NEXT-TO-LEADING POWER CORRECTION IN
PARTICLE SCATTERING NEAR THRESHOLD

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OUTLINE

• Particle scattering near threshold
• Scattering near threshold: NLP logarithms
• Next-to-soft and collinear gluons: factorisation theorems
• NLP logarithms in Drell Yan with two real soft gluons at N3LO

PARTICLE SCATTERING NEAR THRESHOLD
Consider an electroweak annihilation process at the LHC. One has to deal with a multiple scale problem:

\[ d\sigma = \sum_{ij} \int dx_1 \, dx_2 \, f_i/P_1(x_1, \mu_f) \, f_j/P_2(x_2, \mu_f) \, d\hat{\sigma}_{ij}(x_1, x_2, s, Q^2, \mu_f). \]

Define threshold and partonic threshold variable: \( \tau = \frac{Q^2}{s}, \quad z = \frac{Q^2}{\hat{s}}, \quad (z \geq \tau), \quad z \to 1. \)
PARTICLE SCATTERING NEAR THRESHOLD

- Partonic cross sections can be calculated as a **perturbative expansion** in the strong coupling.

\[ \sigma_{ij} = \sigma_{ij}^{(0)} + \frac{\alpha_s}{\pi} \sigma_{ij}^{(1)} + \left( \frac{\alpha_s}{\pi} \right)^2 \sigma_{ij}^{(2)} + \ldots \]

- Calculating additional orders in the perturbative expansion makes the result **more precise**.
- Is this enough? **No!** Near threshold, **large logarithms** of the partonic threshold variable appears:

\[ \frac{d\sigma}{dz} = \sum_{n=0}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^n \sum_{m=0}^{2n-1} c_{nm}^{(-1)} \log^m (1 - z) \left| \frac{1}{1 - z} \right| + \ldots \]

These logarithms **spoil the reliability** of the perturbative expansion, and needs to be **resummed**.

- Resummation of **leading-power (LP)** logarithms is well established, and relies on the **factorisation** and **exponentiation** properties of soft radiation.
The contribution of the threshold limit to a physical observables such as the invariant mass distribution can be significant.

It depends on a phenomenon of dynamical enhancement, which needs to be analysed process by process.

In general, the resummation of large threshold logarithms leads to a more reliable perturbative expansion.
SCATTERING NEAR THRESHOLD: NLP LOGARITHMS

Is this enough? Examining the DY threshold expansion, one realises that the logarithms $\propto c_{nm}^{(-1)}$ are only the leading power terms in a **series expansion**:

$$
\frac{d\sigma}{dz} = \sum_{n=0}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^n \sum_{m=0}^{2n-1} \left\{ c_{nm}^{(-1)} \frac{\log^m (1 - z)}{1 - z} \right\} + c_{nm}^{(0)} \log^m (1 - z) + \ldots
$$

where we have written explicitly the **next-to-leading power (NLP)** logarithms, $\propto c_{nm}^{(0)}$.

Resummation **beyond leading power**: interesting theoretical problem & useful for precision physics.
SCATTERING NEAR THRESHOLD: SOFT AND NEXT-TO-SOFT GLUONS
FACTORISATION OF (NEXT-TO-)SOFT GLUONS

- Let’s examine the emission of soft gluons from an energetic parton: (quark line):

\[
\sim \mathcal{M} \frac{\not p - \not k}{2p \cdot k} \gamma^\mu T^A u(p)
\]

\[
= \mathcal{M} \left( \frac{p^\mu}{p \cdot k} - \frac{k^\mu}{2p \cdot k} + \frac{i k^\alpha \Sigma^\alpha_{\mu}}{p \cdot k} \right) T^A u(p), \quad \Sigma^\alpha_{\mu} = \frac{i}{4} [\gamma^\alpha, \gamma^\mu].
\]

- Emission of soft gluons at leading power (LP) factorises:

\[
\sim \mathcal{M} S u(p), \quad S = \langle 0| \Phi_\beta (-\infty, 0)|0 \rangle,
\]

\[
\Phi_\beta (\lambda_1, \lambda_2) = \mathcal{P} \exp \left\{ i g_s \int_{\lambda_1}^{\lambda_2} d\lambda \beta \cdot A(\lambda) \right\}.
\]

- In general

\[
\sim \mathcal{M} S u(p_1) \bar{v}(p_2) \ldots \bar{u}(p_n),
\]

\[
S = \langle 0| \Phi_1 \ldots \Phi_n |0 \rangle \sim e^{-\sum_G C_G \mathcal{W}_E}.
\]
FACTORISATION OF (NEXT-TO-)SOFT GLUONS

- What happens at next-to-leading power (NLP)?

- Emission of soft gluon at NLP described in terms of "NLP" Wilson lines:

\[
F_p(-\infty, 0) = \mathcal{P} \exp \left[ g \int \frac{d^d k}{(2\pi)^d} A_\mu(k) \left( -\frac{p^\mu}{p \cdot k} + \frac{k^\mu}{2p \cdot k} - k^2 \frac{p^\mu}{2(p \cdot k)^2} - \frac{ik_\nu \Sigma^\nu_\mu}{p \cdot k} \right) + \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d l}{(2\pi)^d} A_\mu(k) A_\nu(l) \left( \frac{\eta^{\mu\nu}}{2p \cdot (k + l)} - \frac{p^{\nu} l^{\mu} p \cdot k + p^{\mu} k^{\nu} p \cdot l}{2(p \cdot l)(p \cdot k) [p \cdot (k + l)]} \right. \\
\left. \quad + \frac{(k \cdot l) p^{\mu} p^{\nu}}{2(p \cdot l)(p \cdot k) [p \cdot (k + l)]} - \frac{i \Sigma^{\mu\nu}}{p \cdot (k + l)} \right) \right].
\]

\[
\tilde{S} = \langle 0 | F_1 \ldots F_n | 0 \rangle \sim e \sum G_e C_{Ge} \mathcal{W}_e + \sum G_{ne} C_{Gne} \mathcal{W}_{ne}.
\]

Laenen, Magnea, Stavenga, White, 2009, 2010
Is this all? No, we did not take into account virtual gluons:

The loop momentum runs over all scaling, cannot be treated as soft:

\[ k = n_+ \cdot \frac{k \cdot n_-}{2} + n_- \cdot \frac{k \cdot n_+}{2} + k_\perp, \]

\[ n_\pm^2 = 0, \quad n_- \cdot n_+ = 2, \quad n_- \sim \frac{p}{s}, \]

hard: \quad k \sim (1, 1, 1)
collinear: \quad k \sim (1, \lambda, \lambda^2)

hard: \quad k \sim (\lambda^2, \lambda, 1)

hard: \quad k \sim (\lambda^2, \lambda^2, \lambda^2), \quad \lambda \ll 1.
AMPLITUDE FACTORISATION AT NLP

- **Goal**: factorise hard, collinear and soft modes;
- describe them in terms of simpler, universal functions:

\[ \partial^\mu M \]

\[ J^\mu \]

\[ \tilde{S}^\mu \]

\[ \mathcal{M} \]

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**References**

Low 1958, Burnett, Kroll 1968

Del Duca 1990, Bonocore, Laenen, Magnea, Melville, LV, White, 2015, 2016

Laenen, Magnea, Stavenga, White, 2009, 2010

Bonocore, Laenen, Magnea, LV, White, 2016
AMPLITUDE FACTORISATION AT NLP: COLLINEAR MODES

- **Collinear modes** can be described by introducing a radiative jet function:

\[ J_{\mu,a}(p,n,k)u(p) = \int d^d y \ e^{-i(p-k)\cdot y} \langle 0 | \Phi_n(\infty,y) \psi(y) j_{\mu,a}(0) | p \rangle, \]

where

\[ j^\mu_a(x) = g \left\{ -\overline{\psi}(x) \gamma^\mu T_a \psi(x) + f^\mu_{bc} \left[ F_{\mu\nu}^c(x) A^\nu_b(x) + \partial^\nu (A^\mu_b(x) A^\nu_c(x)) \right] \right\}. \]

- The radiative jet obeys the Ward identity

\[ k_\mu J^{\mu,a}(p,n,k) = g T^a J(p,n), \quad J(p,n)u(p) = \langle 0 | \Phi_n(\infty,0) \psi(0) | p \rangle, \]

- Warning: \( J^\mu \) contains both **collinear** and **soft modes**: remove soft modes double counting.
For Drell Yan we obtain:

\[
\frac{k^\nu}{p_i \cdot k} \left[ p_{i,\nu} \frac{\partial}{\partial p_i^\mu} - p_{i,\mu} \frac{\partial}{\partial p_i^\nu} \right] = -i k^\nu \frac{L^{(l)}_{\nu \mu}}{p_i \cdot k}
\]

Orbital angular momentum (LBK)

\[
A_{\mu,a}(p_j, k) = \sum_{i=1}^{2} \left( \frac{1}{2} \mathcal{S}_{\mu,a}(p_j, k) + g T_{i,a} \mathcal{G}^{\nu}_{i,\mu} \frac{\partial}{\partial p_i^\nu} + J_{\mu,a}(p_i, n_i, k) \right) A(p_j) - A_{\mu,a}(p_j, k),
\]

for \(n_1 = p_2, n_2 = p_1\).

Spin angular momentum (tree level)

\[
g_s T^A \left[ \frac{p^\mu}{p \cdot k} - \frac{k^\mu}{2p \cdot k} + i k^\alpha \frac{\Sigma^\alpha \mu}{p \cdot k} \right]
\]

Reproduces Drell-Yan up to NNLO
• Better separation of collinear and soft modes (+);
• Immediately generalisable to multiple emission (+);
• Resummation from RGE of operators (+);
• Somewhat more technically involved (-).

Beneke,Garny,Szafron,Yang,2017,
Beneke, Broglio, Garny, Jaskiewicz, Szafron, LV, Wang, in progress
DRELL-YAN AT N3LO AND NLP: DOUBLE REAL EMISSION
DOUBLE REAL EMISSION AT N3LO AND NLP

- Motivation: several reasons:
  
  - Generalisation of $J^\mu$: $J^\mu S^\nu$, $J^{\mu\nu}$?
  
  - Comparison data for resummation

Drell Yan at N3LO unknown beyond leading power
DOUBLE REAL EMISSION AT N3LO AND NLP

Consider the double real emission:

\[
\mathcal{M} = \int \frac{d^d k}{(2\pi)^d} A_{2r,1\nu}(p, \bar{p}, k_1, k_2, k) A^{\dagger}_{2r}(p, \bar{p}, k_1, k_2),
\]

i.e. diagram such as

- What can we learn?
- In the following use invariants

\[
t_2 = (p - k_1)^2 = -2p \cdot k_1, \quad t_3 = (p - k_2)^2 = -2p \cdot k_2,
\]
\[
u_2 = (\bar{p} - k_1)^2 = -2\bar{p} \cdot k_1, \quad u_3 = (\bar{p} - k_2)^2 = -2\bar{p} \cdot k_2,
\]
\[
s_{12} = (k_1 + k_2)^2 = 2k_1 \cdot k_2.
\]
The hard region is completely given in terms of lower-orders information:

\[
M_{\text{LP}, 2r1v, h} &\propto \left( \frac{\mu^2_{\text{MS}}}{-s} \right)^\epsilon \frac{s^3}{t_2 t_3 u_2 u_3} \left[ f_2^H(\epsilon) + \frac{t_2 u_3 + t_3 u_2 - s_{12} s}{2(t_2 + t_3)(u_2 + u_3)} f_1^H(\epsilon) \right], \\
M_{\text{NLP}, 2r1v, h} &\propto \left( \frac{\mu^2_{\text{MS}}}{-s} \right)^\epsilon \frac{s^2(t_2 + t_3 + u_2 + u_3)}{t_2 t_3 u_2 u_3} \\
&\times \left[ f_2^H(\epsilon) + \frac{t_2 u_3 + t_3 u_2 - s_{12} s}{2(t_2 + t_3)(u_2 + u_3)} f_1^H(\epsilon) \right],
\]

compare with single-

\[
M_{\text{LP}, 1r1v, h} &\propto \left( \frac{\mu^2_{\text{MS}}}{-s} \right)^\epsilon \frac{s^2}{tu} f_{h_1}(\epsilon), \\
M_{\text{NLP}, 1r1v, h} &\propto - \left( \frac{\mu^2_{\text{MS}}}{-s} \right)^\epsilon \frac{s(t + u)}{tu} f_{h_2}(\epsilon)
\]

and double-real emission at NNLO:

\[
M_{\text{LP}, 2r0v} &\propto \frac{s^3}{t_2 t_3 u_2 u_3}, \\
M_{\text{NLP}, 2r0v} &\propto \frac{s^2(t_2 + t_3 + u_2 + u_3)}{t_2 t_3 u_2 u_3} \left[ 1 + \frac{t_2 u_3 + t_3 u_2 - s_{12} s}{2(t_2 + t_3)(u_2 + u_3)} \right].
\]

\[\Rightarrow\] Hard region most likely given in terms of the Low-burnett-Kroll theorem: indeed,

\[
f_{h_2}(\epsilon) = s^\epsilon f_{h_1}(\epsilon) \frac{\partial}{\partial s} s^{1-\epsilon} = (1 - \epsilon) f_{h_1}(\epsilon),
\]

where \(f_{h1}(\epsilon)\) is the one-loop function associated with the quark form factor.
DOUBLE REAL EMISSION AT N3LO AND NLP

- Integrating over the phase space we get the differential cross section

\[
\frac{d\sigma_{2r,1v}}{dz} = \frac{1}{4N_c^2} \frac{1}{2s} \left\{ 2\text{Re} \left[ \int \frac{d^d k}{(2\pi)^d} \int d\Phi^{(3)} \delta \left( z - \frac{Q^2}{s} \right) A_{2r,1v}(p, \bar{p}, k_1, k_2, k') A_{2r}^\dagger(p, \bar{p}, k_1, k_2) \right] \right\}.
\]

- In terms of the \(K\) factor:

\[
\left( \frac{\alpha_s}{4\pi} \right)^n K^{(n)}(z) = \frac{1}{\sigma_0} \frac{d\sigma^{(n)}(z)}{dz},
\]

we have

\[
K^{(3),H}_{qq} \bigg|_{c_F^2} = 128 \left\{ \frac{1}{\epsilon^5} (D_0 - 1) + \frac{1}{\epsilon^4} \left( -4D_1 + \frac{3D_0}{2} + 4L - 4 \right) + \frac{1}{\epsilon^3} \left( 8D_2 - 6D_1 + \frac{(8-21\zeta_2)}{2} D_0 - 8L^2 + 16L \\
- \frac{31}{4} + \frac{21}{2} \zeta_2 \right) + \frac{1}{\epsilon^2} \left[ - \frac{32D_3}{3} + 12D_2 + (-16+42\zeta_2) D_1 + \left( 8 - \frac{63}{4} \zeta_2 - 23\zeta_3 \right) D_0 + \frac{32}{3} L^3 - 32L^2 \\
+ (31-42\zeta_2) L - 18 + 42\zeta_2 + 23\zeta_3 \right] \left\{ \frac{32}{3} D_4 - 16D_3 + (32 - 84\zeta_2) D_2 + (-32 + 63\zeta_2 + 92\zeta_3) D_1 \right. \right. \\
+ \left. \left. \left( 16 - 42\zeta_2 - \frac{69}{2} \zeta_3 + \frac{1017}{16} \zeta_4 \right) D_0 - \frac{32}{3} L^4 + \frac{128}{3} L^3 + (-62 + 84\zeta_2) L^2 + (72 - 168\zeta_2 - 92\zeta_3) L - 36 \\
+ \frac{651}{8} \zeta_2 + 92\zeta_3 - \frac{1017}{16} \zeta_4 \right) - \frac{128}{15} D_5 + 16D_4 + \left( -\frac{128}{3} + 112\zeta_2 \right) D_3 + (64 - 126\zeta_2 - 184\zeta_3) D_2 \right. \\
+ \left. \left. \left( -64 + 168\zeta_2 + 138\zeta_3 - \frac{1017}{4} \zeta_4 \right) D_1 + \left( 32 - 84\zeta_2 - 92\zeta_3 + \frac{3051}{32} \zeta_4 - \frac{1053}{5} \zeta_5 + \frac{483}{2} \zeta_3 \zeta_2 \right) D_0 + \frac{128}{15} L^5 \\
- \frac{128}{3} L^4 + \left( \frac{248}{3} - 112\zeta_2 \right) L^3 + (-144 + 336\zeta_2 + 184\zeta_3) L^2 + \left( 144 - \frac{651}{2} \zeta_2 - 368\zeta_3 + \frac{1017}{4} \zeta_4 \right) L \right\}. 
\]

The NLP contribution is a necessary part of information for the calculation of DY at N3LO, and was not known before.
In the collinear region we get

\[\mathcal{M}_{\text{col.}}^{\text{LP}} = 0;\]

\[\mathcal{M}_{\text{col.}}^{\text{NLP}} \propto (\mu_{\text{MS}}^2)^\varepsilon \frac{s^2}{t_2 t_3 u_2 u_3} \left\{ u_2 (-t_2)^{-\varepsilon} + u_3 (-t_3)^{-\varepsilon} \right\} f_1^C (\varepsilon) \]

\[+ \frac{t_3 u_2 + t_2 u_3 - s_{128}}{t_2 + t_3} \left[ \left( (-t_2)^{-\varepsilon} - 2 (-t_2 - t_3)^{-\varepsilon} + (-t_3)^{-\varepsilon} \right) f_2^C (\varepsilon) \right. \]

\[- \left. \left( \frac{t_2}{t_3} (-t_2)^{-\varepsilon} - \frac{(t_2^2 + t_3^2)}{t_2 t_3} (-t_2 - t_3)^{-\varepsilon} + \frac{t_3}{t_2} (-t_3)^{-\varepsilon} \right) f_3^C (\varepsilon) \right\},\]

where

\[f_1^C = -\frac{5}{2} - \frac{2}{\varepsilon} + \varepsilon \left( -3 + \zeta_2 \right) + \varepsilon^2 \left( -4 + \frac{5\zeta_2}{4} + \frac{14\zeta_3}{3} \right) + \ldots,\]

\[f_2^C = -\frac{1}{4\varepsilon} + \frac{1}{8} \varepsilon \left( \frac{3}{4} + \frac{\zeta_2}{8} \right) + \varepsilon^2 \left( 2 - \frac{\zeta_2}{16} + \frac{7\zeta_3}{12} \right) + \ldots,\]

\[f_3^C = \frac{1}{4\varepsilon^2} - \frac{1}{8\varepsilon} - \frac{3}{4} - \frac{\zeta_2}{8} + \varepsilon \left( -2 + \frac{\zeta_2}{16} - \frac{7\zeta_3}{12} \right) + \varepsilon^2 \left( -\frac{9}{2} + \frac{3\zeta_2}{8} + \frac{7\zeta_3}{24} - \frac{47\zeta_4}{64} \right) + \ldots.\]

We find some hints of factorisation, as expected; however, the result seems also to point to a contribution due to a genuinely new radiative jet \(J^{\mu\nu}\).
DOUBLE REAL EMISSION AT N3LO AND NLP

• This integrates to

\[
K_{qq}^{(3),C}\bigg|_{C_F^3} = 32 \left\{ -\frac{1}{\epsilon^4} + \frac{1}{\epsilon^3} \left( 5L - \frac{5}{4} \right) + \frac{1}{\epsilon^2} \left( -\frac{3}{2} - \frac{25}{2}L^2 + \frac{25}{4}L + \frac{21}{2}\zeta_2 \right) \\
+ \frac{1}{\epsilon} \left[ \frac{125L^3}{6} - \frac{125L^2}{8} + \left( \frac{15}{2} - \frac{105\zeta_2}{2} \right)L - 2 + \frac{105}{8}\zeta_2 + 41\zeta_3 \right] - \frac{625}{24}L^4 + \frac{625}{24}L^3 \\
+ \left( -\frac{75}{4} + \frac{525\zeta_2}{4} \right)L^2 + \left( 10 - \frac{525}{8}\zeta_2 - 205\zeta_3 \right)L \right\},
\]

which is consistent with the expectation that virtual collinear radiation contributes only starting at NLL, at NLP.

• This is however not the end of the story..
The double real emission gets a (rather involved) contribution from the soft region, too!

\[ M_{\text{soft}}^{\text{LP}} = 0; \]

\[ M_{\text{soft}}^{\text{NLP}} \propto \left( \frac{\mu_{\text{MS}}^2}{-s_{12}} \right) \epsilon \frac{s^2}{t_2 t_3 u_2 u_3} \]

\[
\begin{aligned}
&\times \left\{ \frac{t_3 f_1^S(\epsilon)}{t_2(t_2 + t_3)^2} \left[ (s_{12}s - t_2 u_3 - t_3 u_2) \left( t_2 + t_3 - t_3^2 F_1 \left( 1, 1, 1 - \epsilon, \frac{t_2}{t_2 + t_3} \right) \right) \right] \\
&+ \frac{f_2^S(\epsilon)}{s s_{12}(t_2 + t_3)} \left[ (t_2 u_3 - t_3 u_2)^2 - s_{12}s(t_2 u_3 + t_3 u_2) \right] \\
&+ \frac{f_3^S(\epsilon)}{s s_{12}t_2(t_2 + t_3)^2} \left[ s_{12}^2 s^2 t_3(t_2 - t_3) + t_3(t_2 + t_3)(t_2 u_3 - t_3 u_2)^2 \\
&+ s_{12}st_2(t_2 + t_3)(t_2 u_3 - 3t_3 u_2) - t_3 \left( s_{12}^2 s^2(t_2 - t_3) + (t_2 + t_3)(t_2 u_3 - t_3 u_2)^2 \\
&- 2s_{12}st_2(t_2 u_3 + t_3 u_2) \right) \right] F_1 \left( 1, 1, 1 - \epsilon, \frac{t_2}{t_2 + t_3} \right) \\
+ \{ t_2, t_3 \leftrightarrow u_2, u_3 \} + \{ t_2, t_3 \leftrightarrow u_3, u_2 \} + \{ t_2, u_2 \leftrightarrow t_3, u_3 \} \right\},
\end{aligned}
\]

where the loop functions \( f_i^S \) are of the type

\[
\begin{aligned}
f_1^S(\epsilon) &= \frac{1}{4\epsilon^2} + \frac{1}{4\epsilon} + \ldots, \\
f_2^S(\epsilon) &= \frac{1}{4\epsilon} + \ldots, \\
f_3^S(\epsilon) &= \frac{1}{4\epsilon} + \ldots.
\end{aligned}
\]

This result is somehow unexpected at first, because at NNLO there is no soft region contribution, in the abelian-like correction.

\[ \text{Bahjat-Abbas, Sinninghe Damsté, LV, White, 2018} \]
The soft region gets a contribution starting at N3LO, because only with two emissions one has a genuinely soft scale $s_{12} = 2k_1.k_2$, absent in case of single emission ($k^2 = 0$).

The soft region arises from a single diagram:

The integration over the phase space gives a rather simple result:

$$K_{qar{q}}^{(3),S} \bigg|_{C_F^3} = 32 \left\{ \frac{1}{\epsilon} \left( \frac{2}{3} \zeta_2 + \frac{1}{3} \zeta_3 \right) - (4\zeta_2 + 2\zeta_3)L \right\}.$$
CONCLUSION

Differential distributions near threshold develop large logarithms, needs special attention → resummation

Resummation is well known at leading power. However the precision goal set by the LHC requires to study NLP logarithms as well.

NLP logarithms are interesting: they allow us to access all-order properties of the amplitude beyond the semi-classical approximation.

Drell-Yan amplitude factorises into universal functions, such as a radiative jet function, and a next-to-soft function.

We have further considered two (next-to-)soft real emission, with the method of regions, to gain insight into the factorisation structure in presence of multiple emissions. The result suggest that a factorised structure hides also at higher orders.

Factorisation theorem for DY at subleading power can be derived as well in SCET; study concerning the resummation of leading NLP logarithms is ongoing.