real emissions, scalar product matrix and the diagonal part of the anomalous dimension matrix without any approximation, and N^d LC approximation involves at most d swaps for each Sudakov operator, this is much more than N^d LC for observables.



	virtuals	reals
(blue)	$(0\text{ flips}) \times 1 \times (\alpha_s N)^n$	$(\mathbf{t}[\ldots]\mathbf{t}\vert_{0\text{ flips}})^{r-1}\ \mathbf{t}[\ldots]\mathbf{t}\vert_{2\text{ flips}} \times 1$ $(\mathbf{t}[\ldots]\mathbf{t}\vert_{0\text{ flips}})^{r-1}\ \mathbf{t}[\ldots]\mathbf{s}\vert_{1\text{ flip}} \times N^{-1}$ $(\mathbf{t}[\ldots]\mathbf{t}\vert_{0\text{ flips}})^{r-1}\ \mathbf{s}[\ldots]\mathbf{s}\vert_{0\text{ flips}} \times N^{-2}$
(orange)	$(1\text{ flip}) \times \alpha_s \times (\alpha_s N)^n$	$(\mathbf{t}[\ldots]\mathbf{t}\vert_{0\text{ flips}})^{r}$ $(\mathbf{t}[\ldots]\mathbf{t}\vert_{0\text{ flips}})^{r-1}\ \mathbf{t}[\ldots]\mathbf{s}\vert_{1\text{ flip}} \times N^{-1}$
(yellow)	$(0\text{ flips}) \times \alpha_s N^{-1} \times (\alpha_s N)^n$	$(\mathbf{t}[\ldots]\mathbf{t}\vert_{0\text{ flips}})^{r}$
(green)	$(0\text{ flips}) \times \alpha_s^2 \times (\alpha_s N)^n$ $(2\text{ flips}) \times \alpha_s^2 \times (\alpha_s N)^n$	$(\mathbf{t}[\ldots]\mathbf{t}\vert_{0\text{ flips}})^{r}$ $(\mathbf{t}[\ldots]\mathbf{t}\vert_{0\text{ flips}})^{r-1}\ \mathbf{t}[\ldots]\mathbf{t}\vert_{2\text{ flips}}$

Subleading contributions III

Subleading colour contributions arise from the hard scattering matrix, from the $1/N_c$ and $1/N_c^2$ terms in both the real emission and virtual evolution operators and from scalar product matrix.

- Pure $1/N_c$ corrections can only originate from interference contributions in the hard process matrix; we will ignore subleading colour contributions from this source here.
- The leading colour contributions from the virtual evolution operator come from Γ, so are all enhanced by powers of $\alpha_s N_c$, and owing to the fact that the leading contribution is diagonal, can easily be accounted to all orders in an exponential. This evolution does not change the colour structure in the amplitude or its conjugate.
- Subleading colour contributions (suppressed by $1/N_c^2$) due to real emissions come from three sources: two flips - $\mathbf{t}[\ldots]\mathbf{t}$, one flip and a factor of $1/N_c$ - e.g $\mathbf{t}[\ldots]\mathbf{s}$, zero flips and a factor of $1/N_c^2$ - $\mathbf{s}[\ldots]\mathbf{s}$.
- An insertion of a perturbation, Σ, comes with a factor of $(\alpha_s N_c)/N_c$ and induces a flip. This can also undo flips induced by real emissions of the type $\mathbf{s}[\ldots]\mathbf{t}$. Whilst we rid ourselves of a factor of $1/N_c$, the $\mathbf{s}$ introduces another.
- A similar reasoning applies to a single ρ perturbation, which contributes at same order.
- With two Σ insertions, we can have a net zero or two flips. Zero flips contributes a $(\alpha_s N_c)^2/N_c^2$ correction, whereas two flips contributes if it compensates a $\mathbf{t}[\ldots]\mathbf{t}$ two-flip real emission.

Grey contributions lead to factors of $(\alpha_s N_c)^2/N_c^4$ and are beyond NLC.

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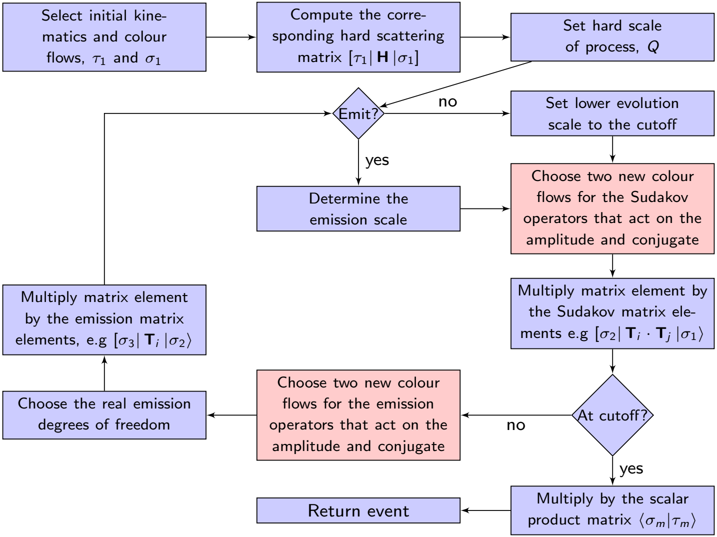
Program structure I

We can Monte Carlo over the intermediate colour states (based heavily on CVolver code [Plätzer, Eur. Phys. JC (2014) 74]) [Work in progress: De Angelis, Forshaw and Plätzer] :



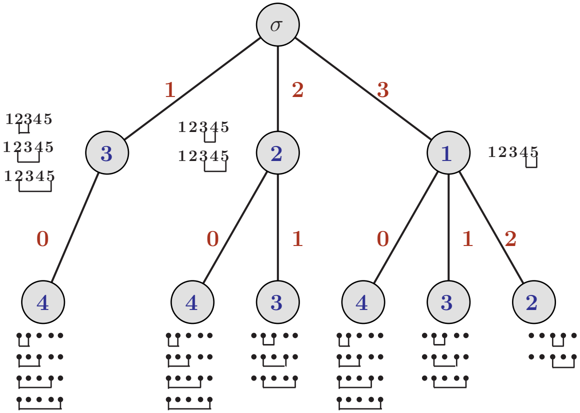
Program structure II

- We choose the new colour flows from the set of all possible colour flows that can be accessed after the action of a Sudakov or emission operator. The main challenge is to account for the independent colour evolution in the amplitude and the conjugate amplitude.
- For emissions, the next pair of chosen colour states, σ_n and τ_n, differ by n, $n+1$ or $n+2$ transpositions, where $n = \#(\sigma_{n-1}, \tau_{n-1})$.
- For virtuals, choose a number of flips to make, p, from a $(1/N_c)^p$ distribution up to d in an attempt to prevent $\#(\sigma_n, \tau_n)$ from becomming too large.

Random level swap

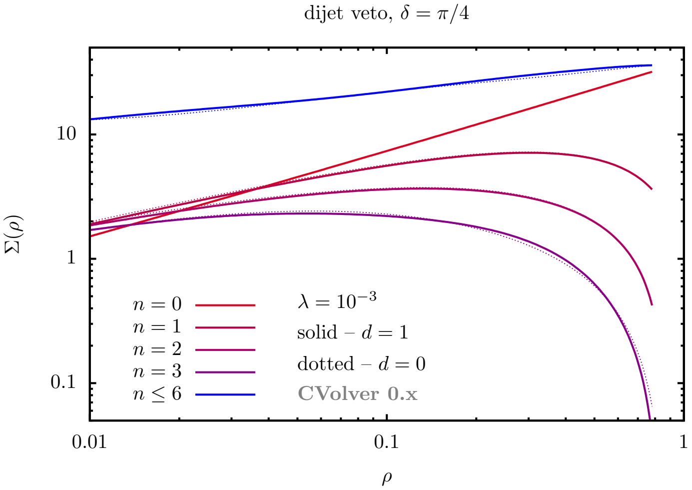
- How to select σ_{n+1} from σ_n such that the number of transpositions between the two, $\#(\sigma_{n+1}, \sigma_n)$, equals L?

For example, $n = 5$, $L = 2$ and $\sigma = |12345\rangle$:



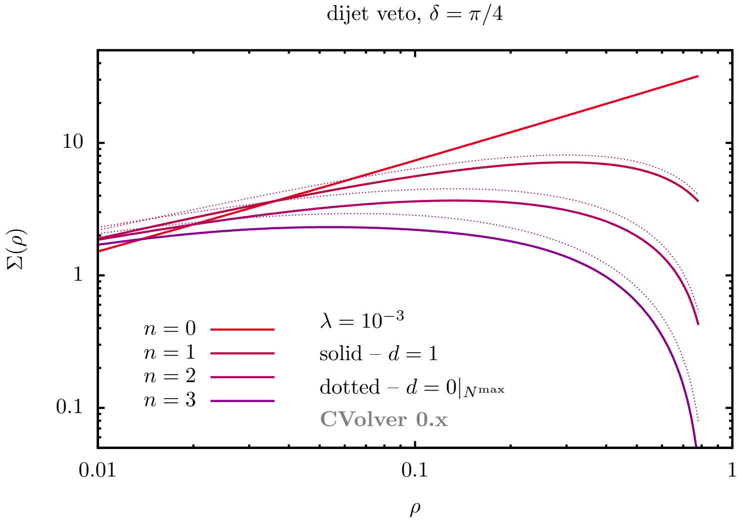
- The sum over all permutations of basis tensors in Eq. 2 is also computationally troublesome.

- We can calculate many distributions at once.
- Using primed definition, Γ'_σ.



Results II

- $d = 0|_{N^{\max}}$ is effectively what a state-of-the-art parton shower can achieve.



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Conclusions

- Iterative form of the algorithm is well suited to a Monte Carlo implementation.
- Colour flow basis facilitates numerical implementation to arbitrary order colour expansion.
- Currently handling soft gluons in e^+e^- but the framework is ready to accomodate a fully-fledged parton shower.
- Want to go beyond the purely soft limit in the LLA and include hard-collinear emissions, NLL soft emissions and go beyond e^+e^-.

Thanks for listening!