(my personal)
User’s guide to lattice QCD

Stephan Dürr

Uni Bern, ITP

<table>
<thead>
<tr>
<th>USA/Can.</th>
<th>Europe</th>
<th>Asia/Aus.</th>
<th>Japan</th>
</tr>
</thead>
<tbody>
<tr>
<td>MILC</td>
<td>UKQCD</td>
<td>Adelaide</td>
<td>CP-PACS</td>
</tr>
<tr>
<td>LHPC</td>
<td>ALPHA</td>
<td>Beijing</td>
<td>JLQCD</td>
</tr>
<tr>
<td>FNAL</td>
<td>QCDSF</td>
<td>Taipei</td>
<td>...</td>
</tr>
<tr>
<td>RBC</td>
<td>SPQR</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>JLab</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
QCD with lattice (cut-off) regulator

lattice spacing: \(0.05 \text{ fm} \leq a \leq 0.20 \text{ fm}\)

\[1 \text{ GeV} \leq a^{-1} \leq 4 \text{ GeV}\]

box length: \(1.5 \text{ fm} \leq L \leq 3.0 \text{ fm}\)

require: \(ma \ll 1\) and \(M_{\text{had}}a \ll 1\)

require: \(M_\pi L > 3\) (note: \(3/M_\pi^{\text{phys}} \approx 4.2 \text{ fm}\))

\[
\begin{array}{ccc}
  u & c & (t) \\
  d & s & b \\
\end{array}
\]

\[\begin{array}{c}
\text{extrapolate} \\
\text{work at “physical” value} \\
\text{extrapolate} \\
\end{array}
\]

\[
\begin{array}{ccc}
  m_u & m_u^{\text{phys}} & \\
  m_d & m_d^{\text{phys}} & \\
  m_b & m_b^{\text{phys}} & m_\infty \\
\end{array}
\]

In QCD with \(N_f\) quarks, \(1+N_f\) observables sacrificed to determine scale and quark masses

S. Dürr, ITP Bern

QCD spectroscopy: quenched/dynamical

Hadronic correlator in $N_f \geq 2$ QCD: $C(t) = \int d^4x \ C(t, x) \ e^{ipx}$ with

$$C(x) = \langle O(x) \ O(0) \rangle = \frac{1}{Z} \int DU D\bar{q} Dq \ O(x) \ O(0) \ e^{-S_G - S_F}$$

where $O(x) = \bar{d}(x) \Gamma u(x)$ and $\Gamma = \gamma_5, \gamma_4 \gamma_5$ for $\pi^\pm$ and $S_G = \beta \sum (1 - \frac{1}{3} \text{ReTr} \ U_{\mu\nu}(x)), \ S_F = \sum \bar{q}(D+m)q$

$$\langle \bar{d}(x) \Gamma_1 u(x) \ \bar{u}(0) \Gamma_2 d(0) \rangle = \frac{1}{Z} \int DU \ \text{det}(D+m)^{N_f} \ e^{-S_G} \times \text{Tr} \left\{ \Gamma_1 (D+m)^{-1}_{x0} \Gamma_2 (D+m)^{-1}_{0x} \right\}$$

\[
\gamma_5 \left[ (D+m)^{-1} \right] \gamma_5
\]

Usually $m_u = m_d$ [isospin $SU(2)$ symmetry good: $m_{ud} \equiv \frac{m_u + m_d}{2}$]

Usually $m_{\text{valence}} \neq m_{\text{sea}}$ (“parametric solver slow-down”)

(A) Quenched QCD: quark loops neglected

(B) Full QCD
\[ C(t) = \text{const} \: e^{-M_\pi t} \left[ 1 + e^{-(M_{\text{exc}} - M_\pi) t} \right] \]

\[ M_{\text{eff}}(t) = \frac{1}{2} \log \left( \frac{C(t-1)}{C(t+1)} \right) \quad 0 \ll t \ll T/2 \]

\[ \text{const} \propto \begin{cases} 
G_\pi^2 & \text{for } \langle PP \rangle \\
F_\pi G_\pi & \text{for } \langle A_0 P \rangle \\
F_\pi^2 & \text{for } \langle A_0 A_0 \rangle 
\end{cases} \]

\[ \xrightarrow{= \text{good plateau signals absence of contamination from excited states}} \]
Overview on systematics

Result of the simulation: exact up to statistical errors and systematics/extrapolations

- **continuum limit**
  
  Symanzik: artefact $\propto a^n$, $n = 2$ for \{clover overlap\}

- **infinite-volume limit**

- **chiral extrapolation** $m_{ud} \rightarrow m_{ud}^{phys}$ [XPT]

- **heavy-quark extrapolation** $m_b \rightarrow m_b^{phys}$ [HQET]

- **unquenching** [new simulations]
  
  - excited states contamination
  - renormalization / matching

$\text{Gflop-year per conf} \simeq 2.8 \left( \frac{L}{3 \text{ fm}} \right)^5 \left( \frac{0.6}{M_\pi/M_\rho} \right)^6 \left( \frac{a^{-1}}{2 \text{ GeV}} \right)^7$

[ A. Ukawa, Lat2001 ]

[standard algorithm]
toroidal boundary conditions allow only for lattice momenta \( p = \frac{2\pi}{L} (n_1, n_2, n_3) \)

relativistic dispersion relation \( E^2 = M_{\text{had}}^2 + p^2 \) slightly distorted on the lattice [more pronounced for rho (upper curve) than pion (lower)]

decay constant in \( \langle 0|A_0(t, p)|\pi(0) \rangle \propto f_\pi e^{-Et} \) has big cut-off effects for large \( p^2 \)
Example (2): \(V\)-formfactor for pions

\[ L^3 \times T = 4^3 \times 8 \text{ geometry} \]

\[ \pi(t, p_f) \]

\[ t': V \text{ with } q = p_f - p_i \]

\[ \pi(0, p_i) \]

\[ \langle \pi(t, p_f) | V_k^{\text{cont}}(t', q) | \pi(0, p_i) \rangle = Z_V \langle | V_k^{\text{latt}}(t', q) | \rangle = \left[ p_f + p_i \right] F_V(Q^2) \]

\[ Q^2 = -(E_f - E_i)^2 + (p_f - p_i)^2 \quad \text{[usually } E_i = M_\pi, \ p_i = 0] \]
Example (3): $K_{\ell 3}$ decays

Experiment: $K^+ \rightarrow \pi^0 \ell^+ \nu_\ell$ and $K^0 \rightarrow \pi^- \ell^+ \nu_\ell$ with $\Gamma_{K_{\ell 3}} \propto |V_{us}|^2 \cdot |f_+(0)|^2$

$$\langle \pi(p')|s\gamma_\mu u|K(p)\rangle = \left[ p + p' - q \frac{M_K^2 - M^2_{\pi}}{q^2} \right] \mu f_+(q^2) + q_\mu \frac{M_K^2 - M^2_{\pi}}{q^2} f_0(q^2)$$

with $q = p - p'$

Constraint $f_+(0) = f_0(0)$:
extrapolating $f_0(q^2 \rightarrow 0)$ is sufficient,
but limits $a \rightarrow 0, m \rightarrow m^\text{phys}_{ud}$ required

$\implies$ with growing $Q^2$ both cut-off effects and noise increase

$$\langle K(p')|\bar{c}\gamma_\mu s|D(p)\rangle = \left[ p + p' - q \frac{M_D^2 - M_K^2}{q^2} \right] \mu f_+(q^2) + q_\mu \frac{M_D^2 - M_K^2}{q^2} f_0(q^2)$$

with $q = p - p'$

$\implies$ systematic bias for large extrapolations (dep. on $Q^2_{\text{min}}$)
Example (4): $K^0\bar{K}^0$ mixing

Continuum: just one operator ($\Delta s=2$)

$$O_\Gamma = \bar{s} \Gamma d s \bar{d}$$

with $\Gamma$-structure $VV+AA = \gamma_\mu \otimes \gamma_\mu + \gamma_\mu \gamma_5 \otimes \gamma_\mu \gamma_5$ contributes to $B_K$:

$$B_K = \frac{\langle \bar{K}_0 | O_{VV+AA} | K_0 \rangle}{\frac{8}{3} M_K^2 f_K^2}$$

$$|\epsilon_K| = \text{const} \ A^2 \lambda^6 \bar{\eta} \left[ \text{const} + \text{const} \ A^2 \lambda^4 (1-\bar{\rho}) \right] B_K^{RGI}$$

Lattice: $A = \frac{G_F}{\sqrt{2}} V_{CKM} \sum_j C_j^{\text{cont}}(\mu) \langle . | O_j^{\text{cont}}(\mu) | . \rangle$

with $\langle . | O_j^{\text{cont}}(\mu) | . \rangle = Z_j(\mu a) \langle . | O_j^{\text{latt}}(a^{-1}) | . \rangle$

or $\langle . | O_j^{\text{cont}}(\mu) | . \rangle = \sum_k Z_{jk}(\mu a) \langle . | O_k^{\text{latt}}(a^{-1}) | . \rangle$

$\rightarrow$ all $O_\Gamma^{\text{latt}}$ with $\Delta s=2$ mix, unless GW-symmetry
Summary

With a phenomenological lattice paper, please check:

- does the “effective mass/matrix-element” look convincing ?
- has the continuum limit been taken ?
- are backgrounds quenched/dynamical ?
- are (some) pions in the “chiral” regime, say $200 \text{ MeV} < M_\pi < 300...500 \text{ MeV}$ ?
- is the “chiral” extrapolation done after $a \to 0$, or with a dedicated finite-$a$ ansatz ?
- are finite-volume effects under control ?
  - for experts: improvement/renormalization/matching non-perturbatively ?
  - for experts: need worry about action/algorithm issues ?

Please don’t:

- throw away high-precision (!) lattice data, just because they are quenched (except for observables which are known to get corrupted by $N_f=0$) !
- select “small cut-off effect” lattices by a cut on the lattice spacing (say $a < 0.1 \text{ fm}$) !
Unquenching (1): topological susceptibility

Idea: $\chi_{\text{top}} = \frac{\langle \nu^2 \rangle}{V}$ probes vacuum structure, depends exclusively on $m \equiv m_{\text{sea}}$

$$\chi_{\text{top}} \rightarrow \begin{cases} 
\frac{\sum}{2/m_{ud} + 1/m_s + \ldots} & \text{for } m \to 0 \\
\chi_{\text{qutop}} & \text{for } m \to \infty
\end{cases}$$

with $\Sigma = \lim_{m_q \to 0} \langle \bar{q}q \rangle \bigg|_{m_{ud}, m_s, \ldots}$

Interpolation:

$$\frac{1}{\chi_{\text{top}}} = \frac{2/m_{ud} + 1/m_s + \ldots}{\Sigma} + \frac{1}{\chi_{\text{qutop}}}$$

[SD, hep-lat/0103011]

C.Bernard et al. [MILC], hep-lat/0308019

Monte Carlo simulation time between topology changes versus quark mass. [T.DeGrand, A.Schaefer, hep-lat/0506021]

Cut-off effects may mask underlying continuum behaviour – in particular in full QCD
Unquenching (2): chiral logs

XPT at NLO with $B = \Sigma/F^2$:

$$M_\pi^2 = 2Bm \left(1 + \frac{Bm}{(4\pi F)^2} \log \left(\frac{2Bm}{\Lambda_3^2}\right)\right)$$

$$F_\pi = F \left(1 - \frac{2Bm}{(4\pi F)^2} \log \left(\frac{2Bm}{\Lambda_4^2}\right)\right)$$

attention: FVE typically mimic chiral logs !!!
Unquenching (3): strategy of PQ data taking

- PQ-QCD is a useful extension of QCD, same low-energy constants as (full) QCD [unlike Q-QCD]
- performing $a \rightarrow 0$ first and $m \rightarrow 0$ in a second step is safe but requires (lots of) precise data
- (crucial) practical issues:
  - renormalization
  - scale setting
  - regime of applicability of XPT