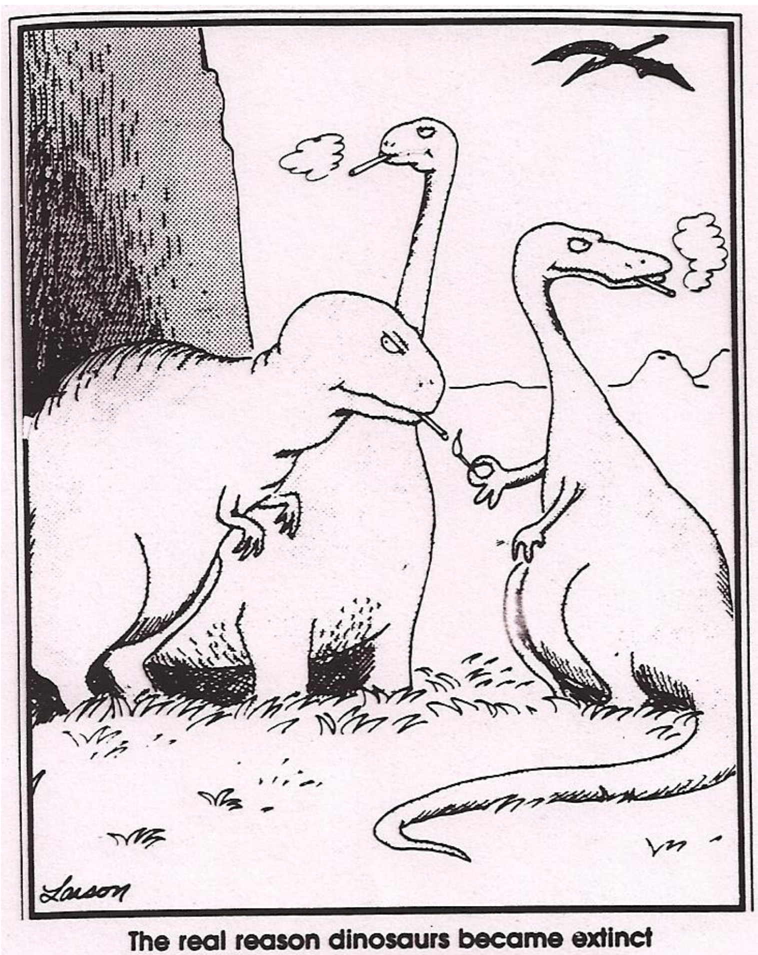
Nonlinear eigenvalue problems
--a talk in fond memory of Lev Lipatov



(Work done in collaboration with A. Fring, J. Komijani, & Q. Wang)

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Linear eigenvalue problems...

$$-\psi''(x) + V(x)\psi(x) = E\psi(x) \qquad\qquad \psi(\pm\infty) = 0$$

(1) Oscillatory in *classically allowed* region (*n*th eigenfunction has *n* nodes)

(2) Monotone decay in *classically forbidden* region

(3) Transition at the boundary (*turning point*)

(4) Unstable with respect to small changes in *E*

Physical solution is *subdominant*

Leading asymptotic behavior of eigenfunctions of

$$-\psi''(x)+V(x)\psi(x)=E\psi(x)$$

for large positive *x*:

$$\psi(x)\sim D[V(x)-E]^{-1/4}\exp\left[-\int^x ds\sqrt{V(s)-E}\right]\quad(x\to\infty)$$

NOTE: There is only *ONE* arbitrary constant even though the differential equation is second order

For *linear* problems *WKB* gives a good approximation for large eigenvalues

$$\int_{x_1}^{x_2} dx\sqrt{E_n - V(x)} \sim (n+1/2)\pi \quad (n \to \infty)$$

*n*th energy level grows like a *constant* times a power of *n*

Example 1: harmonic oscillator

$$V(x) = x^2$$

$$E_n \sim n \quad (n \to \infty)$$

Example 2: anharmonic oscillator

$$V(x) = x^4$$

$$E_n \sim Bn^{4/3} \quad (n \to \infty)$$

$$B = \left[\frac{3\Gamma(3/4)\sqrt{\pi}}{\Gamma(1/4)}\right]^{4/3}$$

Nonlinear eigenvalue problems

Consider an equation like

$$y'(x) = F[xy(x)]$$

-or-

$$y''(x) = F[xy(x)]$$

If the solution $y(x)$ to $f\,(x,y) = 0$ vanishes for large x, does the solution to the differential equation also vanish for large x *<u>in a stable fashion</u>*?

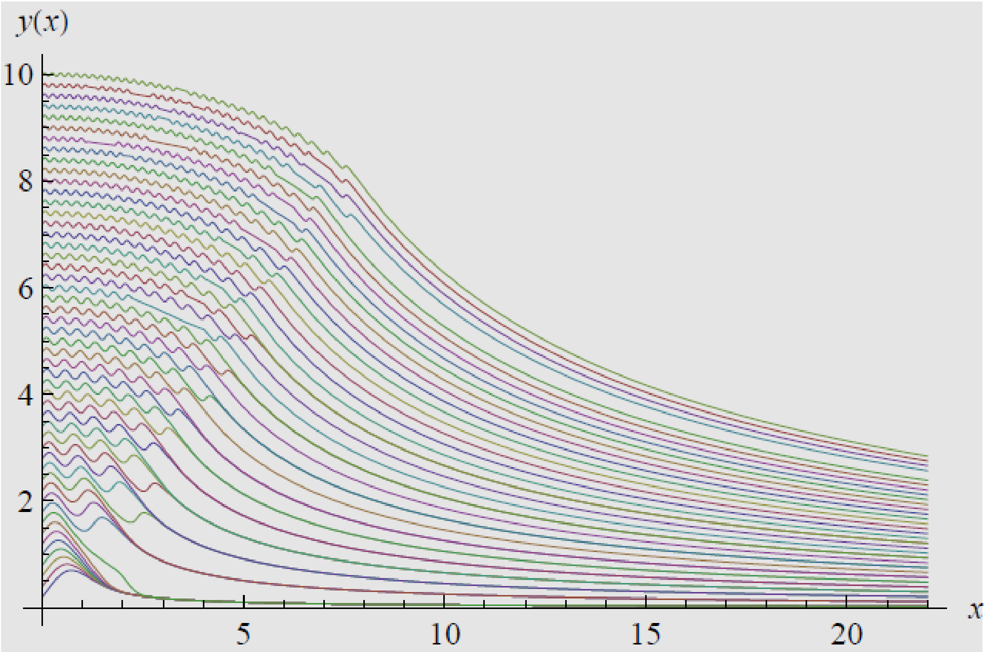
Toy nonlinear eigenvalue problem

$$\boxed{y'(x) = \cos[\pi x y(x)], \quad y(0) = a}$$

Asymptotic behavior for large *x*

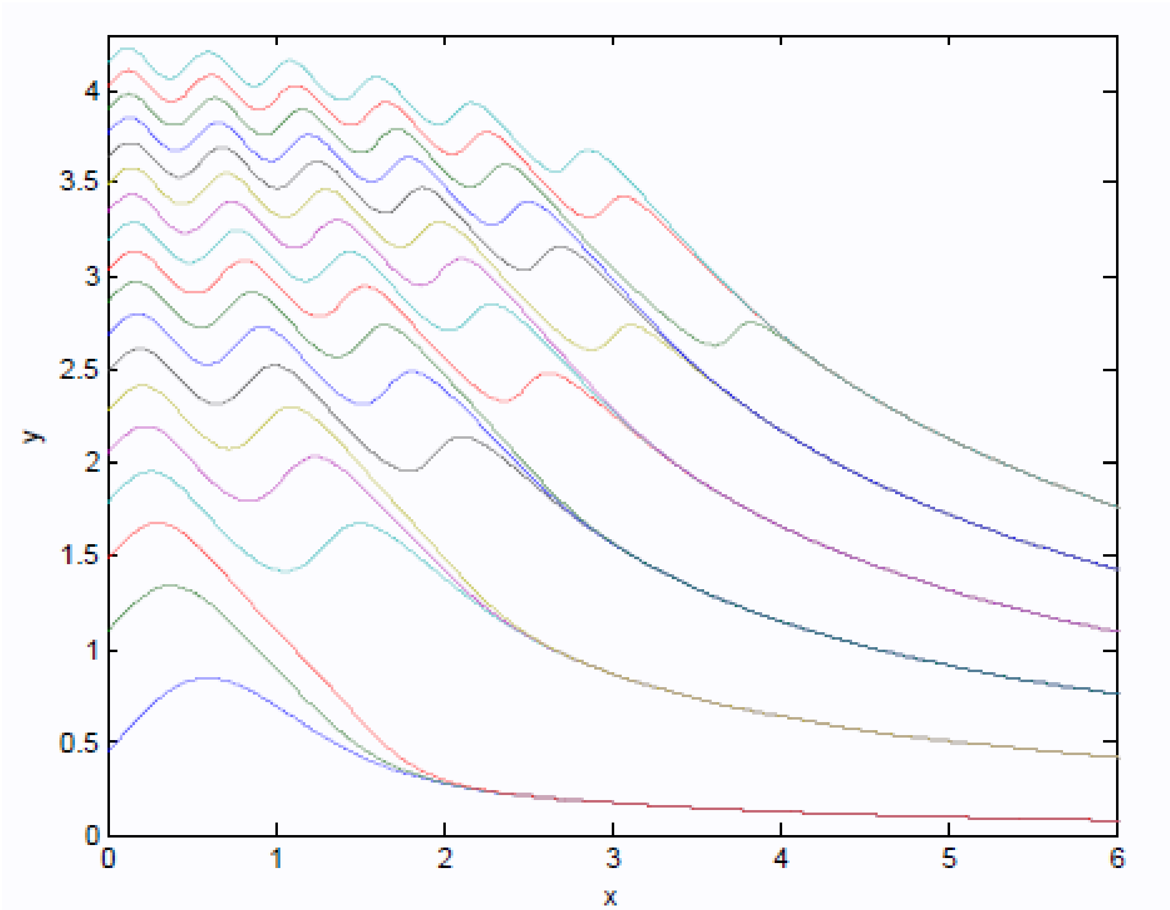
Solution behaves like: $$y(x) \sim \frac{m + 1/2}{x}$$

***m* = 0, 1, 2, 3, ... is an integer**



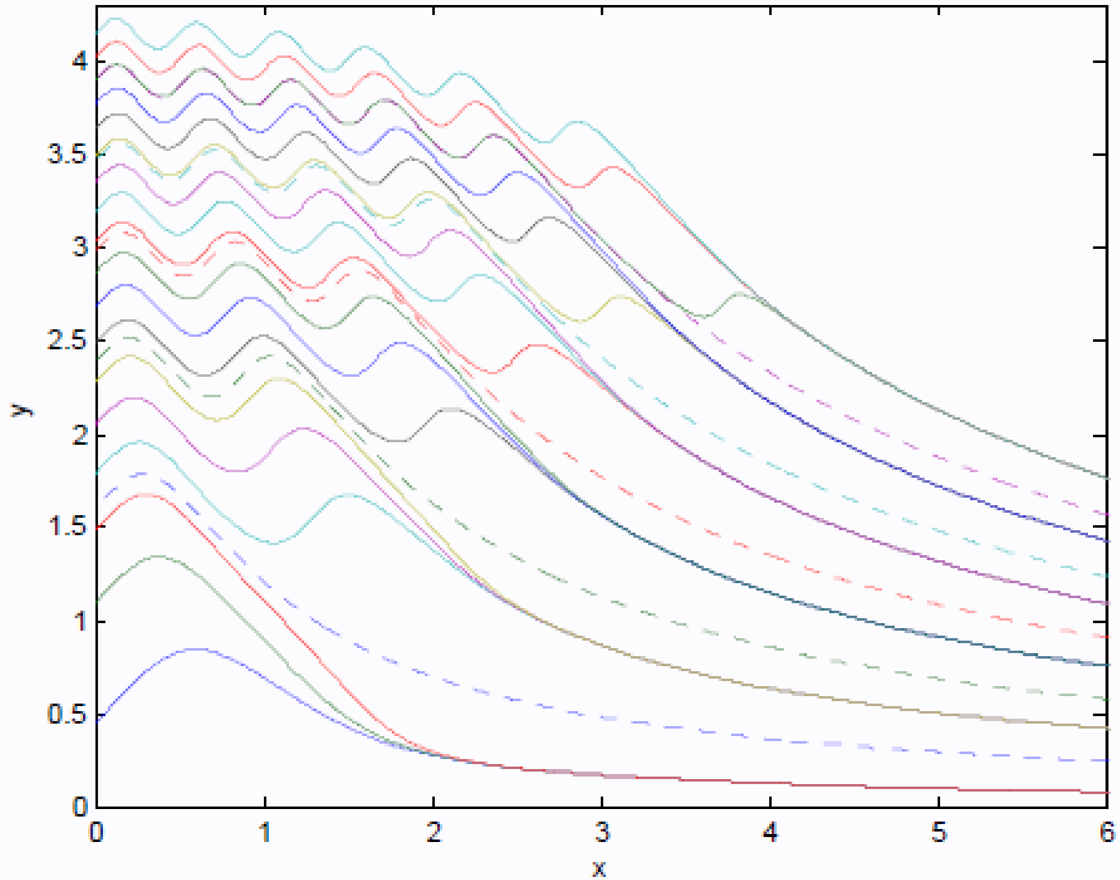
Note: oscillation followed by monotone decay

But there's a ***big*** problem here...



m **= 10**
m **= 8**
m **= 6**
m **= 4**
m **= 2**
m **= 0**

Where are the odd-*m* solutions?!?



$m = 9$
$m = 7$
$m = 5$
$m = 3$
$m = 1$

$$y(0) = a \in \{1.6026, 2.3884, 2.9767, 3.4675, 3.8975, 4.2847, \ldots\}$$

Eigenvalues **correspond to odd-*m* initial values.**
Eigenfunctions **are (*unstable*) *separatrices*, which begin at eigenvalues.**

Note: no arbitrary constant appears in the asymptotic behavior!!

Where is the arbitrary constant?!?

Is it in higher order….?

Higher-order asymptotic behavior for large *x* still contains <u>no arbitrary constant</u>!

$$y(x) \sim \frac{m+1/2}{x} + \sum_{k=1}^{\infty} \frac{c_k}{x^{2k+1}} \quad (x \to \infty)$$

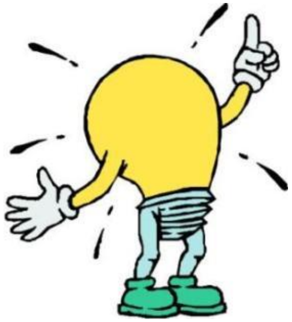
$$\begin{aligned}
c_1 &= \frac{(-1)^m}{\pi}(m+1/2), \\
c_2 &= \frac{3}{\pi^2}(m+1/2), \\
c_3 &= (-1)^m \left[\frac{(m+1/2)^3}{6\pi} + \frac{15(m+1/2)}{\pi^3}\right], \\
c_4 &= \frac{8(m+1/2)^3}{3\pi^2} + \frac{105(m+1/2)}{\pi^4}, \\
c_5 &= (-1)^m \left[\frac{3(m+1/2)^5}{40\pi} + \frac{36(m+1/2)^3}{\pi^3} + \frac{945(m+1/2)}{\pi^5}\right], \\
c_6 &= \frac{38(m+1/2)^5}{15\pi^2} + \frac{498(m+1/2)^3}{\pi^4} + \frac{10395(m+1/2)}{\pi^6}.
\end{aligned}$$

Asymptotics beyond all orders

Difference of two solutions in one bundle: $Y(x) \equiv y_1(x) - y_2(x)$

$$\begin{aligned} Y'(x) &= \cos[\pi x y_1(x)] - \cos[\pi x y_2(x)] \\ &= -2\sin\left[\tfrac{1}{2}\pi x y_1(x) + \tfrac{1}{2}\pi x y_2(x)\right]\sin\left[\tfrac{1}{2}\pi x y_1(x) - \tfrac{1}{2}\pi x y_2(x)\right] \\ &\sim -2\sin\left[\pi\left(m+\tfrac{1}{2}\right)\right]\sin\left[\tfrac{1}{2}\pi x Y(x)\right] \quad (x\to\infty) \\ &\sim -(-1)^m \pi x Y(x) \quad (x\to\infty). \end{aligned}$$

$$Y(x) \sim K\exp\left[-(-1)^m \pi x^2\right] \quad (x\to\infty)$$



Aha! *K* is the invisible arbitrary constant!
Odd-*m* solutions (eigenfunctions) are <u>*unstable*</u>;
even-*m* solutions are *stable*.

We calculated up to *m*=500,001

Let $m = 2n - 1$

For large *n* the *n*th eigenvalue grows like the *square root* of *n* times a constant *A*, and we used Richardson extrapolation to show that

$$A = 1.7817974363...$$

and then we <u>guessed</u> *A*.

Result:

$$a_n \sim A\sqrt{n} \quad (n \to \infty)$$

$$A = 2^{5/6}$$

This is a rather nontrivial problem...

Analytic calculation of the constant *A*

Construct moments of *z*(*t*):

$$A_{n,k}(t) \equiv \int_0^t ds \cos[n\lambda s z(s)] \frac{s^{k+1}}{[z(s)]^k}$$

Moments are associated with a semi-infinite linear one-dimensional random walk in which random walkers become static as they reach n=1

$$2\alpha_{1,k} + \alpha_{2,k-1} = 0, \qquad 2\alpha_{n,k} + \alpha_{n-1,k-1} + \alpha_{n+1,k-1} = 0 \quad (n \geq 3).$$

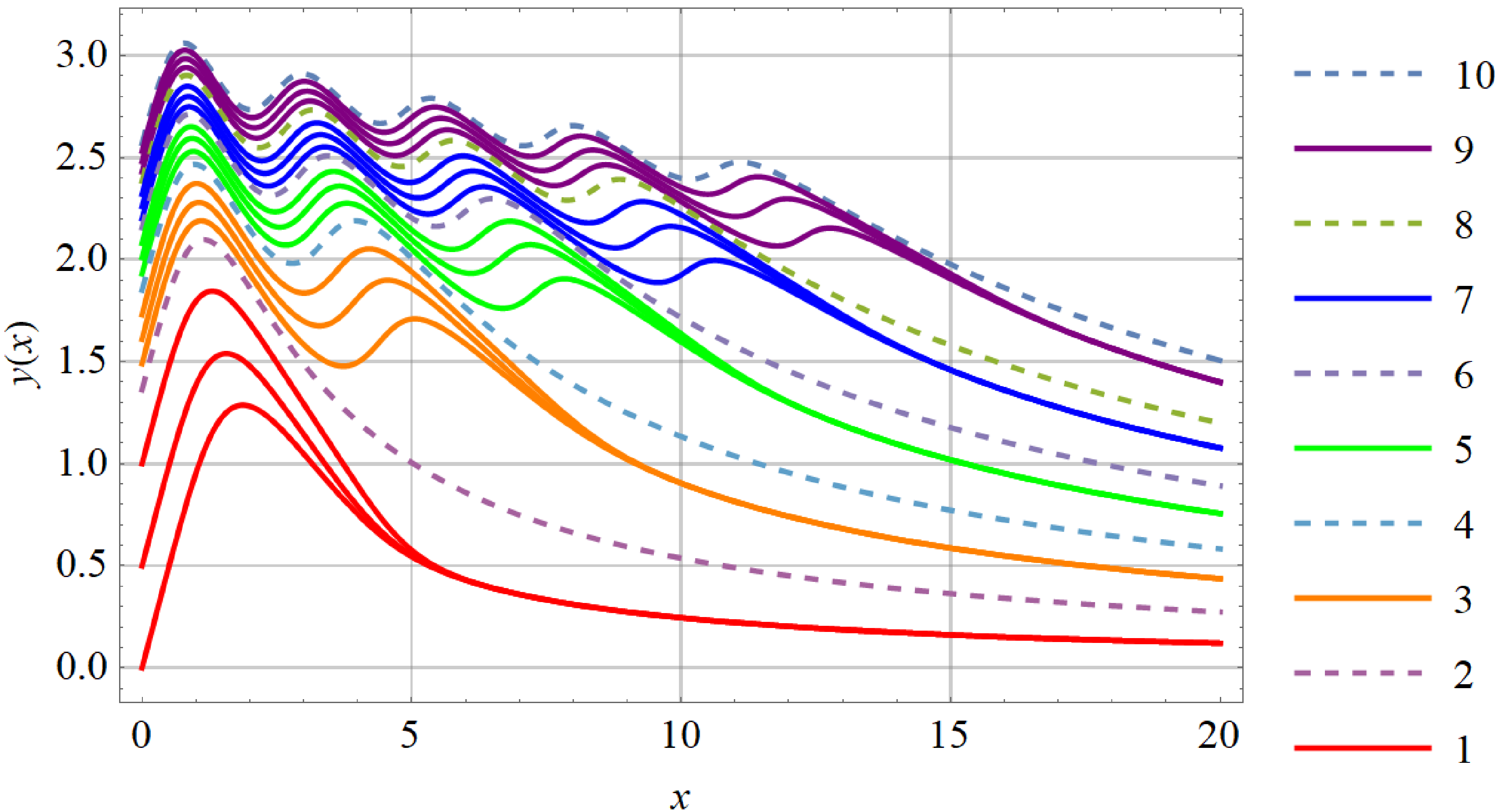
$$2\alpha_{2,k} + \alpha_{3,k-1} = 0,$$

Solve the random walk problem exactly and get $A = 2^{5/6}$

CMB, A. Fring, and J. Komijani
J. Phys. A: Math. Theor. **47**, 235204 (2014)
[arXiv: math-ph/1401.6161]

Bessel function instead of cosine

$$y'(x) = \mathrm{J}_0[xy(x)]$$



Painlevé equations

Paul Painlevé
(1863-1933)

Six Painlevé equations known as Painlevé I – VI

Only spontaneous singularities are poles

Painlevé I

$$\frac{d^2y}{dt^2} = 6y^2 + t$$

Painlevé II

$$\frac{d^2y}{dt^2} = 2y^3 + ty + \alpha$$

Painlevé III

$$ty\frac{d^2y}{dt^2} = t\left(\frac{dy}{dt}\right)^2 - y\frac{dy}{dt} + \delta t + \beta y + \alpha y^3 + \gamma t y^4$$

Painlevé IV

$$y\frac{d^2y}{dt^2} = \tfrac{1}{2}\left(\frac{dy}{dt}\right)^2 + \beta + 2(t^2 - \alpha)y^2 + 4ty^3 + \tfrac{3}{2}y^4$$

Painlevé V

$$\frac{d^2y}{dt^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1}$$

Painlevé VI

$$\frac{d^2y}{dt^2} = \tfrac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)$$

First Painlevé transcendent

$$y''(t) = 6[y(t)]^2 + t, \qquad y(0) = b,\ y'(0) = c$$

If the right side of this equation vanishes, the solution *y*(*x*) must *choose* between two possible asymptotic behaviors as *x* gets large and negative:

$$+\sqrt{-t/6} \text{ or } -\sqrt{-t/6}.$$

Lower square-root branch is *stable*:

$$y(x) \sim -\sqrt{-x} + c(-x)^{-1/8} \cos\left[\tfrac{4}{5}\sqrt{2}(-x)^{5/4} + d\right] \quad (x \to -\infty)$$

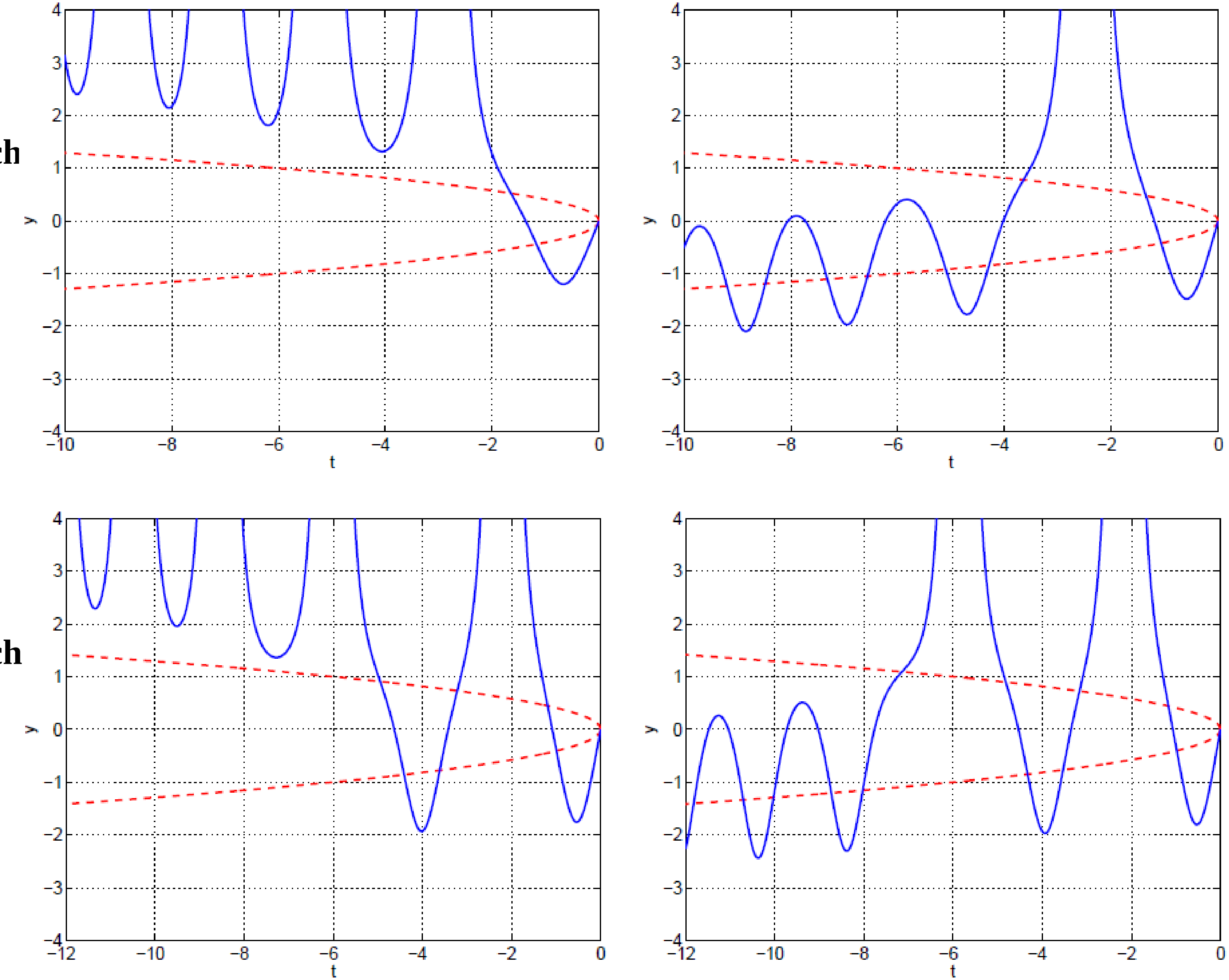
Upper square-root branch is *unstable*:

$$y(x) \sim \sqrt{-x} + c_{\pm}(-x)^{-1/8} \exp\left[\pm\tfrac{4}{5}\sqrt{2}(-x)^{5/4}\right] \quad (x \to -\infty)$$

Two kinds of solutions (NOT eigenfunctions):

Unstable branch

Stable branch



Unstable branch

Stable branch

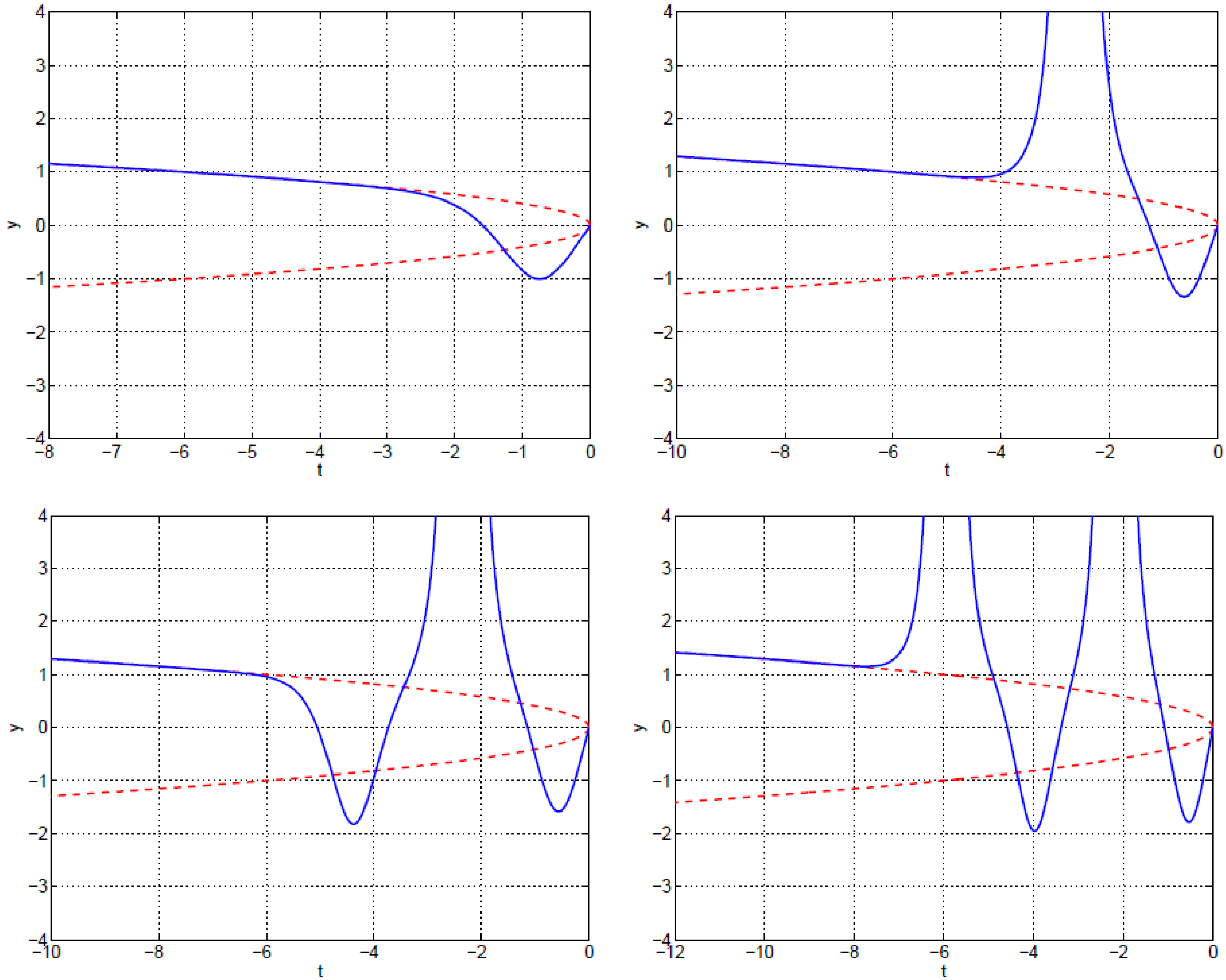
First four separatrix (eigenfunction) solutions:

Unstable

Stable

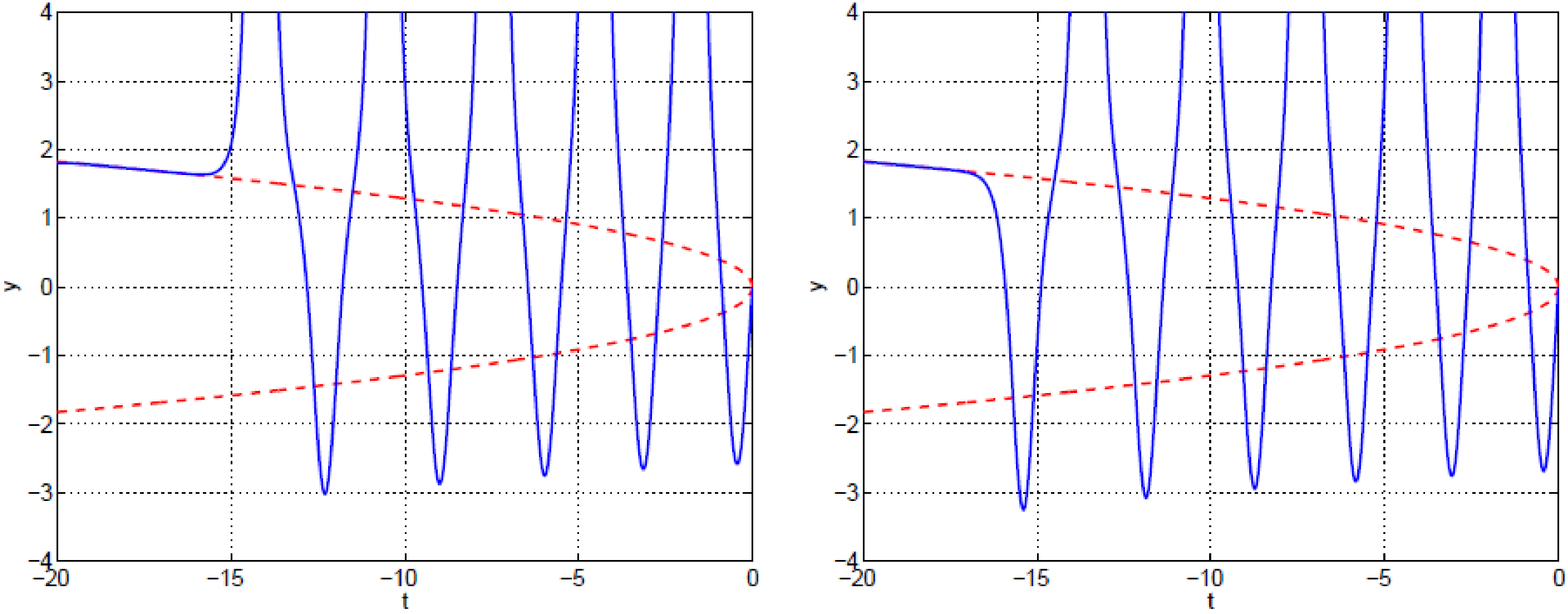
Unstable

Stable



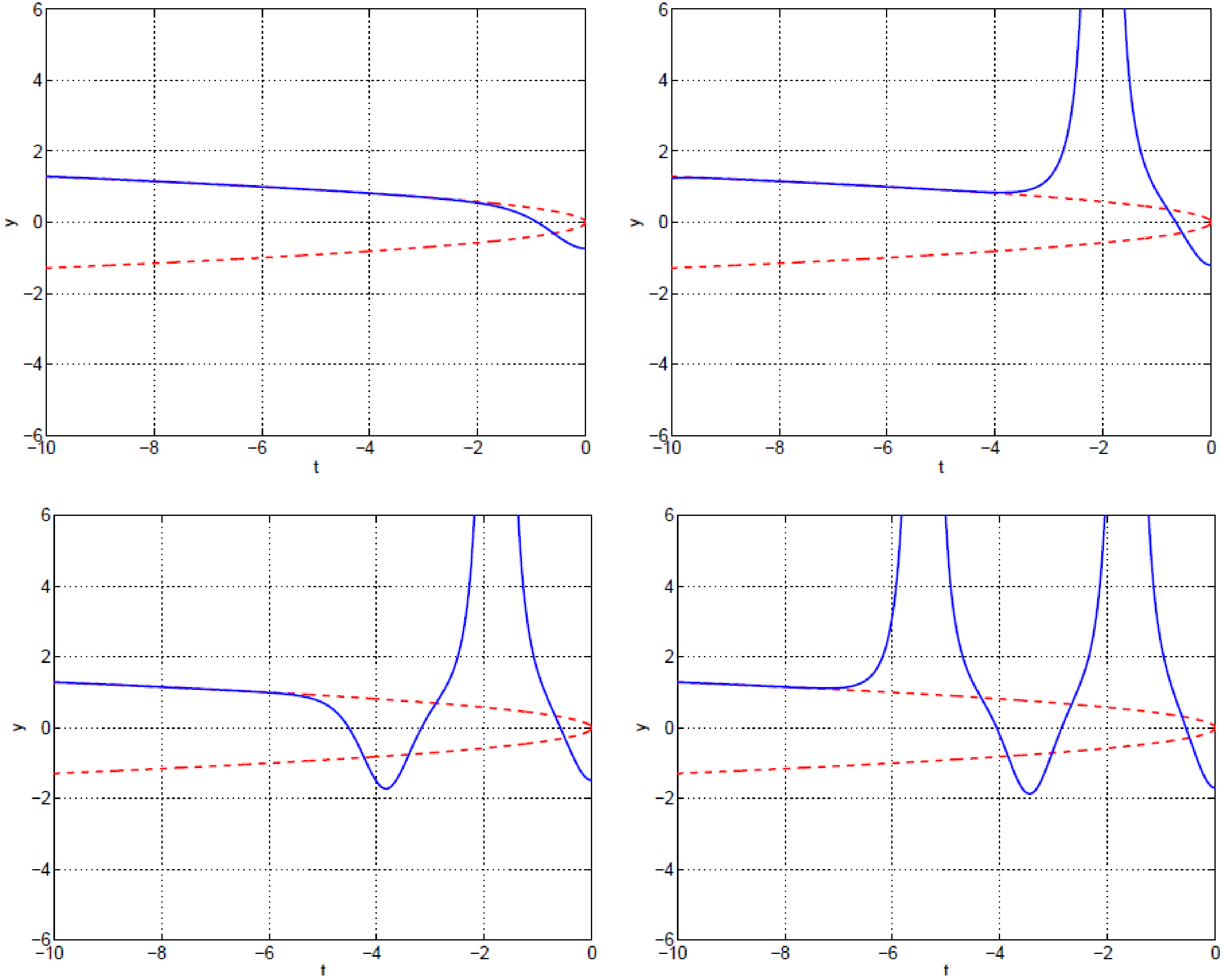
***Initial slope is the eigenvalue*, initial value *y*(0) = 0**

Tenth and eleventh separatrix (eigenfunction) solutions:



***Here, the initial slope* is the eigenvalue, initial value *y*(0) = 0**

First four separatrix solutions with 0 initial slope:



Numerical calculation of eigenvalues

(*nonlinear* semiclassical large-*n* limit)

$$y'(0) = b_n \qquad y(0) = 0$$

$$b_n \sim B_{\mathrm{I}} n^{3/5} \qquad B_{\mathrm{I}} = 2.0921467\underline{4}$$

$$y(0) = c_n \qquad y'(0) = 0$$

$$c_n \sim C_{\mathrm{I}} n^{2/5} \qquad C_{\mathrm{I}} = -1.030484\underline{4}$$

Analytical asymptotic calculation of eigenvalues

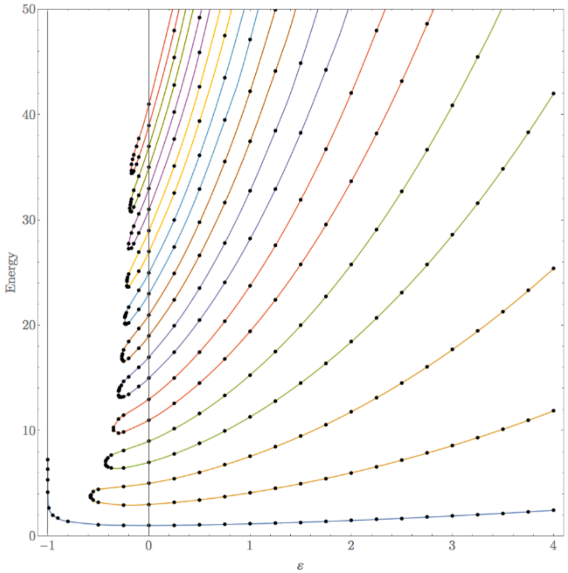
$$B_{\mathrm{I}} = 2\left[\frac{5\sqrt{\pi}\,\Gamma(5/6)}{2\sqrt{3}\,\Gamma(1/3)}\right]^{3/5} \qquad C_{\mathrm{I}} = -\left[\frac{5\sqrt{\pi}\,\Gamma(5/6)}{2\sqrt{3}\,\Gamma(1/3)}\right]^{2/5}$$

Obtained by using WKB to calculate the large eigenvalues of the
cubic PT-symmetric Hamiltonian

$$H = \frac{1}{2}p^2 + 2ix^3$$

A class of ***PT***-symmetric Hamiltonians:

$$\boxed{H = p^2 + x^2(ix)^\varepsilon \qquad (\varepsilon \text{ real})}$$



WKB **works for *PT*-symmetric Hamiltonians:**

$$E_n \sim \left[\frac{\Gamma\left(\frac{3}{2}+\frac{1}{\varepsilon+2}\right)\sqrt{\pi}\,n}{\sin\left(\frac{\pi}{\varepsilon+2}\right)\Gamma\left(1+\frac{1}{\varepsilon+2}\right)}\right]^{\frac{2\varepsilon+4}{\varepsilon+4}} \qquad (n \to \infty)$$

Analytical asymptotic calculation of eigenvalues

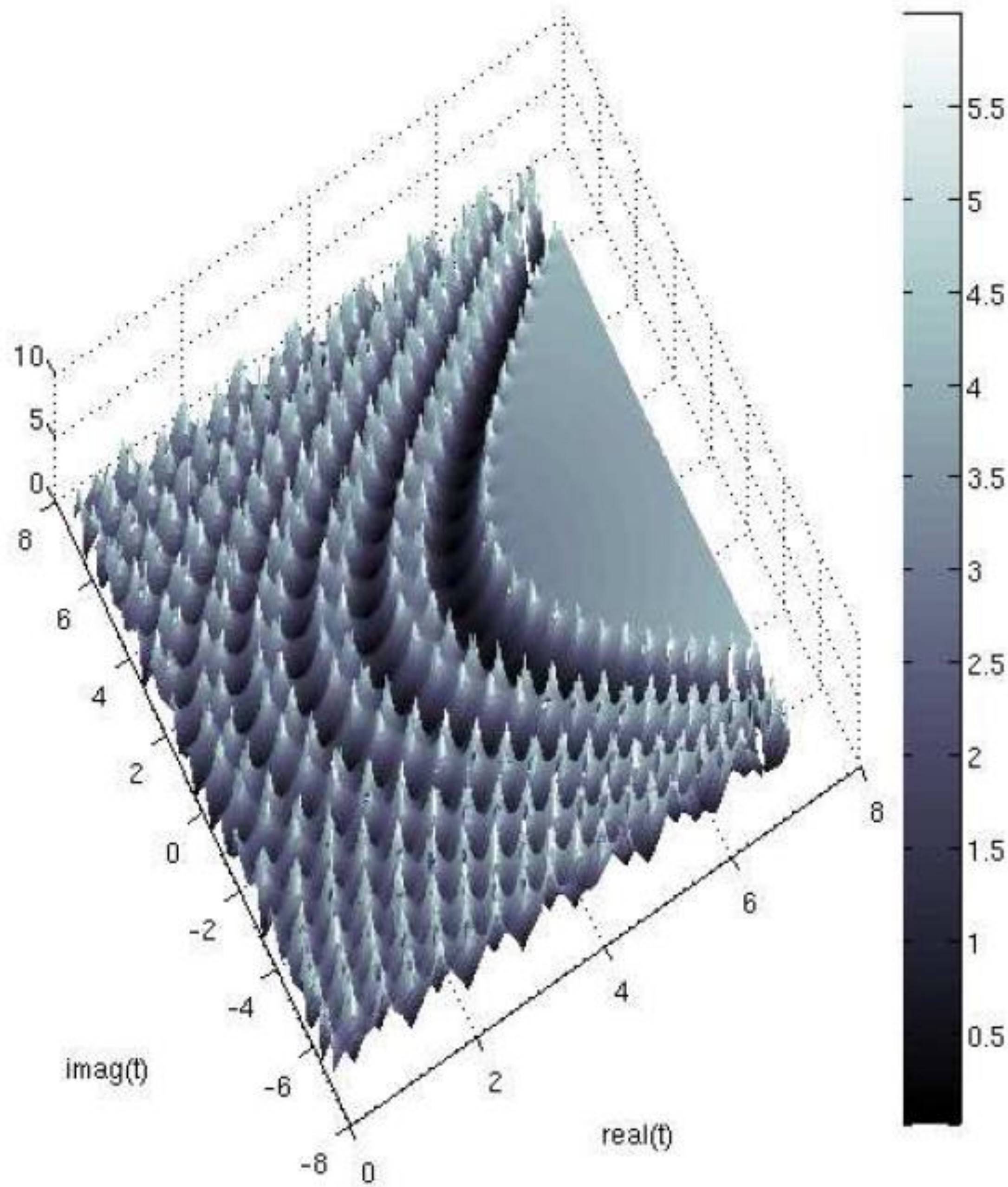
Multiply Painlevé I equation by *y'*(*t*);
Then integrate from *t* = 0 to *t* = *x*:

$$H \equiv \tfrac{1}{2}[y'(x)]^2 - 2[y(x)]^3 = \tfrac{1}{2}[y'(0)]^2 - 2[y(0)]^3 + I(x),$$

where $I(x) = \int_0^x dt\, t y'(t).$

Take |*x*| large at an angle of π/4, *I*(*x*) → 0, and we get the *PT*-symmetric Hamiltonian for ε = 1.

Painlevé I corresponds to ε = 1



10
5
0
8
6
4
2
0
-2
-4
-6
-8
imag(t)
0
2
4
6
8
real(t)
5.5
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5

Second Painlevé transcendent

$$y''(t) = 2[y(t)]^3 + ty(t), \qquad y(0) = b, \; y'(0) = c$$

Now, both solutions

$$+\sqrt{-t/2} \text{ or } -\sqrt{-t/2}$$

are unstable and 0 is stable.

Numerical calculation of eigenvalues

$$y(0) = 0,\ b_n = y'(0)$$

$$c_n = y(0),\ y'(0) = 0$$

$$b_n \sim B_{\rm II} n^{2/3} \quad \text{and} \quad c_n \sim C_{\rm II} n^{1/3}$$

$$B_{\rm II} = 1.862412\underline{8} \qquad C_{\rm II} = 1.2158116\underline{5}$$

$$B_{\rm II} = \left[3\sqrt{2\pi}\,\Gamma\left(\tfrac{3}{4}\right)/\Gamma\left(\tfrac{1}{4}\right)\right]^{2/3} \qquad C_{II} = \left[3\sqrt{\pi}\,\Gamma\left(\tfrac{3}{4}\right)/\Gamma\left(\tfrac{1}{4}\right)\right]^{1/3}$$

Obtained by using WKB to calculate the large eigenvalues of the *quartic PT-symmetric Hamiltonian* $H = p^2 - x^4$

An upside-down potential with real positive eigenvalues!

Painlevé II corresponds to ε = 2

CMB and J. Komijani
J. Physics A: Math. Theor. **48**, 475202 (2015)

Fourth Painlevé transcendent

$$y(t)y''(t) = \frac{1}{2}[y'(t)]^2 + 2t^2[y(t)]^2 + 4t[y(t)]^3 + \frac{3}{2}[y(t)]^4$$

with $y(0) = c$ and $y'(0) = b$.

Large *n* behaviour of eigenvalues: $b_n \sim B_{\rm IV} n^{3/4}$ and $c_n \sim C_{\rm IV} n^{1/2}$.

$B_{\rm IV} = 4.25684\underline{3}$. $C_{\rm IV} = -2.62658\underline{7}$

$$B_{\rm IV} = 2^{3/2}\left[\sqrt{\pi}\,\Gamma\left(\tfrac{5}{3}\right)/\Gamma\left(\tfrac{7}{6}\right)\right]^{3/4}, \qquad C_{\rm IV} = -2\left[\sqrt{\pi}\,\Gamma\left(\tfrac{5}{3}\right)/\Gamma\left(\tfrac{7}{6}\right)\right]^{1/2}$$

Obtained by using WKB to calculate the large eigenvalues of the *sextic PT-symmetric Hamiltonian* $\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{8}\hat{x}^6.$

Note:

Painlevé I, II, and IV correspond to $\varepsilon = 1, 2,$ and 4

This analysis extends to huge classes of equations beyond Painlevé. For example:

Super Painlevé:

$$y''(x) = \frac{2M+2}{(M-1)^2}[y(x)]^M + x[y(x)]^N$$

Hyperfine splitting

$$y'' = \frac{1}{a^2}y^4 + xy^2, \quad \text{with} \quad a \equiv \frac{3}{\sqrt{10}}.$$

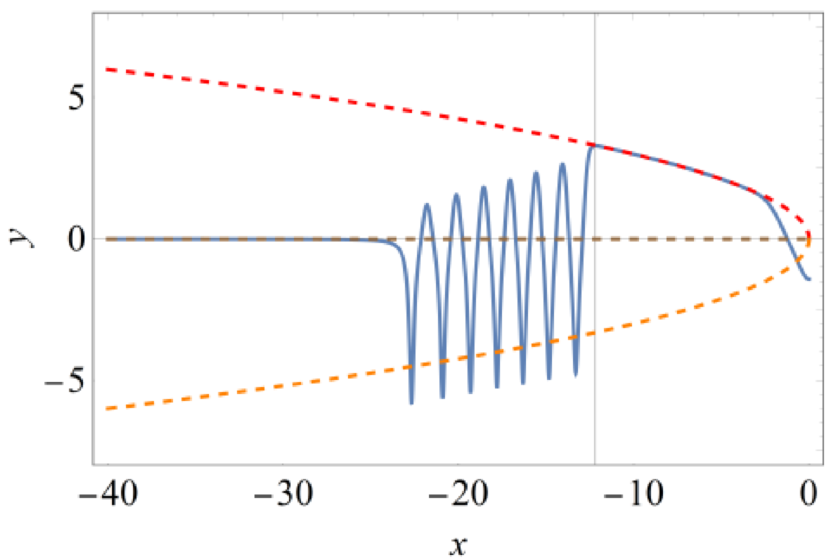
Let $y_{nm}(x) = Y_n(x) + \phi_m(x)$, where $Y_n(x)$ is a separatrix solution with

$$Y_n(x) \sim a\sqrt{-x}, \qquad x \to -\infty.$$

The new hyperfine solutions initially follow $Y_n(x)$.

Then they deviate from $Y_n(x)$ and oscillate *m* times about the curve $-a\sqrt{-x}$.

Finally, they level off for large x as $y_{nm}(x) \sim \frac{12}{x^3}, \qquad x \to -\infty.$



The initial values of ϕ are the hyperfine eigenvalues.

For example, for the lowest eigenfunction Y_0

$$\phi_m(0) \sim 4.1789\,\mathrm{e}^{-9.26201m}, \qquad m \to \infty.$$

The hyperfine oscillation separates at the negative values

$$T_m \sim \left(\frac{7}{4\sqrt{2a}} 9.26201m\right)^{\frac{4}{7}}, \qquad m \to \infty.$$

We hope we have opened a window to a new area of *nonlinear semiclassical* asymptotic analysis

Thanks for listening!

Possible connection with the *power series constant P*???

(Remember the numerical constant A = 1.7818)

W. K. Hayman, *Research Problems in Function theory*
[Athlone Press (University of London), London, 1967]

J. Clunie and P. Erdös, Proc. Roy. Irish Acad. **65**, 113 (1967).
J. D. Buckholtz, Michigan Math. J. **15**, 481 (1968).

$$1 \le P \le 2$$

$$\sqrt{2} \le P \le 2$$

$$1.7 \le P \le 12^{1/4}$$

$$1.7818 \le P \le 1.82$$