Two-loop calculations for a compressible turbulence: Renormalization group analysis of stochastic Navier-Stokes equation

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Physics of Elementary Particles and Statistical Physics

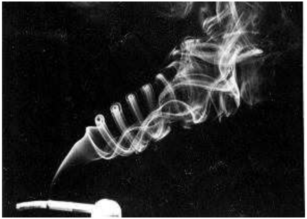
The plan is to give a little overview of the area and then consider a model of compressible turbulence.

In these problems we are concerning with three independent areas: hydrodynamics, statistical physics and high energy physics:

- Navier-Stokes equation describing the moving of the liquid;
- stochastic description of the system;
- functional integration and calculation of Feynman graphs;
- renormalization group (RG).

The problems under consideration are turbulent motion of the gas and liquid or turbulent advection of impirity fields.

Turbulence: pictures



Very complex trajectories of the particles, so huge difficulties in mathematical describing (Milleniun problem for 1 000 000$!).

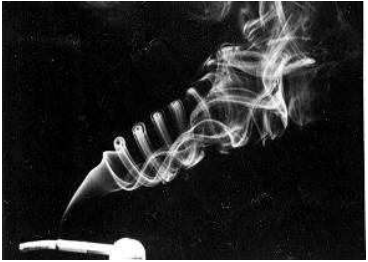
Turbulence

Incompressible version of the Navier-Stokes equation

$$\partial_t v_i + (v_k \partial_k) v_i = \nu \partial^2 v_i - \partial_i p$$

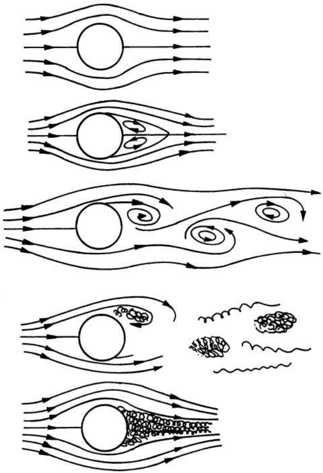
Relevant parameter is **Reynolds** number

$$\mathrm{Re} = \frac{L_0 V_0}{\nu} = \frac{\text{inertia}}{\text{dissipation}}$$



$\mathrm{Re} \to \infty$: **fully developed turbulence**, characterized by **statistical** restoration of symmetries (homogeneity in time and space, isotropy).

Turbulence: behavior of the system when $\mathrm{Re} \to \infty$



Fully developed turbulence

Fully developed turbulence is characterized by

- velocity field $V_i(\mathbf{x}, t) = v_i(\mathbf{x}, t) + u_i(\mathbf{x}, t)$, where $u_i(\mathbf{x}, t)$ is laminar component, $v_i(\mathbf{x}, t)$ is small stochastic (unregular) component;
- statistical characteristics (objects of interest) are correlation and response functions (Green functions in quantum field theory);

According to the Kolmogorov's hypotesis the key objects are

- W and L: the power of external energy and related with it large scale; for trophosphere $L \sim 5$ km.
- ν и l: viscosity and related with it small scale; for trophosphere $l \sim 1$ cm.

Fully developed turbulence: $\mathrm{Re} \gg 1 \Rightarrow L \gg l \Rightarrow$

we may deal with inertial range $l \ll r \ll L$.

Kolmogorov's theory "K41"

Phenomenological theory: consider structure functions

$$S_n(\mathbf{r}) = \left\langle \left[v_r(t, \mathbf{x}) - v_r(t, \mathbf{x}') \right]^n \right\rangle .$$

Kolmogorov's hypotesis No.1: in the region $r \ll L$ distribution depends on power of external energy W and **does not** depend on any detailes, in particular it does not depend on L.

Kolmogorov's hypotesis No.2: in the region $r \gg l$ distribution **does not** depends on viscosity ν and small scale l.

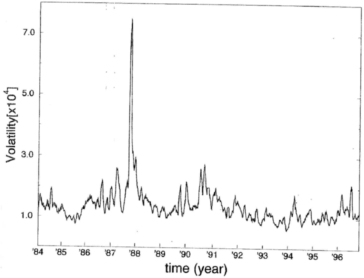
From two hypotesis it follows that in the inertial range $l \ll r \ll L$

$$S_p(\mathbf{r}) = C_p \, (Wr)^{\zeta_p}$$

with exacts exponents $\zeta_p = p/3$ and universal amplitudes (Kolmogorov's constants) C_p.

Intermittency and anomalous scaling

Intermittency: rare configurations may give nonzero contribution to the statistics.



This phenomena is connected with strong fluctuations of energy flux and leads to violation of classical theory K41:

$$S_p(\mathbf{r}) \cong (Wr)^{p/3} \, (r/L)^{\gamma_p}$$

with singular dependence of L and infinite set of independent exponents γ_p.

The goal is to calculate γ_p within a controllable scheme.

Stochastic formulation of the problem

Turbulence is modelled by an external force (random variable) f_i which mimics the input of energy into the system:

$$\partial_t v_i + (v_k \partial_k) v_i = \nu \partial^2 v_i - \partial_i p + f_i; \quad \partial_i v_i = 0.$$

The force f_i supposed to be Gaussian with zero mean and correlation function

$$\langle f_i(t, \boldsymbol{x}) f_j(t', \boldsymbol{x}') \rangle = \frac{\delta(t - t')}{(2\pi)^d} \int_{k>m} d\boldsymbol{k} P_{ij}(\boldsymbol{k}) d(k) e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')},$$

$$d(k) = g_0 \nu_0^3 k^{4-d-\varepsilon}.$$

Here $P_{ij}(\boldsymbol{k}) = \delta_{ij} - k_i k_j / k^2$ is the transverse projector,
g_0 is a coupling constant,
ε is an ultraviolet regularizator (free parameter).

Quantum Field Theory

Theorem: any stochastic equation of the type

$$\partial_t \phi(x) = U(x, \phi) + f(x), \quad \left\langle f(x) f(x') \right\rangle = D(x, x'),$$

where $\phi(x) = \phi(t, \mathbf{x})$ is a random field, $U(x, \phi)$ is a t-local functional depending on the fields and their derivatives, $f(x)$ is a random force, **is equivalent to quantum field model** of the double set of fields $\widetilde{\phi} = \{\phi, \phi'\}$ and action functional

$$S[\varphi] = \underbrace{\frac{1}{2} \varphi' D \varphi'}_{\text{noise term}} + \varphi' \underbrace{\left[-\partial_t \varphi + U \right]}_{\text{dynamics}}.$$

Quantum Field Theory

For the incompressible Navier-Stokes equation this means that

$$S(\varphi) = \frac{v_i' D_{ik} v_k'}{2} + v_i' \left[-\partial_t v_i - v_j \partial_j v_i + \nu_0 \partial^2 v_i \right].$$

What does it mean:

- statistical average is equivalent to functional integration with weight $\exp S[\phi]$;
- classical random field $\to$ quantum field;
- we may use all techniques from quantum field theory: Feynman graphs, renormalization group, operator expansion, ...

Navier-Stokes equation for compressible fluid

Compressible fluid. Equation of motion in this case is

$$\rho(\partial_t + v_k\partial_k)v_i = \nu_0(\delta_{ik}\partial^2 - \partial_i\partial_k)v_k + \mu_0\partial_i\partial_k v_k - \partial_i p + f_i,$$

$$(\partial_t + v_k\partial_k)\phi = -c_0^2\partial_i v_i; \quad \phi = c_0^2 \ln(\rho/\overline{\rho}).$$

For random force f_i we choose

$$\langle f_i(t,\boldsymbol{x})f_j(t',\boldsymbol{x}') \rangle = \frac{\delta(t-t')}{(2\pi)^d}\int_{k>m} d\boldsymbol{k}\, D_{ij}(\boldsymbol{k})e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')},$$

$$D_{ij}(\boldsymbol{k}) = g_{10}\nu_0^3 k^{4-d-y}\left\{P_{ij}(\boldsymbol{k}) + \alpha Q_{ij}(\boldsymbol{k})\right\}.$$

Here $P_{ij}(\boldsymbol{k}) = \delta_{ij} - k_i k_j/k^2$ is the transverse projector;
$Q_{ij}(\boldsymbol{k}) = k_i k_j/k^2$ is the longitudinal projector;
g_{10} is a coupling constant;
y is an UV regularizator.

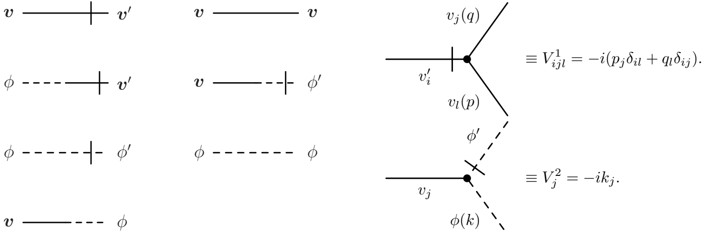
Action functional for compressible fluid

The actional functional has the form

$$S(\varphi) = v_i'\Big\{-\partial_t v_i - v_j\partial_j v_i + \nu_0(\delta_{ik}\partial^2 - \partial_i\partial_k)v_k + u_0\nu_0\partial_i\partial_k v_k - \partial_i\phi\Big\}$$

$$+\,\phi'\Big\{-\partial_t\phi + v_j\partial_j\phi + v_0\nu_0\partial^2\phi - c_o^2(\partial_i v_i)\Big\} + \frac{v_i' D^f_{ik} v_k'}{2}.$$

This model corresponds to a standard Feynman diagrammatic technique with two the triple vertices and seven bare propagators:



The term $\phi' v_0\nu_0\partial^2\phi$ has been added because of RG procedure.

Renormalization procedure: Index of diverdency

Index of diverdency: if $\delta_\Gamma \geq 0$ function Γ requires a counterterm;

$$\delta_\Gamma = d + 2 - \sum N_\Phi d_\Phi$$

with summation over all fields Φ entering into the function Γ.

$$\delta_{v'v} = 2, \quad \text{diverge for any } d;$$
$$\delta_{\phi\phi'} = 2, \quad \text{diverge for any } d;$$
$$\delta_{v'\phi} = 1, \quad \text{diverge for any } d;$$
$$\delta_{\phi' v} = 1, \quad \text{diverge for any } d;$$
$$\delta_{v'v'} = -d + 4, \quad \text{diverge for } d = 2, 3, 4;$$
$$\delta_{v'v'v} = -d + 3, \quad \text{diverge for } d = 2, 3;$$
$$\delta_{v'v'v'} = -2d + 5, \quad \text{diverge for } d = 2;$$
$$\delta_{v'v'v'v'} = -3d + 6, \quad \text{diverge for } d = 2.$$

Renormalization procedure at $d = 3$, $d = 4$, and $d = 2$

$\underline{d = 3}$

Consider function $\langle v'v' \rangle$: $\delta_{v'v'} = -d + 4 = -1$, but it is impossible to construct a scalar counterterm containing two vector fields and one derivative.

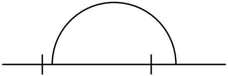
Consider function $\langle v'v'v \rangle$: $\delta_{v'v'} = -d + 3 = 0$, but it is impossible to construct a scalar counterterm containing three vector fields and zero derivatives.

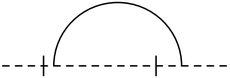
So: we left only with functions $\langle v'v \rangle$, $\langle \phi'\phi \rangle$, $\langle v'\phi \rangle$, and $\langle \phi' v \rangle$.

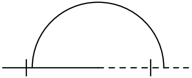
$\underline{d = 4}$: additional function $\langle v'v' \rangle$ under consideration.

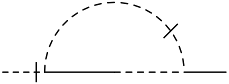
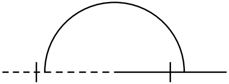
$\underline{d = 2}$: additional functions $\langle v'v' \rangle$, $\langle v'v'v \rangle$, $\langle v'v'v' \rangle$, and $\langle v'v'v'v' \rangle$ under consideration.

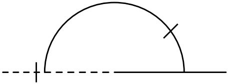
Renormalization procedure: Feynman graphs at $d = 3$

$$\Gamma_{v'v} = i\omega - (\delta_{ij}p^2 - p_i p_j)Z_1\nu - p_i p_j Z_2 u\nu +$$  ,

$$\Gamma_{\phi\phi'} = i\omega - p^2 Z_3 v\nu +$$  ,

$$\Gamma_{v'\phi} = -iZ_4 p_i +$$  ,

$$\Gamma_{\phi'v} = -iZ_5 p_i c^2 +$$  $+$  $+$

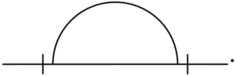
$+$  .

Renormalization procedure: Feynman graphs at $d = 4$

Like $d = 3$ case plus one more:

$$\Gamma_{v'v'} = g_1 \nu^3 p^{4-d-y} Z_6 \left\{ P_{ij}(\mathbf{p}) + \alpha Q_{ij}(\mathbf{p}) \right\} + g_2 \nu^3 \delta_{ij} Z_7 +$$

$$+ \frac{1}{2}$$



For this reason function $D_{ij}(\boldsymbol{k})$ should be midified and secong coupling constant g_{20} should be introduced:

$$D_{ij}(\boldsymbol{k}) = g_{10} \nu_0^3 k^{4-d-y} \left\{ P_{ij}(\boldsymbol{k}) + \alpha Q_{ij}(\boldsymbol{k}) \right\} \quad \Longrightarrow$$

$$D_{ij}(\boldsymbol{k}) = g_{10} \nu_0^3 k^{4-d-y} \left\{ P_{ij}(\boldsymbol{k}) + \alpha Q_{ij}(\boldsymbol{k}) \right\} + g_{20} \nu_0^3.$$

$d = 2$ case: under consideration.

Fixed points and asymptotic

From remormalization group (RG) it follows, that in the case of one charge the asymptotic behaviour of the invariant charge $\bar{g}$ is

$$\bar{g}\,(s) \cong g^* + const \cdot s^{\omega},$$

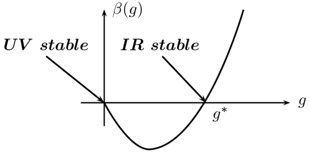
where $s = 1/\mu r$, μ is the renormalization mass, g^* is fixed point:

$$\beta_g(g^*) = 0.$$

IR asymptotic behaviour ($s \to 0 \Leftrightarrow r \to \infty$): $\omega = \beta'(g^*) > 0$.

In the case of many charges $\beta_i(g_j^*) = 0$ and $\Omega_{ik} = \partial\beta_i/\partial g_k$ at the point $g_j = g_j^*$ has to be positive.

Def $\beta(\bar{g}) : \dfrac{d\bar{g}}{d\ln\mu} = -\beta(\bar{g});$



$\beta(\bar{g})$ is responsible for evolution of the coupling constant $\bar{g}$.

Results: Fixed points and asymptotic at $d = 3$

Depending on the values of parameters the model possesses two different fixed points:

- Gaussian,

$$g_1^* = 0; \quad u^* \text{ and } v^* \text{ are undetermined.}$$

- Non-local regime,

$$g_1^* = \frac{16y}{9}; \quad u^* = 1, \quad v^* = 1.$$

These fixed points define possible types of inertial range behavior, i.e., possible values of exponents γ_p.

Advection of a passive scalar at $d = 3$

A passive advection of a scalar density field $\theta(x)$:

$$\partial_t \theta + \partial_i (v_i \theta) = \kappa_0 \partial^2 \theta + f_\theta,$$

where velocity v_i obeys Navier-Stokes equation. Crtical dimensions in non-local regime are

$$\Delta \left[\theta^n\right] = -n + \frac{ny}{6} - \frac{2\alpha y}{9} n(n-1).$$

It is an answer. Problems: $\Delta\left[\theta^n\right] \to \infty$ if $\alpha \to \infty$; α is just a parameter in $D_{ij}(\boldsymbol{k})$

$$D_{ij}(\boldsymbol{k}) = g_{10} \nu_0^3 k^{4-d-y} \left\{ P_{ij}(\boldsymbol{k}) + \alpha Q_{ij}(\boldsymbol{k}) \right\}.$$

Results: Fixed points and asymptotic at $d = 4$

Depending on the values of parameters the model possesses three different fixed points:

- Gaussian,

$$g_1^* = 0, \quad g_2^* = 0; \quad u^* \text{ and } v^* \text{ are undetermined.}$$

- Local regime,

$$g_1^* = 0, \quad g_2^* = \frac{8\varepsilon}{3}; \quad u^* = 1, \quad v^* = 1.$$

- Non-local regime,

$$g_1^* = \frac{16y(2y - 3\varepsilon)}{9[y(2+\alpha) - 3\varepsilon]}, \quad g_2^* = \frac{16\alpha y^2}{9[y(2+\alpha) - 3\varepsilon]}; \quad u^* = v^* = 1.$$

Resume: one more "local" regime is found + answers in well known non-local regime should be modified.

Advection of a passive scalar at $d = 4$

Crtical dimensions in (well-known) non-local regime are

$$\Delta\left[\theta^n\right] = -n + \frac{ny}{6} - \frac{2n(n-1)}{3}\frac{\alpha y(y-\varepsilon)}{y(\alpha+2)-3\varepsilon},$$

where $\varepsilon = 4 - d$. No patologies at large α!

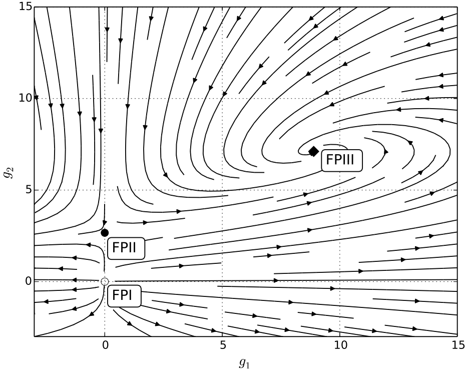
Expanding of this expression in y at fixed value $\varepsilon = 1$ (which corresponds to $d = 3$) gives

$$\Delta\left[\theta^n\right] = \underbrace{-n + \frac{ny}{6} - \frac{2\alpha y}{9}n(n-1)}_{\text{coincide with } d=3} + \mathcal{O}(y^2).$$

Resume: the result of the double y and $\varepsilon = 4 - d$ expansion near $d = 4$ may be considered as a certain partial infinite resummation of the ordinary y expansion near $d = 3$.

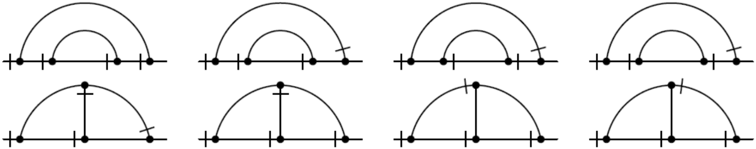
RG flow: Numerical simulation

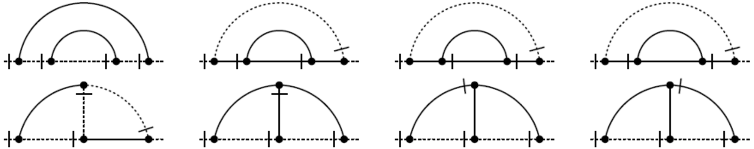
Numerical simulation of the RG flow in the plane g_1 and g_2; three fixed points exist; stable (IR-attractive) fixed point which defines the inertial range asymptotic behavior (at some concrete values of parameters) is non-local point FPIII.



Two-loop approximation at $d = 3$

- 8 diagrams for $\Gamma_{v'v}$.
- 8 diagrams for $\Gamma_{\phi'\phi}$.





Results: Two-loop approximation at $d = 3$

- Numerical calculations using Mathematica.
- At $d = 3$ we have obtained following prediction for non-trivial scaling regime

$$g_1^* = \frac{16y}{9}; \quad u^* = 1, \quad v^* = 1$$

$$\Downarrow\Downarrow\Downarrow$$

$$g^* \approx 2y + \frac{-2.00625\alpha^2 - 4.8847\alpha + 4.4206}{5\alpha + 12}\, y^2$$

$$u^* \approx 1 + \frac{0.125797\alpha^2 - 0.83854\alpha - 0.188233}{5\alpha + 12}\, y$$

$$v^* \approx 1 + \frac{0.217295\alpha^3 + 1.72474\alpha^2 - 1.27116\alpha - 6.9228}{(\alpha + 6)(5\alpha + 12)}\, y$$

Conclusion No. 1: Compressible Navier-Stokes equation

- We applied the field theoretic renormalization group to the analysis of the stochastic Navier-Stokes equation of a compressble fluid.
- Coordinates of non-trivial fixed point has been found.
- Simple analysis near $d = 3$ shows us only two scaling regimes: Gaussian and non-local, whereas analysis near $d = 4$ providing three stable fixed points in the IR region: Gaussian, local and non-local.
- Numerical checks and complete calculation of anomalous dimensions are needed.
- Analysis near $d = 2$ and two-loop calculations are in progress.

Conclusion No. 2: Quantum Fields Methods in Turbulence

In this type of problems methods of **quantum field theory** (functional integration, calculation of Feynman graphs and renormalization group) are applied to the **models of turbulent motion**.

- The goal is to justify the anomalous scaling, i.e., deviations from the classical K41 theory, using a controllable scheme.
- The key point is the possibility to reformulate initial stochastic problem into some quantum field theory.
- Feynman graphs are divergent. Renormalization group allows us to work with these objects and, moreover, provides the leading term of inertial range asymptotic behavior.

Conclusion No. 2: Quantum Fields Methods in Turbulence

This techniques well works for any stochastic equations of the type

$$\partial_t \phi(x) = U(x, \phi) + f(x),$$

where $U(x, \phi)$ is a t-local functional, so we may use them to explore different systems:

- turbulent advection of impurity fields;
- systems with anisotropy and helicity;
- magnetic hydrodynamics;
- Kardar-Parisi-Zhang or similar models describing erosion of landscapes;
- percolation model;
- etc.

Thank you for your attention!