Nonlinear eigenvalue problems
--a talk in fond memory of Lev Lipatov

(Work done in collaboration with A. Fring, J. Komijani, & Q. Wang)

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Linear eigenvalue problems...

\[-\psi''(x) + V(x)\psi(x) = E\psi(x)\]
\[\psi(\pm\infty) = 0\]

(1) **Oscillatory** in *classically allowed* region (\(n\)th eigenfunction has \(n\) nodes)

(2) **Monotone decay** in *classically forbidden* region

(3) **Transition** at the boundary (\textit{turning point})

(4) **Unstable** with respect to small changes in \(E\)
Physical solution is *subdominant*

Leading asymptotic behavior of eigenfunctions of

\[-\psi''(x) + V(x)\psi(x) = E\psi(x)\]

for large positive \(x\):

\[
\psi(x) \sim D[V(x) - E]^{-1/4} \exp \left[ - \int_x^\infty ds \sqrt{V(s) - E} \right] \quad (x \rightarrow \infty)
\]

NOTE: There is only **ONE** arbitrary constant even though the differential equation is second order
For linear problems \textit{WKB} gives a good approximation for \textit{large} eigenvalues.

\[
\int_{x_1}^{x_2} dx \sqrt{E_n - V(x)} \sim (n + 1/2)\pi \quad (n \to \infty)
\]

\textbf{Example 1: harmonic oscillator}

\[V(x) = x^2\]

\[E_n \sim n \quad (n \to \infty)\]

\textbf{Example 2: anharmonic oscillator}

\[V(x) = x^4\]

\[E_n \sim Bn^{4/3} \quad (n \to \infty)\]

\[B = \left[\frac{3\Gamma(3/4)\sqrt{\pi}}{\Gamma(1/4)}\right]^{4/3}\]
Consider an equation like

\[ y'(x) = F[xy(x)] \]

-or-

\[ y''(x) = F[xy(x)] \]

If the solution \( y(x) \) to \( f(x,y) = 0 \) vanishes for large \( x \), does the solution to the differential equation also vanish for large \( x \) \text{ in a stable fashion?}
Toy nonlinear eigenvalue problem

\[ y'(x) = \cos[\pi xy(x)], \quad y(0) = a \]
Asymptotic behavior for large $x$

Solution behaves like: \[ y(x) \sim \frac{m + 1/2}{x} \]

$m = 0, 1, 2, 3, \ldots$ is an integer

*Note:* oscillation followed by monotone decay
But there’s a *big* problem here...

Where are the odd-$m$ solutions?!?
Eigenvalues correspond to odd-$m$ initial values. Eigenfunctions are (unstable) separatrices, which begin at eigenvalues.
Note: no arbitrary constant appears in the asymptotic behavior!!

Where is the arbitrary constant?!?

Is it in higher order....?
Higher-order asymptotic behavior for large $x$ still contains no arbitrary constant!

$$y(x) \sim \frac{m + 1/2}{x} + \sum_{k=1}^{\infty} \frac{c_k}{x^{2k+1}} \quad (x \to \infty)$$

$$c_1 = \frac{(-1)^m}{\pi} (m + 1/2),$$

$$c_2 = \frac{3}{\pi^2} (m + 1/2),$$

$$c_3 = (-1)^m \left[ \frac{(m + 1/2)^3}{6\pi} + \frac{15(m + 1/2)}{\pi^3} \right],$$

$$c_4 = \frac{8(m + 1/2)^3}{3\pi^2} + \frac{105(m + 1/2)}{\pi^4},$$

$$c_5 = (-1)^m \left[ \frac{3(m + 1/2)^5}{40\pi} + \frac{36(m + 1/2)^3}{\pi^3} + \frac{945(m + 1/2)}{\pi^5} \right],$$

$$c_6 = \frac{38(m + 1/2)^5}{15\pi^2} + \frac{498(m + 1/2)^3}{\pi^4} + \frac{10395(m + 1/2)}{\pi^6}.$$
Asymptotics beyond all orders

Difference of two solutions in one bundle: \( Y(x) \equiv y_1(x) - y_2(x) \)

\[
Y'(x) = \cos[\pi xy_1(x)] - \cos[\pi xy_2(x)] \\
= -2 \sin \left[ \frac{1}{2} \pi xy_1(x) + \frac{1}{2} \pi xy_2(x) \right] \sin \left[ \frac{1}{2} \pi xy_1(x) - \frac{1}{2} \pi xy_2(x) \right] \\
\sim -2 \sin \left[ \pi \left( m + \frac{1}{2} \right) \right] \sin \left[ \frac{1}{2} \pi xY(x) \right] \quad (x \to \infty) \\
\sim -(-1)^m \pi xY(x) \quad (x \to \infty).
\]

\[
Y(x) \sim K \exp \left[ -(-1)^m \pi x^2 \right] \quad (x \to \infty)
\]

Aha! \( K \) is the invisible arbitrary constant!
Odd-\( m \) solutions (eigenfunctions) are \underline{unstable};
even-\( m \) solutions are \underline{stable}. 
We calculated up to \( m=500,001 \)

Let \( m = 2n - 1 \)

For large \( n \) the \( n \)th eigenvalue grows like the square root of \( n \) times a constant \( A \), and we used Richardson extrapolation to show that

\[ A = 1.7817974363... \]

and then we guessed \( A \).
Result:

\[ a_n \sim A \sqrt{n} \quad (n \to \infty) \]

\[ A = 2^{5/6} \]

This is a rather nontrivial problem...
Construct moments of $z(t)$:

$$A_{n,k}(t) \equiv \int_{0}^{t} ds \cos[n\lambda s z(s)] \frac{s^{k+1}}{[z(s)]^k}$$

Moments are associated with a semi-infinite linear one-dimensional random walk in which random walkers become static as they reach $n=1$

$$2\alpha_{1,k} + \alpha_{2,k-1} = 0, \quad 2\alpha_{n,k} + \alpha_{n-1,k-1} + \alpha_{n+1,k-1} = 0 \quad (n \geq 3).$$

$$2\alpha_{2,k} + \alpha_{3,k-1} = 0,$$

Solve the random walk problem exactly and get $A = 2^{5/6}$

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Bessel function instead of cosine

\[ y'(x) = J_0 \left[ x y(x) \right] \]
Painlevé equations

Paul Painlevé
(1863-1933)

Six Painlevé equations known as Painlevé I – VI

Only spontaneous singularities are poles
Painlevé I \[ \frac{d^2 y}{dt^2} = 6y^2 + t \]

Painlevé II \[ \frac{d^2 y}{dt^2} = 2y^3 + ty + \alpha \]

Painlevé III \[ ty \frac{d^2 y}{dt^2} = t \left( \frac{dy}{dt} \right)^2 - y \frac{dy}{dt} + \delta t + \beta y + \alpha y^3 + \gamma ty^4 \]

Painlevé IV \[ y \frac{d^2 y}{dt^2} = \frac{1}{2} \left( \frac{dy}{dt} \right)^2 + \beta + 2(t^2 - \alpha)y^2 + 4ty^3 + \frac{3}{2}y^4 \]

Painlevé V \[ \frac{d^2 y}{dt^2} = \left( \frac{1}{2y} + \frac{1}{y - 1} \right) \left( \frac{dy}{dt} \right)^2 - \frac{1}{t} \frac{dy}{dt} \]
\[ + \frac{(y - 1)^2}{t^2} \left( \alpha y + \frac{\beta}{y} \right) + \gamma \frac{y}{t} + \delta \frac{y(y + 1)}{y - 1} \]

Painlevé VI \[ \frac{d^2 y}{dt^2} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y - 1} + \frac{1}{y - t} \right) \left( \frac{dy}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{y - t} \right) \frac{dy}{dt} \]
\[ + \frac{y(y - 1)(y - t)}{t^2(t - 1)^2} \left( \alpha + \beta \frac{t}{y^2} + \gamma \frac{t - 1}{(y - 1)^2} + \delta \frac{t(t - 1)}{(y - t)^2} \right) \]
First Painlevé transcendent

\[ y''(t) = 6[y(t)]^2 + t, \quad y(0) = b, \quad y'(0) = c \]

If the right side of this equation vanishes, the solution \( y(x) \) must \textit{choose} between two possible asymptotic behaviors as \( x \) gets large and negative:

\[ +\sqrt{-t/6} \text{ or } -\sqrt{-t/6}. \]

Lower square-root branch is \textit{stable}:

\[ y(x) \sim -\sqrt{-x} + c(-x)^{-1/8} \cos \left[ \frac{4}{5} \sqrt{2}(-x)^{5/4} + d \right] \quad (x \to -\infty) \]

Upper square-root branch is \textit{unstable}:

\[ y(x) \sim \sqrt{-x} + c_{\pm}(-x)^{-1/8} \exp \left[ \pm \frac{4}{5} \sqrt{2}(-x)^{5/4} \right] \quad (x \to -\infty) \]
Two kinds of solutions (NOT eigenfunctions):

Unstable branch

Stable branch

Unstable branch

Stable branch
First four separatrix (eigenfunction) solutions:

Initial slope is the eigenvalue, initial value $y(0) = 0$
Tenth and eleventh separatix (eigenfunction) solutions:

Here, the initial slope is the eigenvalue, initial value $y(0) = 0$
First four separatrix solutions with 0 initial slope:
Numerical calculation of eigenvalues

(nonlinear semiclassical large-$n$ limit)

$$y'(0) = b_n \quad y(0) = 0$$
$$b_n \sim B_1 n^{3/5} \quad B_1 = 2.09214674$$

$$y(0) = c_n \quad y'(0) = 0$$
$$c_n \sim C_1 n^{2/5} \quad C_1 = -1.0304844$$
Analytical asymptotic calculation of eigenvalues

\[ B_1 = 2 \left[ \frac{5\sqrt{\pi} \Gamma(5/6)}{2\sqrt{3}\Gamma(1/3)} \right]^{3/5} \]

\[ C_1 = - \left[ \frac{5\sqrt{\pi} \Gamma(5/6)}{2\sqrt{3}\Gamma(1/3)} \right]^{2/5} \]
Obtained by using WKB to calculate the large eigenvalues of the cubic PT-symmetric Hamiltonian

\[ H = \frac{1}{2}p^2 + 2ix^3 \]

A class of PT-symmetric Hamiltonians:

\[ H = p^2 + x^2(ix)^\varepsilon \quad (\varepsilon \text{ real}) \]

WKB works for PT-symmetric Hamiltonians:

\[ E_n \sim \left[ \frac{\Gamma \left( \frac{3}{2} + \frac{1}{\varepsilon+2} \right) \sqrt{\pi} n^{\frac{2\varepsilon+4}{\varepsilon+4}}}{\sin \left( \frac{\pi}{\varepsilon+2} \right) \Gamma \left( 1 + \frac{1}{\varepsilon+2} \right)} \right] (n \to \infty) \]
Multiply Painlevé I equation by \( y'(t) \);
Then integrate from \( t = 0 \) to \( t = x \):

\[
H = \frac{1}{2}[y'(x)]^2 - 2[y(x)]^3 = \frac{1}{2}[y'(0)]^2 - 2[y(0)]^3 + I(x),
\]

where \( I(x) = \int_0^x dt \, t y'(t) \).

Take \( |x| \) large at an angle of \( \pi/4 \), \( I(x) \to 0 \), and we get the \( PT \)-symmetric Hamiltonian for \( \varepsilon = 1 \).

Painlevé I corresponds to \( \varepsilon = 1 \)
Second Painlevé transcendent

\[ y''(t) = 2[y(t)]^3 + ty(t), \quad y(0) = b, \quad y'(0) = c. \]

Now, both solutions

\[ +\sqrt{-\frac{t}{2}} \text{ or } -\sqrt{-\frac{t}{2}} \]

are unstable and 0 is stable.
Numerical calculation of eigenvalues

\[
y(0) = 0, \quad b_n = y'(0) \\
c_n = y(0), \quad y'(0) = 0 \\
b_n \sim B_{II} n^{2/3} \quad \text{and} \quad c_n \sim C_{II} n^{1/3}
\]

\[
B_{II} = 1.8624128 \\
C_{II} = 1.21581165 \\
B_{II} = \left[3\sqrt{2\pi}\Gamma \left(\frac{3}{4}\right) / \Gamma \left(\frac{1}{4}\right)\right]^{2/3} \\
C_{II} = \left[3\sqrt{\pi}\Gamma \left(\frac{3}{4}\right) / \Gamma \left(\frac{1}{4}\right)\right]^{1/3}
\]

Obtained by using WKB to calculate the large eigenvalues of the **quartic PT-symmetric Hamiltonian**

\[
H = p^2 - x^4
\]

An upside-down potential with real positive eigenvalues!

\[
\text{Painlevé II corresponds to } \varepsilon = 2
\]

CMB and J. Komijani

Fourth Painlevé transcendent

\[ y(t)y''(t) = \frac{1}{2}[y'(t)]^2 + 2t^2[y(t)]^2 + 4t[y(t)]^3 + \frac{3}{2}[y(t)]^4 \]

with \( y(0) = c \) and \( y'(0) = b \).

Large \( n \) behaviour of eigenvalues:

\[ b_n \sim B_{IV} n^{3/4} \quad \text{and} \quad c_n \sim C_{IV} n^{1/2}. \]

\[ B_{IV} = 4.256843, \quad C_{IV} = -2.626587. \]

\[ B_{IV} = 2^{3/2} \left[ \sqrt{\pi} \Gamma \left( \frac{5}{3} \right) / \Gamma \left( \frac{7}{6} \right) \right]^{3/4}, \quad C_{IV} = -2 \left[ \sqrt{\pi} \Gamma \left( \frac{5}{3} \right) / \Gamma \left( \frac{7}{6} \right) \right]^{1/2}. \]

Obtained by using WKB to calculate the large eigenvalues of the sextic \( PT \)-symmetric Hamiltonian

\[ \hat{H} = \frac{1}{2} \hat{p}^2 + \frac{1}{8} \hat{x}^6. \]
Note:

Painlevé I, II, and IV correspond to $\varepsilon = 1, 2, \text{ and } 4$
This analysis extends to huge classes of equations beyond Painlevé. For example:

Super Painlevé:

\[ y''(x) = \frac{2M + 2}{(M - 1)^2}[y(x)]^M + x[y(x)]^N \]
Hyperfine splitting

\[ y'' = \frac{1}{a^2} y^4 + xy^2, \quad \text{with} \quad a = \frac{3}{\sqrt{10}}. \]

Let \( y_{nm}(x) = Y_n(x) + \phi_m(x) \), where \( Y_n(x) \) is a separatrix solution with

\[ Y_n(x) \sim a\sqrt{-x}, \quad x \to -\infty. \]

The new hyperfine solutions initially follow \( Y_n(x) \).

Then they deviate from \( Y_n(x) \) and oscillate \( m \) times about the curve \( -a\sqrt{-x} \).

Finally, they level off for large \( x \) as \( y_{nm}(x) \sim \frac{12}{x^3}, \quad x \to -\infty \).
The initial values of $\phi$ are the hyperfine eigenvalues.

For example, for the lowest eigenfunction $Y_0$,

$$\phi_m(0) \sim 4.1789 e^{-9.26201m}, \quad m \to \infty.$$ 

The hyperfine oscillation separates at the negative values

$$T_m \sim \left( \frac{7}{4\sqrt{2a}}9.26201m \right)^{\frac{4}{7}}, \quad m \to \infty.$$
We hope we have opened a window to a new area of nonlinear *semiclassical* asymptotic analysis.
Possible connection with the power series constant $P$???

(Remember the numerical constant $A = 1.7818$)

W. K. Hayman, *Research Problems in Function theory*  
[Athlone Press (University of London), London, 1967]


\[ 1 \leq P \leq 2 \]

\[ \sqrt{2} \leq P \leq 2 \]

\[ 1.7 \leq P \leq 12^{1/4} \]

\[ 1.7818 \leq P \leq 1.82 \]