Broken Approximate Scale Symmetry and Dilaton in a Cold Fermi Gas

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We want to study a quantum field theory with:

- Approximate scale invariance
- Spontaneous symmetry breaking
- A condensate with scaling dimension

\[ [D, \varphi] = \Delta \varphi \]

- Dilaton as a pseudo-Goldstone boson

Motivation:

- Standard Model Higgs as a dilaton (For example, in walking technicolor theories of dynamical breaking of electroweak symmetry. )
- Dilaton is a pseudo-Goldstone boson for spontaneous breaking of approximate conformal symmetry.
- The Coulomb branch of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory has a massless dilaton.
- Can we find a quantum field theory model which exhibits this phenomenon which is under complete analytic control?
- Yes, in large $N$ limit of a non-relativistic Fermi gas in 2 space, one time dimensions. Strong coupling, controlled by $\frac{1}{N}$ expansion.
- How is the dilaton manifest there?
Example: \( O(N) \)-invariant \( \frac{\lambda}{N^2} (\vec{\phi}^2)^3 \)-theory in 3 dimensions:

- Classical scale invariance, broken by beta function

\[
\beta_\lambda \sim \frac{1}{N}
\]

which is small when \( N \) is large

- \( N \to \infty \): Bardeen-Bander-Moshe phase transition
  \(<: \vec{\phi}^2 :> \neq 0 \) breaks scale symmetry
  \( \exists \) a massless dilaton

- When \( N < \infty \), dilaton becomes a tachyon
  BBM phase is unstable
Scale invariant 2-body interaction potential:

\[ i\hbar \frac{\partial}{\partial t} \psi(\vec{x}_1, \vec{x}_2, t) = \left( -\frac{\hbar^2}{2m} \nabla_1^2 - \frac{\hbar^2}{2m} \nabla_2^2 - g\delta^2(\vec{x}_1 - \vec{x}_2) \right) \psi(\vec{x}_1, \vec{x}_2, t) \]

- Classical scale invariance: \( \vec{x}_1 \rightarrow \Theta \vec{x}_1, \vec{x}_2 \rightarrow \Theta \vec{x}_2, t \rightarrow \Theta^2 t \)
- \( g > 0 \), attractive interaction
- Quantum mechanical problem always needs an extra dimensionful parameter to define the Schrödinger equation in the s-wave channel (self-adjoint extension).
- The attractive interaction always has a bound state. Dimensionful parameter = energy of s-wave bound state = binding energy of Cooper pair
- An attractive Fermi gas should always be a superfluid with a condensate of Cooper pairs
Many-particle theory:

- (Euclidean) Lagrangian density ($\hbar = 1, 2m = 1$, anti-commuting $\psi, \psi^\dagger$)

$$L(\psi, \psi^\dagger) = \psi_a^\dagger(\vec{x}, t)\dot{\psi}_a(\vec{x}, t) + \vec{\nabla}\psi_a^\dagger(\vec{x}, t) \cdot \vec{\nabla}\psi_a(\vec{x}, t)$$

$$- \mu \psi_a^\dagger(\vec{x}, t)\psi_a(\vec{x}, t) - \frac{g}{2N} (\psi_a^\dagger(\vec{x}, t)\psi_a(\vec{x}, t))^2$$

- Euclidean action

$$S[\psi, \psi^\dagger] = \int dt \int d^2 x \ L(\psi, \psi^\dagger)$$

- $\Phi =$ grand canonical potential density

$$e^{-\Phi \mathcal{V}} = \int d\psi d\psi^\dagger \exp (-S[\psi, \psi^\dagger])$$

where $\mathcal{V} =$ space-time volume.
Many-particle theory:

- (Euclidean) Lagrangian density \( (\hbar = 1, 2m = 1, \) anti-commuting \( \psi, \psi^\dagger ) \)

\[
\mathcal{L}(\psi, \psi^\dagger) = \psi^\dagger_a(\vec{x}, t) \psi_a(\vec{x}, t) + \vec{\nabla} \psi^\dagger_a(\vec{x}, t) \cdot \vec{\nabla} \psi_a(\vec{x}, t)
\]

\[
- \mu \psi^\dagger_a(\vec{x}, t) \psi_a(\vec{x}, t) - \frac{g}{2N} \left( \psi^\dagger_a(\vec{x}, t) \psi_a(\vec{x}, t) \right)^2
\]

- when \( \mu = 0 \) \( \exists \) classical scale invariance, \( S = \int dt \int d^2x \mathcal{L}(\psi, \psi^\dagger) \)
  is invariant under \( \psi(\vec{x}, t) \rightarrow e^\Theta \psi(e^\Theta \vec{x}, e^{2\Theta} t) \)

- Tune \( \mu \rightarrow 0 \) and \( g \rightarrow g^* \) \( \beta(g) \sim 1/N \)

- Use (strong) attractive interaction to retain non-zero density
  \( \rho = \frac{1}{N} \sum_{a=1}^{N} < \psi^\dagger_a(\vec{x}, t) \psi_a(\vec{x}, t) > \)

- \( \rho \rightarrow e^{2\Theta} \rho \) breaks (approximate) scale symmetry

- dilaton as a collective mode
Leading order at large N:

\[ \mathcal{L} = \psi^+ \psi + \nabla^2 \psi^+ \nabla^2 \psi - u \psi^+ \psi - \frac{g}{2N} (\psi^+ \psi)^2 \]

\[ = \psi^+ \psi + \nabla^2 \psi^+ \nabla^2 \psi - (u + g \langle \rho \rangle) \psi^+ \psi + \frac{N^2 \langle \rho \rangle^2}{2} \]

\[ \mathcal{M}_{\text{eff}} = u + g \langle \rho \rangle \]

\[ = N \langle \psi \rangle \langle \nabla^2 \psi \rangle \langle \nabla^2 \psi \rangle + N^2 \langle \rho \rangle \langle \nabla^2 \psi \rangle \langle \nabla^2 \psi \rangle + N^3 \langle \rho \rangle^2 \frac{\langle \nabla^2 \psi \rangle^2}{N} \]

\[ V_{\text{eff}} = \frac{N^4 \langle \rho \rangle^3}{N^3} + \cdots \]

\[ \Phi = \int \frac{d^4 x}{2} \left[ \frac{\partial \Phi}{\partial t} - \frac{(u + g \langle \rho \rangle)^2}{8 \pi} + o(1/N) \right] \]
$N \to \infty$:

Determine $\rho$ by finding minimum of grand canonical potential

$$
\Phi = N \left[ g \frac{\rho^2}{2} - \frac{(\mu + g\rho)^2}{8\pi} + \mathcal{O}(1/N) \right] 
$$

$$
\mu = (4\pi - g)\rho + \mathcal{O}(1/N) \ , \ g = 4\pi - \frac{\mu}{\rho} + \mathcal{O}(1/N)
$$

Quantum critical behaviour at double scaling limit

$$
\mu \to 0 \ , \ g \to g^* = 4\pi + \mathcal{O}(1/N) \text{ with } \rho \neq 0
$$

When $g \to 4\pi$,

$$
P = N \frac{\rho^2}{2} \left[ 4\pi - g + \mathcal{O}(1/N) \right] \sim g^* - g
$$

$N = \infty$, $P = 0$ for any $\rho \to$ infinite compressibility,

$$
\kappa = \frac{1}{(N\rho)^2} \frac{d(N\rho)}{d\mu} = \kappa_0 \frac{g^*}{g^* - g} \text{ $\kappa_0$ for free Fermi gas}
$$
Dilaton is a collective mode in the density-density correlation function;

In the $\omega << k\sqrt{4\pi \rho} << 4\pi \rho$ regime,

$$\langle \rho \rho \rangle = \frac{1}{Ng} \sqrt{\frac{p}{\pi}} |\vec{k}| \cdot \Gamma(k) = \left( \frac{g^*}{g} - 1 \right) \sqrt{\frac{p}{\pi}} |\vec{k}|$$

$$\Gamma(k) \rightarrow 0 \text{ when } g \rightarrow g^* = 4\pi + O(1/N)$$

Dilaton pole $\sim 1/\omega$

Non-relativistic Goldstone bosons:

Type I : $\omega \sim |k|$ antiferromagnet

Type II : $\omega \sim k^2$ ferromagnet

Type III : $\omega \sim 0$ dilaton

Fourier transform of $\frac{1}{i\omega}$ is the step function $\theta(t)$
Phase diagram:
ULTRA-VIOLET DIVERGENCE IN THE FERMION-FERMION CHANNEL

\[ \sim \frac{g}{N} \]

\[ \sim \frac{g^2}{N^2} \int \frac{dw \, dk}{(iw+k^2)(-iw+k^2)} \frac{1}{\ln \frac{\Lambda^2}{EF}} \]
**Scale anomaly:**

beta function for $g$ in fermion-fermion channel, is of order $1/N$

$$
\beta(g(\Lambda^2)) = \Lambda^2 \frac{d}{d\Lambda^2} g(\Lambda^2) = -\frac{g^2}{8\pi N} + \mathcal{O}(\frac{1}{N^2})
$$

At $N = \infty$, scale invariance is exact.
At large but finite $N$, scale invariance is only approximate.

$$
g(\mu) = \frac{g(\Lambda^2)}{1 + \frac{g(\Lambda^2)}{8\pi N} \ln \frac{\mu}{\Lambda^2}} + \mathcal{O}(\frac{1}{N^2})
$$

Infrared Landau pole at

$$
\mu_B = \Lambda^2 e^{-8\pi N/g(\Lambda^2)}
$$

Binding energy of Cooper pair.
Next-to-leading order:

\[ \mathcal{V}_{\text{eff}} \quad \text{to order } N^0 \]

\[ \sim N^4 \frac{g^4}{N^4} \sim 1 \]

*Divergent diagram*

\[ \sim N^2 \frac{g^2}{N^2} \ln \Lambda \]

\[ \sim \frac{2}{g^2} \quad \longrightarrow \quad \ln \Lambda \]

\[ \sim N^2 \frac{g}{N^2} \ln \Lambda \]

*Divergence removed by coupling constant renormalization*

\[ \sim g^2 (\mu + g\Lambda)^2 \ln \frac{\Lambda}{\mu + g\Lambda} \]
Corrections to the large $N$ limit:

- example of Coleman-Weinberg mechanism for dynamical symmetry breaking
- use renormalization group to re-sum logarithmically singular terms to all orders
- replace $g$ by the running coupling constant $g(\epsilon_F)$
- Renormalization group improved grand canonical potential

$$
\Phi = N \left[ \frac{g \rho^2}{2} - \frac{(\mu + g \rho)^2}{8\pi} \left( 1 - \frac{\varphi(g)}{4\pi N} \right) + \mathcal{O}(1/N^2) \right]
$$

- $\frac{\partial \Phi}{\partial \rho} = 0$ yields

$$
\mu = \rho \left[ 4\pi - g + \frac{\varphi(g)}{N} - \frac{2\pi \beta(g)}{g} + \mathcal{O}(1/N^2) \right]
$$
Pressure and compressibility:

\[ P = N \frac{\rho^2}{2} [g^* - g] \]

\[ g^* = 4\pi \left( 1 + \frac{\varphi(4\pi)}{4\pi N} - \frac{\beta(4\pi)}{4\pi} + \mathcal{O}(1/N^2) \right) \]

Lower critical density

\[ g(4\pi \rho_{\text{crit.}}) = g^* , \quad \rho_{\text{crit.}} = \frac{\Lambda^2}{4\pi} e^{-8\pi N \left( \frac{1}{g(\Lambda^2)} - \frac{1}{g^*} \right)} = e^{2N m_B} \]

critical chemical potential is of order \(1/N\) and negative,

\[ \mu_{\text{crit.}} = 2\pi \rho_{\text{crit.}} \frac{\beta(g^*)}{g^*} = -\frac{\pi}{N} \rho_{\text{crit.}} \quad (1) \]

Compressibility

\[ \kappa = \kappa_0 \frac{4\pi}{g^* - g(\tilde{\mu}) + \frac{1}{2N} g(\tilde{\mu})} = \kappa_0 \frac{2N + \ln \frac{\rho}{\rho_{\text{crit.}}}}{1 + \ln \frac{\rho}{\rho_{\text{crit.}}}} \]
Phase diagram:
Summary and conclusions:

- Interesting strong coupling phase of N-component 2-dimensional Fermi gas with attractive interaction

- Characterized by large compressibility $\kappa \sim 2N\kappa_0$, weakly damped dilaton $<\rho\rho>\sim \frac{1}{\Gamma-i\omega}$, $\Gamma \sim |k|/N$

- weakly bound Cooper pair. Bose condensation of Cooper pairs would further stabilize the finite density phase.

- but condensation unstable at small temperature in the range $m_B < k_B T < \rho$ or other small randomizing effects

- ongoing numerical investigations about how large $N$ has to be in order to see this behaviour

- up to $N=10$ achievable experimentally (but difficult to control attractive interactions)