Causal evolution of probability measures

Tomasz Miller

Joint project with Michał Eckstein (Univ. Gdańsk & CC, Cracow, Poland)

Warsaw University of Technology & Copernicus Center (Cracow)

7th ICNFP, Kolymvári, Greece, 5th July 2018
Spacetime — the arena of classical and quantum physics

A spacetime $\mathcal{M}$ is a smooth manifold, which is

- 4-dimensional,
- connected,
- Lorentzian, i.e. it is endowed with a map $g$

$$\mathcal{M} \ni p \mapsto g_p : T_p \mathcal{M} \times T_p \mathcal{M} \to \mathbb{R}$$

such that

- $\forall p \in \mathcal{M}$ $g_p$ is a symmetric bilinear map of signature $(-+++)$,
- $\forall X, Y \in \mathfrak{X}(\mathcal{M})$ the map $p \mapsto g_p(X_p, Y_p)$ is smooth.

- time-oriented (will explain in a second)

Points of $\mathcal{M}$ are suggestively called events.
Preliminaries, motivations and goals

Spacetime — the arena of classical and quantum physics

A spacetime $\mathcal{M}$ is a smooth manifold, which is

- 4-dimensional,
- connected,
- Lorentzian, i.e. it is endowed with a map $g$ such that

$$\mathcal{M} \ni p \mapsto g_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$$

such that

- $\forall p \in \mathcal{M}$ $g_p$ is a symmetric bilinear map of signature $(-+++)$,
- $\forall X,Y \in \mathcal{X}(\mathcal{M})$ the map $p \mapsto g_p(X_p,Y_p)$ is smooth.

- time-oriented (will explain in a second)

Points of $\mathcal{M}$ are suggestively called events.
Spacetime — the arena of classical and quantum physics

A spacetime $\mathcal{M}$ is a smooth manifold, which is
- 4-dimensional,
- connected,
- Lorentzian, i.e. it is endowed with a map $g$

$$\mathcal{M} \ni p \mapsto g_p : T_p \mathcal{M} \times T_p \mathcal{M} \to \mathbb{R}$$

such that
- $\forall p \in \mathcal{M}$ $g_p$ is a symmetric bilinear map of signature $(-+++)$,
- $\forall X, Y \in \mathcal{X}(\mathcal{M})$ the map $p \mapsto g_p(X_p, Y_p)$ is smooth.
- time-oriented (will explain in a second)

Points of $\mathcal{M}$ are suggestively called events.
Preliminaries, motivations and goals

Spacetime — the arena of classical and quantum physics

A spacetime $\mathcal{M}$ is a smooth manifold, which is

- 4-dimensional,
- connected,
- Lorentzian, i.e. it is endowed with a map $g$

\[ \mathcal{M} \ni p \mapsto g_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R} \]

such that

- $\forall p \in \mathcal{M}$ $g_p$ is a symmetric bilinear map of signature $(-+++)$,
- $\forall X, Y \in \mathcal{X}(\mathcal{M})$ the map $p \mapsto g_p(X_p, Y_p)$ is smooth.

- time-oriented (will explain in a second)

Points of $\mathcal{M}$ are suggestively called events.
Spacetime — the arena of classical and quantum physics

A spacetime $\mathcal{M}$ is a smooth manifold, which is

- 4-dimensional,
- connected,
- Lorentzian, i.e. it is endowed with a map $g$

\[ \mathcal{M} \ni p \mapsto g_p : T_p \mathcal{M} \times T_p \mathcal{M} \to \mathbb{R} \]

such that

- $\forall p \in \mathcal{M}$ $g_p$ is a symmetric bilinear map of signature $(- + + +)$,
- $\forall X, Y \in \mathfrak{X}(\mathcal{M})$ the map $p \mapsto g_p(X_p, Y_p)$ is smooth.

- time-oriented (will explain in a second)

Points of $\mathcal{M}$ are suggestively called events.
A *spacetime* $\mathcal{M}$ is a smooth manifold, which is

- 4-dimensional,
- connected,
- Lorentzian, i.e. it is endowed with a map $g$

$$\mathcal{M} \ni p \mapsto g_p : T_p \mathcal{M} \times T_p \mathcal{M} \to \mathbb{R}$$

such that

- $\forall p \in \mathcal{M}$ $g_p$ is a symmetric bilinear map of signature $(-+++)$,
- $\forall X, Y \in \mathcal{X}(\mathcal{M})$ the map $p \mapsto g_p(X_p, Y_p)$ is smooth.

*time-oriented* (will explain in a second)

Points of $\mathcal{M}$ are suggestively called *events*. 
Classification of tangent vectors

Vector $v \in T_p \mathcal{M}$ is called

- **timelike** if $g_p(v, v) < 0$,
- **null** if $g_p(v, v) = 0$,
- **spacelike** if $g_p(v, v) > 0$,
- **causal** if $g_p(v, v) \leq 0$ and $v \neq 0$,

The set of timelike vectors tangent at $p$ has two components. A *time-orientation* is a continuous choice of one of the components, containing (by definition) future-directed timelike vectors.

Above terms extend naturally to vector fields and to piecewise $C^1$ curves.
Preliminaries, motivations and goals

Classification of tangent vectors

Vector $v \in T_p\mathcal{M}$ is called

- **timelike** if $g_p(v, v) < 0$,
- **null** if $g_p(v, v) = 0$,
- **spacelike** if $g_p(v, v) > 0$,
- **causal** if $g_p(v, v) \leq 0$ and $v \neq 0$.

The set of timelike vectors tangent at $p$ has two components. A *time-orientation* is a continuous choice of one of the components, containing (by definition) future-directed timelike vectors.

Above terms extend naturally to vector fields and to piecewise $C^1$ curves.
Classification of tangent vectors

Vector $v \in T_p M$ is called

- **timelike** if $g_p(v, v) < 0$,
- **null** if $g_p(v, v) = 0$,
- **spacelike** if $g_p(v, v) > 0$,
- **causal** if $g_p(v, v) \leq 0$ and $v \neq 0$,

The set of **causal** vectors tangent at $p$ has two components. A **time-orientation** is a **continuous** choice of one of the components, containing (by definition) **future-directed causal vectors**.

Above terms extend naturally to vector fields and to piecewise $C^1$ curves.
Preliminaries, motivations and goals

Classification of tangent vectors

Vector $v \in T_p \mathcal{M}$ is called
- **timelike** if $g_p(v, v) < 0$,
- **null** if $g_p(v, v) = 0$,
- **spacelike** if $g_p(v, v) > 0$,
- **causal** if $g_p(v, v) \leq 0$ and $v \neq 0$,

The set of **causal** vectors tangent at $p$ has two components. A **time-orientation** is a **continuous** choice of one of the components, containing (by definition) **future-directed causal vectors**.

Above terms extend naturally to **vector fields** and to **piecewise $C^1$ curves**.
Causal precedence relation \( \preceq (J^+) \) between events

\( p \preceq q \) if \( \exists \) a piecewise \( C^1 \) fut-dir causal curve \( \gamma \) from \( p \) to \( q \) (or \( p = q \)).

Question: How would one extend \( \preceq \) onto probability measures on a given spacetime?

Subquestion: Why probability measures?

- Pointlike events are idealizations!
- Besnard, Franco, Eckstein: causal relation between the states on a (possibly noncommutative) algebra \( A \).
- If \( A = C_0^\infty(M) \), then:
  - States on \( A \) \( \rightarrow \) Borel probability measures \( \mu \) on \( M \): \( f \mapsto \int_M f \, d\mu \)
  - Pure states on \( A \) \( \rightarrow \) Dirac measures \( \delta_p \) for \( p \in M \): \( f \mapsto f(p) \)

Tomasz Miller (WUT & CC)
Causal precedence relation $\leq (J^+)$ between events

$p \leq q$ if $\exists$ a piecewise $C^1$ fut-dir causal curve $\gamma$ from $p$ to $q$ (or $p = q$).

**Question**: How would one extend $\leq$ onto probability measures on a given spacetime?

**Subquestion**: Why probability measures?

- Pointlike events are idealizations!
- Besnard, Franco, Eckstein: causal relation between the states on a (possibly noncommutative) algebra $\mathcal{A}$.
- If $\mathcal{A} = C_0^\infty(\mathcal{M})$, then:
  - States on $\mathcal{A} \rightsquigarrow$ Borel probability measures $\mu$ on $\mathcal{M}$, $f \mapsto \int_M f \, d\mu$
  - Pure states on $\mathcal{A} \rightsquigarrow$ Dirac measures $\delta_p$ for $p \in \mathcal{M}$, $f \mapsto f(p)$
  - Events.
Causal precedence relation $\preceq (J^+)$ between events

$p \preceq q$ if $\exists$ a piecewise $C^1$ fut-dir causal curve $\gamma$ from $p$ to $q$ (or $p = q$).

**Question**: How would one extend $\preceq$ onto probability measures on a given spacetime?

**Subquestion**: Why probability measures?

- Pointlike events are idealizations!
- Besnard, Franco, Eckstein: causal relation between the states on a (possibly noncommutative) algebra $\mathcal{A}$.
- If $\mathcal{A} = C_0^\infty(\mathcal{M})$, then:
  - States on $\mathcal{A} \rightsquigarrow$ Borel probability measures $\mu$ on $\mathcal{M}$, $f \mapsto \int_{\mathcal{M}} f d\mu$.
  - Pure states on $\mathcal{A} \rightsquigarrow$ Dirac measures $\delta_p$ for $p \in \mathcal{M}$, $f \mapsto f(p)$.
Preliminaries, motivations and goals

Causal precedence relation $\preceq (J^+) \text{ between events}$

$p \preceq q$ if $\exists$ a piecewise $C^1$ fut-dir causal curve $\gamma$ from $p$ to $q$ (or $p = q$).

**Question**: How would one extend $\preceq$ onto probability measures on a given spacetime?

**Subquestion**: Why probability measures?

- Pointlike events are idealizations!
- Besnard, Franco, Eckstein: causal relation between the states on a (possibly noncommutative) algebra $A$.
- If $A = C_0^\infty(M)$, then:
  - States on $A \sim\sim$ Borel probability measures $\mu$ on $M$, $f \mapsto \int_M f d\mu$ and pointanal events
  - Pure states on $A \sim\sim$ Dirac measures $\delta_p$ for $p \in M$, $f \mapsto f(p)$ and events.
Causal precedence relation $\preceq (J^+)$ between events

$p \preceq q$ if $\exists$ a piecewise $C^1$ fut-dir causal curve $\gamma$ from $p$ to $q$ (or $p = q$).

Question: How would one extend $\preceq$ onto probability measures on a given spacetime?

Subquestion: Why probability measures?

- Pointlike events are idealizations!
- Besnard, Franco, Eckstein: causal relation between the states on a (possibly noncommutative) algebra $\mathcal{A}$.
- If $\mathcal{A} = C^\infty_0(\mathcal{M})$, then:
  - States on $\mathcal{A} \rightarrow$ Borel probability measures $\mu$ on $\mathcal{M}$, $f \mapsto \int f \, d\mu$ → nonlocal events.
  - Pure states on $\mathcal{A} \rightarrow$ Dirac measures $\delta_p$ for $p \in \mathcal{M}$, $f \mapsto f(p)$ → events.
Causal precedence relation $\preceq (J^+)$ between events

$p \preceq q$ if $\exists$ a piecewise $C^1$ fut-dir causal curve $\gamma$ from $p$ to $q$ (or $p = q$).

**Question**: How would one extend $\preceq$ onto probability measures on a given spacetime?

**Subquestion**: Why probability measures?

- Pointlike events are idealizations!
- Besnard, Franco, Eckstein: causal relation between the states on a (possibly noncommutative) algebra $\mathcal{A}$.
- If $\mathcal{A} = C_0^\infty(\mathcal{M})$, then:
  - States on $\mathcal{A} \leftrightarrow$ Borel probability measures $\mu$ on $\mathcal{M}$ $f \mapsto \int_\mathcal{M} f \, d\mu$ $\leftrightarrow$ \textit{“nonlocal events”}.
  - Pure states on $\mathcal{A} \leftrightarrow$ Dirac measures $\delta_p$ for $p \in \mathcal{M}$, $f \mapsto f(p)$ $\leftrightarrow$ events.
Causal precedence relation $\preceq (J^+)$ between events

$p \preceq q$ if $\exists$ a piecewise $C^1$ fut-dir causal curve $\gamma$ from $p$ to $q$ (or $p = q$).

**Question**: How would one extend $\preceq$ onto probability measures on a given spacetime?

**Subquestion**: Why probability measures?

- Pointlike events are idealizations!
- Besnard, Franco, Eckstein: causal relation between the states on a (possibly noncommutative) algebra $A$.
- If $A = C_0^\infty(M)$, then:
  - States on $A \leftrightarrow$ Borel probability measures $\mu$ on $M$. $f \mapsto \int_M f d\mu$ ↔ “nonlocal events”.
  - Pure states on $A \leftrightarrow$ Dirac measures $\delta_p$ for $p \in M$, $f \mapsto f(p)$ ↔ events.
Causal precedence relation \( \preceq \) \((J^+)\) between events

\( p \preceq q \) if \( \exists \) a piecewise \( C^1 \) fut-dir causal curve \( \gamma \) from \( p \) to \( q \) (or \( p = q \)).

**Question:** How would one extend \( \preceq \) onto probability measures on a given spacetime?

**Subquestion:** Why probability measures?

- Pointlike events are idealizations!
- Besnard, Franco, Eckstein: causal relation between the states on a (possibly noncommutative) algebra \( A \).
- If \( A = C_0^{\infty}(M) \), then:
  - States on \( A \) \( \leftrightarrow \) Borel probability measures \( \mu \) on \( M \). \( f \mapsto \int_M f \, d\mu \)
  - “nonlocal events”.
  - Pure states on \( A \) \( \leftrightarrow \) Dirac measures \( \delta_p \) for \( p \in M \), \( f \mapsto f(p) \)
  - events.
Preliminaries, motivations and goals

Causal precedence relation \( \preceq (J^+) \) between events

\[ p \preceq q \text{ if } \exists \text{ a piecewise } C^1 \text{ fut-dir causal curve } \gamma \text{ from } p \text{ to } q \text{ (or } p = q). \]

**Question:** How would one extend \( \preceq \) onto probability measures on a given spacetime?

**Subquestion:** Why probability measures?

- Pointlike events are idealizations!
- Besnard, Franco, Eckstein: causal relation between the states on a (possibly noncommutative) algebra \( A \).
- If \( A = C_0^\infty (\mathcal{M}) \), then:
  - States on \( A \leftrightarrow \text{Borel probability measures } \mu \text{ on } \mathcal{M}, \quad f \mapsto \int_{\mathcal{M}} f d\mu \leftrightarrow \text{“nonlocal events”}. \)
  - Pure states on \( A \leftrightarrow \text{Dirac measures } \delta_p \text{ for } p \in \mathcal{M}, \quad f \mapsto \delta_p(f) \leftrightarrow \text{events}. \)
Preliminaries, motivations and goals

The causal precedence relation $\preceq$ on $\mathcal{M}$:

$$p \preceq q \iff \exists \text{ fut-dir causal curve } \gamma \text{ from } p \text{ to } q \text{ (or } p = q)$$

- **Question:** What does it mean that $\mu \preceq \nu$ for $\mu, \nu \in \mathcal{P}(\mathcal{M})$?

- Measures can be spread also in the timelike direction.
The **causal precedence** relation $\preceq$ on $\mathcal{M}$:

$$p \preceq q \iff \exists \text{ fut-dir causal curve } \gamma \text{ from } p \text{ to } q \ (\text{or } p = q)$$

- **Question:** What does it mean that $\mu \preceq \nu$ for $\mu, \nu \in \mathcal{P}(\mathcal{M})$?

- Measures can be spread also in the timelike direction.
What does it mean that $\mu \preceq \nu$? [M. Eckstein, — ’17]

Let $\mathcal{M}$ be a spacetime. Then for any $\mu, \nu \in \mathcal{P}(\mathcal{M})$

$$\mu \preceq \nu \overset{\text{def}}{\iff} \exists \omega \in \mathcal{P}(\mathcal{M}^2) \text{ such that:}$$

- $\forall B - \text{Borel} \quad \omega(B \times \mathcal{M}) = \mu(B), \quad \omega(\mathcal{M} \times B) = \nu(B),$
- $\omega(J^+) = 1,$

where $J^+ := \{(p, q) \in \mathcal{M}^2 \mid p \preceq q\}$.

- $\omega$ can be called a causal coupling or a causal transference plan.
- For $\mu = \delta_p, \nu = \delta_q$, the only coupling is $\omega = \delta_{(p,q)}$ and so $\delta_p \preceq \delta_q \iff p \preceq q$. 

Tomasz Miller (WUT & CC)
What does it mean that $\mu \preceq \nu$? [M. Eckstein, — ’17]

Let $\mathcal{M}$ be a spacetime. Then for any $\mu, \nu \in \mathcal{P}(\mathcal{M})$

\[
\mu \preceq \nu \iff \exists \omega \in \mathcal{P}(\mathcal{M}^2) \text{ such that:}
\]

- $\forall B - \text{Borel } \omega(B \times \mathcal{M}) = \mu(B), \ \omega(\mathcal{M} \times B) = \nu(B),$
- $\omega(J^+) = 1,$

where $J^+ := \{(p, q) \in \mathcal{M}^2 \mid p \preceq q\}$.

- $\omega$ can be called a causal coupling or a causal transference plan.
- For $\mu = \delta_p$, $\nu = \delta_q$, the only coupling is $\omega = \delta_{(p, q)}$ and so $\delta_p \preceq \delta_q$ iff $p \preceq q.$
What does it mean that $\mu \preceq \nu$? [M. Eckstein, — ’17]

Let $\mathcal{M}$ be a spacetime. Then for any $\mu, \nu \in \mathcal{P}(\mathcal{M})$

$$
\mu \preceq \nu \iff \exists \omega \in \mathcal{P}(\mathcal{M}^2) \text{ such that: }
$$

- $\forall \mathcal{B} - \text{Borel} \quad \omega(\mathcal{B} \times \mathcal{M}) = \mu(\mathcal{B}), \quad \omega(\mathcal{M} \times \mathcal{B}) = \nu(\mathcal{B})$,
- $\omega(J^+) = 1$,

where $J^+ := \{(p, q) \in \mathcal{M}^2 \mid p \preceq q\}$.

- $\omega$ can be called a causal coupling or a causal transference plan.
- For $\mu = \delta_p$, $\nu = \delta_q$, the only coupling is $\omega = \delta_{(p,q)}$ and so $\delta_p \preceq \delta_q$ iff $p \preceq q$. 
Each infinitesimal part of the probability measure should travel along a future-directed causal curve.
Causality for probability measures

Each infinitesimal part of the probability measure should travel along a future-directed causal curve.
If $\mathcal{M}$ prohibits causal loops and $J^+$ is closed:

$$\mu \preceq \nu \iff \text{for any compact } K \subseteq \text{supp } \mu \quad \mu(K) \leq \nu(J^+(K))$$
Causal time-evolution of a pointlike particle

A curve $\gamma : I \to M$ with $\gamma(t) = (t, x(t))$ is a worldline of a physical particle if

$$\forall s, t \in I \quad s \leq t \implies \gamma(s) \preceq \gamma(t).$$

Causal time-evolution of a probability measure

A map $\mu : I \to \mathcal{P}(M)$, $t \mapsto \mu_t$ such that $\text{supp}\mu_t \subseteq \{t\} \times \mathbb{R}^3$ for all $t \in I$ is a causal evolution of a measure if

$$\forall s, t \in I \quad s \leq t \implies \mu_s \preceq \mu_t.$$
Causal time-evolution of a pointlike particle

A curve $\gamma : I \rightarrow \mathcal{M}$ with $\gamma(t) = (t, x(t))$ is a worldline of a physical particle if

$$\forall s, t \in I \quad s \leq t \Rightarrow \gamma(s) \preceq \gamma(t).$$

Causal time-evolution of a probability measure

A map $\mu : I \rightarrow \mathcal{P}(\mathcal{M})$, $t \mapsto \mu_t$ such that $\text{supp} \mu_t \subseteq \{t\} \times \mathbb{R}^3$ for all $t \in I$ is a causal evolution of a measure if

$$\forall s, t \in I \quad s \leq t \Rightarrow \mu_s \preceq \mu_t.$$
Causal time-evolution of measures ($\mathcal{M} - \text{Minkowski}$)

Causal time-evolution of a pointlike particle

A curve $\gamma : I \to \mathcal{M}$ with $\gamma(t) = (t, x(t))$ is a worldline of a physical particle if

$$\forall s, t \in I \quad s \leq t \implies \gamma(s) \preceq \gamma(t).$$

Causal time-evolution of a probability measure

A map $\mu : I \to \mathcal{P}(\mathcal{M})$, $t \mapsto \mu_t$ such that $\text{supp}\, \mu_t \subseteq \{t\} \times \mathbb{R}^3$ for all $t \in I$ is a causal evolution of a measure if

$$\forall s, t \in I \quad s \leq t \implies \mu_s \preceq \mu_t.$$
Fix a Cauchy temporal function $T$.

### Causal time-evolution of a pointlike particle

A curve $\gamma : I \to \mathcal{M}$ such that $T(\gamma(t)) = t$ is a worldline of a physical particle if

$$\forall s, t \in I \quad s \leq t \Rightarrow \gamma(s) \preceq \gamma(t).$$

### Causal time-evolution of a probability measure

A map $\mu : I \to \mathcal{P}(\mathcal{M})$, $t \mapsto \mu_t$ such that $\text{supp} \mu_t \subseteq T^{-1}(t)$ for all $t \in I$ is a causal evolution of a measure if

$$\forall s, t \in I \quad s \leq t \Rightarrow \mu_s \preceq \mu_t.$$
Causal time-evolution of measures \((\mathcal{M} - \text{glob. hyperbolic})\)
Causal time-evolution of measures ($\mathcal{M}$ – glob. hyperbolic)
Causal time-evolution of measures (\(\mathcal{M} \ - \ \text{glob. hyperbolic}\))

**Theorem [— ’17]**

Fix a Cauchy temporal function \(\mathcal{T}\).

Consider a map \(t \mapsto \mu_t \in \mathcal{P}(\mathcal{M})\) satisfying \(\text{supp} \ \mu_t \subseteq \mathcal{T}^{-1}(t)\) for all \(t \in I\).

TFAE:

- The map \(t \mapsto \mu_t\) is causal, i.e.
  \[ \forall s, t \in I \quad s \leq t \Rightarrow \mu_s \preceq \mu_t. \]

- There exists a **probability measure** on the space of worldlines, from which one can recover \(\mu_t\) for all \(t \in I\).

The “space of worldlines” is suitably topologized so as to ensure **Polishness** (= separability + complete metrizability).

Adapted from Penrose’s “Road to Reality”
Theorem [--- '17]

Fix a Cauchy temporal function $T$. Consider a map $t \mapsto \mu_t \in \mathcal{P}(\mathcal{M})$ satisfying $\text{supp} \mu_t \subseteq T^{-1}(t)$ for all $t \in I$. TFAE:

- The map $t \mapsto \mu_t$ is causal, i.e.
  \[ \forall s, t \in I \quad s \preceq t \implies \mu_s \preceq \mu_t. \]

- There exists a probability measure on the space of worldlines, from which one can recover $\mu_t$ for all $t \in I$.

The "space of worldlines" is suitably topologized so as to ensure Polishness (= separability + complete metrizability).

Adapted from Penrose's "Road to Reality"
Theorem [M. Eckstein, — ’17]

Suppose $\rho(t, x)$ satisfies the continuity equation $\partial_t \rho + \nabla \cdot \rho \mathbf{v} = 0$ with a velocity field such that $\| \mathbf{v}(t, x) \| \leq 1$. Then $\mu_t$ defined via

$$d\mu_t = \delta_t \otimes \rho(t, x) \, d^3 x$$

evolves causally.

- Wave-packet formalism: $\rho(t, x) := |\psi(t, x)|^2$, where $i\hbar \partial_t \psi = \hat{H} \psi$

  $\Rightarrow$ M. Eckstein and —, Phys. Rev. A 2017 95, 032106
Relationship with the continuity equation ($\mathcal{M}$ – Minkowski)

Theorem [M. Eckstein, — ’17]

Suppose $\rho(t, x)$ satisfies the continuity equation $\partial_t \rho + \nabla \cdot \rho \mathbf{v} = 0$ with a velocity field such that $\|\mathbf{v}(t, x)\| \leq 1$. Then $\mu_t$ defined via

$$d\mu_t = \delta_t \otimes \rho(t, x) d^3x$$

evolves causally.

- Wave-packet formalism: $\rho(t, x) := |\psi(t, x)|^2$, where $i\hbar \partial_t \psi = \hat{H} \psi$

$\rightsquigarrow$ M. Eckstein and —, Phys. Rev. A 2017 95, 032106
Suppose $\rho(t, x)$ satisfies the continuity equation $\partial_t \rho + \nabla \cdot \rho \mathbf{v} = 0$ with a velocity field such that $\|\mathbf{v}(t, x)\| \leq 1$. Then $\mu_t$ defined via

$$d\mu_t = \delta_t \otimes \rho(t, x) \, d^3x$$

evolves causally.

- Wave-packet formalism: $\rho(t, x) := |\psi(t, x)|^2$, where $i\hbar \partial_t \psi = \hat{H} \psi$

$\leadsto$ M. Eckstein and —, Phys. Rev. A 2017 95, 032106
**Conjecture**

Fix a Cauchy temporal function $\mathcal{T}$. Suppose $\mu_t$ (such that $\text{supp } \mu_t \subseteq \mathcal{T}^{-1}(t)$) satisfies:

$$\forall \Phi \in C^\infty_c(\mathcal{T}^{-1}(I)) \quad \int_I \int_{\mathcal{M}} X\Phi \, d\mu_t \, dt = 0 \quad (\star)$$

with a certain **causal** vector field $X$. Then $\mu_t$ evolves causally.

**Converse result (preliminary!)**

Fix a Cauchy temporal function $\mathcal{T}$. Suppose $\mu_t$ evolves causally. Then there exists a **causal** vector field $X$ such that $(\star)$ holds.

$X$ is generally only $L^2$-regular.

**Conjecture**

Fix a Cauchy temporal function $\mathcal{T}$. Suppose $\mu_t$ (such that $\text{supp } \mu_t \subseteq \mathcal{T}^{-1}(t)$) satisfies:

$$
\forall \Phi \in C_\infty^c(\mathcal{T}^{-1}(I)) \quad \int_I \int_{\mathcal{M}} X\Phi \, d\mu_t \, dt = 0
$$

(⋆)

with a certain **causal** vector field $X$. Then $\mu_t$ evolves causally.

**Converse result (preliminary!)**

Fix a Cauchy temporal function $\mathcal{T}$. Suppose $\mu_t$ evolves causally. Then there exists a **causal** vector field $X$ such that (⋆) holds.

$X$ is generally only $L^2$-regular.
Conjecture

Fix a Cauchy temporal function $\mathcal{T}$. Suppose $\mu_t$ (such that $\text{supp } \mu_t \subseteq \mathcal{T}^{-1}(t)$) satisfies:

$$\forall \Phi \in C^\infty_c(\mathcal{T}^{-1}(I)) \quad \int_I \int_{\mathcal{M}} X\Phi \, d\mu_t \, dt = 0 \quad (\star)$$

with a certain causal vector field $X$. Then $\mu_t$ evolves causally.

Converse result (preliminary!)

Fix a Cauchy temporal function $\mathcal{T}$. Suppose $\mu_t$ evolves causally. Then there exists a causal vector field $X$ such that $(\star)$ holds.

$X$ is generally only $L^2$-regular.
Conclusions and take-home messages

- The causal relation ≤ can be naturally extended onto $\mathcal{P}(\mathcal{M})$ – the space of Borel probability measures on $\mathcal{M}$.
- One can use thus extended relations to describe the causal evolution of probability measures in glob. hyperbolic spacetimes.
  - Time-evolution of a pointlike particle ↩️ single worldline.
  - Time-evolution of a nonlocal object ↩️ prob. measure on the space of worldlines.
- Causal time-evolution of measures is intimately related to the continuity equation $\partial_t \mu_t + \nabla \cdot \mu_t v = 0$ with a subluminal velocity field $v$. 
Conclusions and take-home messages

- The causal relation $\preceq$ can be naturally extended onto $\mathcal{P}(\mathcal{M})$ – the space of Borel probability measures on $\mathcal{M}$.
- One can use thus extended relations to describe the causal evolution of probability measures in glob. hyperbolic spacetimes.
  - Time-evolution of a pointlike particle $\iff$ single worldline.
  - Time-evolution of a nonlocal object $\iff$ prob. measure on the space of worldlines.
- Causal time-evolution of measures is intimately related to the continuity equation $\partial_t \mu_t + \nabla \cdot \mu_t v = 0$ with a subluminal velocity field $v$. 
Conclusions and take-home messages

- The causal relation $\preceq$ can be naturally extended onto $\mathcal{P}(\mathcal{M})$ – the space of Borel probability measures on $\mathcal{M}$.
- One can use thus extended relations to describe the causal evolution of probability measures in glob. hyperbolic spacetimes.
  - Time-evolution of a pointlike particle $\Leftrightarrow$ single worldline.
  - Time-evolution of a nonlocal object $\Leftrightarrow$ prob. measure on the space of worldlines.
- Causal time-evolution of measures is intimately related to the continuity equation $\partial_t \mu_t + \nabla \cdot \mu_t v = 0$ with a subluminal velocity field $v$. 
The causal relation $\preceq$ can be naturally extended onto $\mathcal{P}(\mathcal{M})$ – the space of Borel probability measures on $\mathcal{M}$.

One can use thus extended relations to describe the **causal evolution of probability measures** in glob. hyperbolic spacetimes.

- Time-evolution of a **pointlike** particle $\leftrightarrow$ **single** worldline.
- Time-evolution of a **nonlocal** object $\leftrightarrow$ **prob. measure** on the space of worldlines.

Causal time-evolution of measures is intimately related to the continuity equation $\partial_t \mu_t + \nabla \cdot \mu_t v = 0$ with a subluminal velocity field $v$. 

Tomasz Miller (WUT & CC)
Conclusions and take-home messages

• The causal relation $\preceq$ can be naturally extended onto $\mathcal{P}(M)$ – the space of Borel probability measures on $M$.

• One can use thus extended relations to describe the causal evolution of probability measures in glob. hyperbolic spacetimes.
  
  • Time-evolution of a pointlike particle $\leftrightarrow$ single worldline.
  • Time-evolution of a nonlocal object $\leftrightarrow$ prob. measure on the space of worldlines.

• Causal time-evolution of measures is intimately related to the continuity equation $\partial_t \mu_t + \nabla \cdot \mu_t \mathbf{v} = 0$ with a subluminal velocity field $\mathbf{v}$. 

Tomasz Miller (WUT & CC)
Thank you for your attention!

M. Eckstein and T. Miller, *Causality for nonlocal phenomena*, Annales Henri Poincaré 2017, **18**(9), 3049–3096,


M. Eckstein and T. Miller, *Causal evolution of wave packets*, Physical Review A 2017 **95**, 032106,

T. Miller, *On the causality and K-causality between measures*, Universe 2017 3(1):27,
Q: How to topologize sets of (fut-dir) causal curves?
A (naïve): Induce topology from $C(I, M)$ (the compact-open top.)

Too large a space! Various parameterizations of an unparameterized curve treated as distinct elements!

Two ways out:
- Take a quotient modulo (continuous strictly increasing) reparameterizations $\Leftrightarrow$ focus on unparameterized curves, and use the $C^0$-topology.
- Choose the “canonical” parameterization of each curve — e.g. the arc-length parameterization — and use the compact-open topology.
Spaces of causal curves parameterized “in accordance with $\mathcal{T}$”

$\mathcal{M}$ – stably causal spacetime, $\mathcal{T}$ – time function, $I$ – interval.

$C^I_T$ := the space of all fut-dir causal curves $\gamma \in C(I, \mathcal{M})$ such that

$$\exists c_\gamma > 0 \ \forall s,t \in I \quad \mathcal{T}(\gamma(t)) - \mathcal{T}(\gamma(s)) = c_\gamma(t - s),$$

dowered with the compact-open topology induced from $C(I, \mathcal{M})$.

- $C^I_T$ is separable and completely metrizable (i.e. Polish).
- $\mathcal{C}$ := the space of all compact unparameterized causal curves with the $C^0$-topology. **Theorem:** $C^{[a,b]}_T \cong \mathcal{C}$ and hence:
  - $\mathcal{C}$ is Polish!
  - $C^{[a,b]}_T \cong C^{[c,d]}_T$.

$\mathcal{M}$ – glob. hyperbolic, $\mathcal{T}_1, \mathcal{T}_2$ – Cauchy temporal functions.

**Theorem:** $C^{\mathbb{R}}_{T_1} \cong C^{\mathbb{R}}_{T_2}$. 

**Bonus:** Polish spaces of causal curves
Theorem [— ’17]

Fix a Cauchy temporal function $\mathcal{T}$. Consider a map $t \mapsto \mu_t \in \mathcal{P}(\mathcal{M})$ satisfying $\text{supp} \mu_t \subseteq \mathcal{T}^{-1}(t)$ for all $t \in I$.

TFAE:

- The map $t \mapsto \mu_t$ is causal, i.e.

  $$\forall s, t \in I \quad s \leq t \Rightarrow \mu_s \preceq \mu_t.$$ 

- $\exists \sigma \in \mathcal{P}(C^I_{\mathcal{T}})$ such that

  $$(\text{ev}_t) \# \sigma = \mu_t,$$

  where $\text{ev}_t : C^I_{\mathcal{T}} \to \mathcal{M}$, $\gamma \mapsto \gamma(t)$.

Adapted from R. Penrose’s “Road to Reality”