Mass gap in non-perturbative quantization à la Heisenberg

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W. Heisenberg, 50th, nonlinear Dirac equations as a fundamental equation for electron. Nonperturbative quantization = operator equation = infinite equations set for all Green functions.
Outline:

- Initial equations.
- Approximations.
- Spinball, energy spectrum, mass gap.
- Spinball + quantum monopole, energy spectrum, mass gap.
- Conclusions
The operator Yang - Mills - Dirac equations

\[ D_\nu \hat{F}^{B\mu \nu} = \frac{g\hbar c}{2} \hat{\psi} \gamma^\mu \lambda^B \hat{\psi}, \]

\[ i\hbar \gamma^\mu \left[ \partial_\mu \hat{\psi}(x) - ig^2 \lambda^B \hat{A}_\mu^B(x) \hat{\psi}(x) \right] - m_q c \hat{\psi}(x) = 0, \]
The Yang - Mills - Dirac operator equations are equivalent to an infinite set of equations for the Green functions, i.e.,

\[ \langle D_v \hat{F}^{B \mu \nu}(x) \rangle = \frac{g \hbar c}{2} \langle \hat{\psi} \gamma^\mu \lambda^B \hat{\psi} \rangle, \]

\[ \langle i \hbar \gamma^\mu D_\mu \hat{\psi}(x) - mc \hat{\psi}(x) \rangle = 0, \]

\[ \langle \hat{A}_{\alpha_1}^{B_1}(x_1) D_v \hat{F}^{A \mu \nu}(x) \rangle = \frac{g \hbar c}{2} \langle \hat{A}_{\alpha_1}^{B_1}(x_1) \hat{\psi} \gamma^\mu \lambda^B \hat{\psi} \rangle, \]

\[ \ldots = 0, \]

\[ \langle \hat{A}_{\alpha_1}^{B_1}(x_1) \ldots \hat{A}_{\alpha_n}^{B_n}(x_n) D_v \hat{F}^{A \mu \nu}(x) \rangle = 0 \]

This represents an infinite set of differential equations that must be solved for any particular physical system; if this can be done, we end up with an infinite set of Green functions which should contain all the physical information about the field operators and the quantum states of the system.
It then follows that we can identify the set of nonperturbative quantum states with the set of Green functions. This is the main observation that allows us to formally define nonperturbative quantum states in terms of Green functions.
The strategy:

- We assume that in some physical situations all $SU(3)$ degrees of freedom can be decomposed into two groups. In the first group, the gauge fields

\[ \hat{A}^a_\mu = \langle \hat{A}^a_\mu \rangle + i \delta \hat{A}^a_\mu \in SU(2) \times U(1) \subset SU(3). \]

In the second group, the gauge fields

\[ \hat{A}^m_\mu \in SU(3)/\left(SU(2) \times U(1)\right) \]

are pure quantum ones in the sense that $\langle \hat{A}^m_\mu \rangle = 0$. 
Decomposition for 2-point Green function:

\[ \langle \hat{A}^{m\mu}(y) \hat{A}^{n\nu}(x) \rangle \approx C^{mn\mu\nu} \phi(y)\phi(x) \]
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Decomposition for 4-point Green function:

\[ G^{mnpq\alpha\beta\mu\nu}(x,x,x,x) \approx C^{mnpq\alpha\beta\mu\nu} \phi^2(x) \left[ M^2 - \phi^2(x) \right] , \]
Approximations

- Decomposition for 2-point Green function:

\[ \langle \hat{A}^{m\mu}(y)\hat{A}^{n\nu}(x) \rangle \approx C^{mn\mu\nu}\phi(y)\phi(x) \]

- Decomposition for 4-point Green function:

\[ G^{mnpq\alpha\beta\mu\nu}(x,x,x,x) \approx C^{mnpq\alpha\beta\mu\nu}\phi^2(x)\left[M^2 - \phi^2(x)\right], \]

- Decomposition for 3-point Green function:

\[ \langle \hat{\psi}(y)\lambda^m\gamma^\mu\hat{A}_\mu(x)\hat{\psi}\gamma_k(x) \rangle \approx \Lambda \left[\bar{\zeta}(y)\zeta(x)\right]\left(\bar{\zeta}(x)\zeta(x)\right)\phi(x), \]
Virtual quarks

\[ \langle \hat{\psi} \rangle = 0, \]

Dispersion of virtual quarks

\[ \langle \hat{\bar{\psi}} \gamma^\mu \lambda^a \hat{\psi} \rangle = \bar{\zeta} \gamma^\mu \lambda^a \zeta, \]
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- Stationarity
Three equation approximation

\[ D_\nu F^{a\mu\nu} - \left[ (m^2)^{ab\mu\nu} - (\mu^2)^{ab\mu\nu} \right] A^b_\nu = \frac{g \hbar c}{2} \left( \bar{\zeta} \gamma^\mu \lambda^a \zeta \right), \]

\[ \Box \phi - (m_\phi^2)^{ab\mu\nu} A^a_\nu A^b_\mu \phi - \lambda \phi (M^2 - \phi^2) = \Lambda^2 \frac{g \hbar c}{2} \left( \bar{\zeta} \zeta \right)^2, \]

\[ i \hbar \gamma^\mu \left( \partial_\mu \zeta - i \frac{g}{2} \lambda^a A^a_\mu \zeta \right) + \Lambda_1 \frac{g \hbar}{2} \phi \zeta \left( \bar{\zeta} \zeta \right) - m_q c \zeta = 0 \]
The solution we are searching in the form

\[
A_i^a = \frac{f(r) - 1}{gr^2} \varepsilon_{iaj} x^j, \quad a = 1, 2, 3
\]

\[
A_t^a = 0,
\]

\[
A_t^\delta = \frac{\chi(r)}{g},
\]

\[
\phi = \frac{\xi(r)}{g},
\]

\[
\zeta^T = \frac{e^{-i\frac{E_t}{\hbar}}}{gr \sqrt{2}} \left\{ \begin{pmatrix} 0 \\ -u(r) \\ 0 \end{pmatrix}, \begin{pmatrix} u(r) \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} iv(r) \sin \theta e^{-i\varphi} \\ -iv(r) \cos \theta \\ 0 \end{pmatrix}, \begin{pmatrix} -iv(r) \cos \theta \\ -iv(r) \sin \theta e^{i\varphi} \\ 0 \end{pmatrix} \right\}
\]
\[-f'' + f \left( \frac{f^2 - 1}{x^2} \right) - m^2 (1 - f) \overline{\xi}^2 + \frac{\overline{u} \overline{v}}{x} = -\overline{\mu}^2 (1 - f),\]

\[\tilde{\chi}'' + \frac{2}{x} \tilde{\chi}' = \frac{1}{2\sqrt{3}} \frac{\overline{u}^2 + \overline{v}^2}{x^2},\]

\[\xi'' + \frac{2}{x} \xi' = \xi \left[ \frac{(1 - f)^2}{2x^2} + \tilde{\lambda} \left( \overline{\xi}^2 - \tilde{M}^2 \right) \right] - \frac{\Lambda}{4\tilde{g}^2} \frac{\left( \overline{u}^2 - \overline{v}^2 \right)^2}{x^4},\]

\[\tilde{v}' + \frac{f \tilde{v}}{x} = \tilde{u} \left( -\tilde{m}_q + \tilde{E} + \frac{m^2 \Lambda}{\tilde{g}^2} \xi \frac{u^2 - v^2}{x^2} + \frac{\tilde{\chi}}{2\sqrt{3}} \right),\]

\[\tilde{u}' - \frac{f \tilde{u}}{x} = \tilde{v} \left( -\tilde{m}_q - \tilde{E} + \frac{m^2 \Lambda}{\tilde{g}^2} \xi \frac{u^2 - v^2}{x^2} - \frac{\tilde{\chi}}{2\sqrt{3}} \right).\]
Non-linear Dirac equation

\[ \tilde{v}' + \frac{\tilde{v}}{\tilde{x}} = \tilde{u} \left( -1 + \tilde{E} + \tilde{\Lambda} \frac{u^2 - v^2}{x^2} \right), \]

\[ \tilde{u}' - \frac{\tilde{u}}{\tilde{x}} = \tilde{v} \left( -1 - \tilde{E} + \tilde{\Lambda} \frac{u^2 - v^2}{x^2} \right) \]
Figure: Profiles $\tilde{v}(x)/x$, $\tilde{u}(x)/x$ for spinball solution.

Figure: Profile of energy density $\varepsilon(x)$ for spinball.
Dimensional energy density

\[ \tilde{\varepsilon} = \frac{m_q^3}{\tilde{g}^2} \left[ \frac{\tilde{E} \tilde{u}^2 + \tilde{\nu}^2}{2} - \frac{\tilde{\Lambda}}{4 \tilde{g}^2} \frac{(\tilde{u}^2 - \tilde{\nu}^2)^2}{\tilde{x}^4} \right] = \frac{m_q^3}{\tilde{g}^2} \tilde{\varepsilon} \]
Mass gap in spinball

\begin{align*}
E^- &= 0.1 \\
E^- &= 0.2 \\
E^- &= 0.4 \\
E^- &= 0.6 \\
E^- &= 0.8 \\
E^- &= 0.9
\end{align*}

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\bar{E}^- &= 0.1 \\
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\bar{E}^- &= 0.4 \\
\bar{E}^- &= 0.6 \\
\bar{E}^- &= 0.8 \\
\bar{E}^- &= 0.9
\end{align*}


The set of equations for the spinball-plus-quantum-monopole system

\[-f'' + \frac{f \left( f^2 - 1 \right)}{x^2} - m^2 \left( 1 - f \right) \xi'' + \xi = -\mu^2 \left( 1 - f \right),\]

\[\xi'' + \frac{2}{x} \xi' = \xi \left[ \frac{(1 - f)^2}{2x^2} + \lambda \left( \xi^2 - M^2 \right) \right] - \frac{\tilde{g}^2 \tilde{\Lambda} \left( \tilde{u}^2 - \tilde{v}^2 \right)^2}{8} \left( \tilde{u}^2 - \tilde{v}^2 \right)^2,\]

\[\tilde{v}' + \frac{f \tilde{v}}{x} = \tilde{u} \left( -\tilde{m}_q + \tilde{E} + m^2 \tilde{\Lambda} \frac{\tilde{u}^2 - \tilde{v}^2}{x^2} \xi \right),\]

\[\tilde{u}' - \frac{f \tilde{u}}{x} = \tilde{v} \left( -\tilde{m}_q - \tilde{E} + m^2 \tilde{\Lambda} \frac{\tilde{u}^2 - \tilde{v}^2}{x^2} \xi \right).\]
The dimensionless energy density for the spinball-plus-quantum-monopole system

\[ \tilde{\varepsilon} = \frac{1}{\tilde{g}^2} \left\{ \frac{f'^2}{x^2} + \frac{(f^2 - 1)^2}{2x^4} - \tilde{\mu}^2 \frac{(f - 1)^2}{x^2} \right\} + \]

\[ 2m^2 \left[ \tilde{\xi}'^2 + \frac{(f - 1)^2}{2x^2} \tilde{\xi}^2 + \frac{\tilde{\lambda}}{2} \left( \tilde{\xi}^2 - \tilde{M}^2 \right)^2 \right] \}

\[ + \tilde{E} \frac{\tilde{u}^2 + \tilde{v}^2}{x^2} + m^2 \frac{\tilde{\Lambda}}{2} \tilde{\xi} \frac{(\tilde{u}^2 - \tilde{v}^2)^2}{x^4}. \]
Three-dimensional and contour plots for the energy:
Conclusions

- The energy spectra for the spinball and for the spinball-plus-quantum-monopole system are obtained. It was shown that they possess the mass gaps.
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- Solutions describing virtual quarks and gauge fields in a bag have been found.
- It was shown that the bags are created due to the Meissner effect, when the coset condensate expels the gauge fields.
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- For the quantum-monopole/dyon systems, it was shown that color magnetic and electric fields decrease asymptotically according to exponential and power laws, respectively.
- The nonlinear Dirac equation has been used as an approximate description of an infinite set of equations for all Green functions of the spinor equation.
Turbulent river = flux tube

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Thanks for your attention!