Kinematically Dependent Renormalization

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This is an attempt to shed some new light on non-renormalizable interactions with the aim to make sense of them at least in some cases.

As an example we consider maximally supersymmetric gauge theory in $D=8$ dimensions and focus on the on-shell scattering amplitudes.

The reason is that this case was studied in detail in collaboration with A. Borlakov, D.Tolkachev and D.Vlasenko.


and has important advantages.

All analysis in performed within dimensional regularization.
Motivation

Maximal SYM

- $D=4$ $N=4$
- $D=6$ $N=2$
- $D=8$ $N=1$
- $D=10$ $N=1$

Partial or total cancellation of UV divergences (all bubble and triangle diagrams cancel)

- First UV divergent diagrams at $D=4+6/L$
- Conformal or dual conformal symmetry
- Common structure of the integrands

References:
- Bern, Dixon & Co 10
- Arkani-Hamed 12
- Drummond, Henn, Korchemsky, Sokatchev 10
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D=4 N=4
D=6 N=2
D=8 N=1
D=10 N=1

Partial or total cancellation of UV divergences (all bubble and triangle diagrams cancel)

First UV divergent diagrams at D=4+6/L

Conformal or dual conformal symmetry

Common structure of the integrands

All of them can be obtained from 10dim superstring by compactification on a torus

Bern, Dixon & Co 10
Drummond, Henn, Korchemsky, Sokatchev 10
Arkani-Hamed 12
Motivation

Maximal SYM

D=4 N=4
D=6 N=2
D=8 N=1
D=10 N=1

D=4 N=8 Supergravity

Partial or total cancellation of UV divergences (all bubble and triangle diagrams cancel)
First UV divergent diagrams at D=4+6/L
Conformal or dual conformal symmetry
Common structure of the integrands

On-shell finite up to 8 loops
Similar to higher dim SYM

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Motivation

Maximal SYM

- \( D=4 \ N=4 \)
- \( D=6 \ N=2 \)
- \( D=8 \ N=1 \)
- \( D=10 \ N=1 \)

Partial or total cancellation of UV divergences (all bubble and triangle diagrams cancel)

First UV divergent diagrams at \( D=4+6/L \)

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Common structure of the integrands

Object: Helicity Amplitudes on mass shell with arbitrary number of legs and loops

The case: Planar limit \( N_c \to \infty, g_{YM}^2 \to 0 \) and \( g_{YM}^2 N_c \) - fixed

The aim: to get all loop (exact) result

D=4 N=8 Supergravity

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Study of higher dim SYM gives insight into quantum gravity
UV divergences in all Loops

<table>
<thead>
<tr>
<th>Dimension (D)</th>
<th>Number of Supersymmetry (N)</th>
<th>UV Divergences</th>
<th>IR Divergences</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>4</td>
<td>No UV div</td>
<td>IR div on shell</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>UV div from 3 loops</td>
<td>No IR div</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>UV div from 1 loop</td>
<td>No IR div</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>UV div from 1 loop</td>
<td>No IR div</td>
</tr>
</tbody>
</table>

All these theories are non-renormalizable by power counting.

The coupling $g^2$ has dimension $[g^2] = \frac{1}{M^{D-4}}$

The aim: to get all loop (exact) result for the leading (at least) divs
Perturbation Expansion for the 4-point Amplitudes for any D

\[ A_4 / A_{4}^{\text{tree}} \]

No bubbles
No Triangles

First UV div at L=[6/(D-4)] loops

IR finite

Universal expansion for any D in maximal SYM due to Dual conformal invariance
In renormalizable theories the leading divergences can be found from the 1-loop term due to the renormalization group, in particular, for a single coupling theory the coefficient of $1/\epsilon^n$ in $n$ loops is

$$\mathcal{R}' \mathcal{G} = \sum_n \frac{a_n^{(n)}}{\epsilon^n} \quad a_n^{(n)} = (a_1^{(1)})^n$$
In renormalizable theories the leading divergences can be found from the 1-loop term due to the renormalization group, in particular, for a single coupling theory the coefficient of \( \frac{1}{\varepsilon^n} \) in \( n \) loops is

\[
\mathcal{R}' G = \sum_n \frac{a_n^{(n)}}{\varepsilon^n}, \quad a_n^{(n)} = (a_1^{(1)})^n
\]

In non-renormalizable theories the leading divergences can be also found from 1-loop due to locality and R-operation

\[
\mathcal{R}' G = 1 - \sum_{\gamma} K \mathcal{R}'_{\gamma} + \sum_{\gamma, \gamma'} K \mathcal{R}'_{\gamma} K \mathcal{R}'_{\gamma'} - \ldots,
\]
In renormalizable theories the leading divergences can be found from the 1-loop term due to the renormalization group, in particular, for a single coupling theory the coefficient of \( \frac{1}{\epsilon^n} \) in \( n \) loops is

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In non-renormalizable theories the leading divergences can be also found from 1-loop due to locality and \( R \)-operation

\[
\mathcal{R}' G = 1 - \sum_{\gamma} K\mathcal{R}_{\gamma} + \sum_{\gamma, \gamma'} K\mathcal{R}_{\gamma}' K\mathcal{R}_{\gamma'}' - \ldots,
\]

\[
\mathcal{R}' G_n = \frac{A^{(n)}_n (\mu^2)^n \epsilon^n}{\epsilon^n} + \frac{A^{(n)}_{n-1} (\mu^2)^{(n-1)} \epsilon^n}{\epsilon^n} + \ldots + \frac{A^{(n)}_1 (\mu^2)^\epsilon}{\epsilon^n}
\]

\[
+ \frac{B^{(n)}_n (\mu^2)^n \epsilon^{n-1}}{\epsilon^{n-1}} + \frac{B^{(n)}_{n-1} (\mu^2)^{(n-1)} \epsilon^{n-1}}{\epsilon^{n-1}} + \ldots + \frac{B^{(n)}_1 (\mu^2)^\epsilon}{\epsilon^{n-1}}
\]

+ lower order terms

**Leading pole**

**SubLeading pole**

\( A^{(n)}_1, B^{(n)}_1 \)

\( 1 \)-loop graph

\( B^{(n)}_2 \)

\( 2 \)-loop graph
SubLeading Divergences from Generalized «Renormalization Group»

- In non-renormalizable theories the leading divergences can be also found from 1-loop due to locality and R-operation

All terms like \((\log \mu^2)^m / \epsilon^k\) should cancel

\[
A^{(n)}_n = (-1)^{n+1} \frac{A_1^{(n)}}{n},
\]

\[
B^{(n)}_n = (-1)^n \left( \frac{2}{n} B_2^{(n)} + \frac{n-2}{n} B_1^{(n)} \right)
\]

\[
\mathcal{K}_R' G_n = \sum_{k=1}^{n} \left( \frac{A^{(n)}_k}{\epsilon^n} + \frac{B^{(n)}_k}{\epsilon^{n-1}} \right) \equiv \frac{A^{(n)'}_n}{\epsilon^n} + \frac{B^{(n)'}_n}{\epsilon^{n-1}}.
\]

\[
A^{(n)'}_n = (-1)^{n+1} A^{(n)}_n = \frac{A^{(n)}_1}{n},
\]

\[
B^{(n)'}_n = \left( \frac{2}{n(n-1)} B_2^{(n)} + \frac{2}{n} B_1^{(n)} \right)
\]

Leading pole from 1 loop diagrams

SubLeading pole from 2 loop diagrams

Just like in renormalizable theories one can deduce the leading, subheading, etc divergences from 1, 2, etc diagrams
Kinematically dependent renormalization

One-loop box

\[ st = \frac{st}{3! \epsilon} \]

Two-loop box

\[ s^2 t + st^2 = \frac{st}{3! 4!} \left( \frac{s^2 + t^2}{\epsilon^2} + \frac{27/4 s^2 + 1/3 st + 27/4 t^2}{\epsilon} \right) \]

\[ R' = 1 - \frac{1}{3! \epsilon} - \frac{s}{4! \epsilon} \]

Totally defined by 1 loop

Independent term

This is true to all orders of PT like in renormalizable theories via the locality of the counterterms due to the R-operation
**Ladder diagrams (leading divs)**

**D=8 N=1**

Leading poles

\[ A_n^{(n)} = s^{n-1} A_n \]

\[ nA_n = -\frac{2}{4!} A_{n-1} + \frac{2}{5!} \sum_{k=1}^{n-2} A_k A_{n-1-k}, \quad n \geq 3 \]

1 loop box

\[ A_1 = \frac{1}{3!} \]

\[ A_2 = -\frac{1}{3!4!} \]

\[ A_3 = \frac{2}{33!4!4!} + \frac{2}{35!3!3!} \]
Ladder diagrams (leading divs)

**D=8 N=1**

Horizontal boxes

\[ A_n^{(n)} = s^{n-1} A_n \]

\[ nA_n = -\frac{2}{4!} A_{n-1} + \frac{2}{5!} \sum_{k=1}^{n-2} A_k A_{n-1-k}, \quad n \geq 3 \]

\[ A_1 = 1/6 \]

1 loop box
Ladder diagrams (leading divs)

D=8 N=1

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\[ A_n^{(n)} = s^{n-1} A_n \]

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\[ A_1 = 1/6 \quad 1 \text{ loop box} \]

Summation

\[ \Sigma_m(z) = \sum_{n=m}^{\infty} A_n (-z)^n \]
Ladder diagrams (leading divs)

**D=8 N=1**

Horizontal boxes

\[ A_n^{(n)} = s^{n-1} A_n \]

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\[ A_1 = \frac{1}{6} \]

1 loop box

**Summation**

\[ \Sigma_m(z) = \sum_{n=m}^{\infty} A_n (-z)^n \]

\[ -\frac{d}{dz} \Sigma_3 = -\frac{2}{4!} \Sigma_2 + \frac{2}{5!} \Sigma_1 \Sigma_1. \quad \Sigma_3 = \Sigma_1 + A_1 z - A_2 z^2, \quad \Sigma_2 = \Sigma_1 + A_1 z, \quad A_1 = \frac{1}{3!}, \quad A_2 = -\frac{1}{3!4!} \]

**Diff eqn**

\[ \frac{d}{dz} \Sigma_A = -\frac{1}{3!} + \frac{2}{4!} \Sigma_A - \frac{2}{5!} \Sigma_A^2 \]

\[ z = g^2 s^2 / \epsilon \]
Ladder diagrams (leading divs)

**D=8 N=1**

**Horizontal boxes**

\[
A_n^{(n)} = s^{n-1} A_n
\]

\[
n A_n = -\frac{2}{4!} A_{n-1} + \frac{2}{5!} \sum_{k=1}^{n-2} A_k A_{n-1-k}, \quad n \geq 3
\]

\[
A_1 = \frac{1}{6}
\]

**1 loop box**

\[
\Sigma_m(z) = \sum_{n=m}^{\infty} A_n (-z)^n
\]

\[
-\frac{d}{dz} \Sigma_3 = -\frac{2}{4!} \Sigma_2 + \frac{2}{5!} \Sigma_1 \Sigma_1.
\]

\[
\Sigma_3 = \Sigma_1 + A_1 z - A_2 z^2, \quad \Sigma_2 = \Sigma_1 + A_1 z, \quad A_1 = \frac{1}{3!}, \quad A_2 = -\frac{1}{3!4!}
\]

\[
\Sigma_A \equiv \Sigma_1
\]

**Diff eqn**

\[
\frac{d}{dz} \Sigma_A = -\frac{1}{3!} + \frac{2}{4!} \Sigma_A - \frac{2}{5!} \Sigma_A^2
\]

\[z = g^2 s^2 / \epsilon\]

\[
\Sigma_A(z) = -\sqrt{\frac{5}{3}} \frac{4 \tan(z/(8\sqrt{15}))}{1 - \tan(z/(8\sqrt{15})) \sqrt{5/3}} = \sqrt{10} \frac{\sin(z/(8\sqrt{15}))}{\sin(z/(8\sqrt{15}) - z_0)}
\]

\[
\Sigma(z) = -(z/6 + z^2/144 + z^3/2880 + 7z^4/414720 + \ldots) \quad \text{with} \quad z_0 = \arcsin(\sqrt{3}/8)
\]
All loop Exact Recurrence Relation

D=8 N=1

s-channel term $S_n(s, t)$  t-channel term $T_n(s, t) \quad T_n(s, t) = S_n(t, s)$

Exact relation for ALL diagrams

\[
nS_n(s, t) = -2s^2 \int_0^1 dx \int_0^x dy \, y(1 - x) \left( S_{n-1}(s, t') + T_{n-1}(s, t') \right) |_{t' = tx + yu}
\]

\[
+ s^4 \int_0^1 dx \, x^2 (1 - x)^2 \sum_{k=1}^{n-2} \sum_{p=0}^{2k-2} \frac{1}{p! (p + 2)!} \frac{d^p}{dt'^p} (S_k(s, t') + T_k(s, t')) \times \n\]

\[
S_1 = \frac{1}{12}, \quad T_1 = \frac{1}{12} \quad \times \frac{d^p}{dt'^p} (S_{n-1-k}(s, t') + T_{n-1-k}(s, t')) |_{t' = -sx} \ (t sx (1 - x))^p
\]
All loop Exact Recurrence Relation

$D=8 \ N=1$

s-channel term $S_n(s, t)$ t-channel term $T_n(s, t)$ $T_n(s, t) = S_n(t, s)$

Exact relation for ALL diagrams

\[
nS_n(s, t) = -2s^2 \int_0^1 dx \int_0^x dy \ y(1 - x) \ (S_{n-1}(s, t') + T_{n-1}(s, t'))|_{t'=tx+yu}
\]
\[
+ s^4 \int_0^1 dx \ x^2(1 - x)^2 \sum_{k=1}^{n-2} \sum_{p=0}^{2k-2} \frac{1}{p!(p+2)!} \frac{d^p}{dt'^p} (S_k(s, t') + T_k(s, t')) \times
\]
\[
S_1 = \frac{1}{12}, \ T_1 = \frac{1}{12} \times \frac{d^p}{dt'^p} (S_{n-1-k}(s, t') + T_{n-1-k}(s, t'))|_{t'=-sx} (tsx(1 - x))^p
\]

summation
All loop Exact Recurrence Relation

\[ D=8 \quad N=1 \]

s-channel term \[ S_n(s, t) \] t-channel term \[ T_n(s, t) \]

\[ T_n(s, t) = S_n(t, s) \]

Exact relation for ALL diagrams

\[
nS_n(s, t) = -2s^2 \int_0^1 dx \int_0^x dy \, y(1-x) \left( S_{n-1}(s, t') + T_{n-1}(s, t') \right)|_{t'=tx+yu}
\]

\[
+ \quad s^4 \int_0^1 dx \, x^2 (1-x)^2 \sum_{k=1}^{n-2} \sum_{p=0}^{2k-2} \frac{1}{p!(p+2)!} \frac{d^p}{dt'^p} (S_k(s, t') + T_k(s, t')) \times
\]

\[
S_1 = \frac{1}{12}, \quad T_1 = \frac{1}{12} \quad \times \frac{d^p}{dt'^p} (S_{n-1-k}(s, t') + T_{n-1-k}(s, t'))|_{t'=-sx} \left( tsx(1-x) \right)^p
\]

summation \[ \Sigma_3(s, t, z) = \Sigma_1(s, t, z) - S_2(s, t)z^2 + S_1(s, t)z, \quad \Sigma_2(s, t, z) = \Sigma_1(s, t, z) + S_1(s, t)z \]
All loop Exact Recurrence Relation

\[ D=8 \quad N=1 \]

**s-channel term** \( S_n(s, t) \) **t-channel term** \( T_n(s, t) \)

\[ T_n(s, t) = S_n(t, s) \]

**Exact relation for ALL diagrams**

\[
nS_n(s, t) = -2s^2 \int_0^1 dx \int_0^x dy \, y(1-x) \left( S_{n-1}(s, t') + T_{n-1}(s, t') \right)|_{t'=tx+yu} \\
+ \frac{1}{s^4} \int_0^1 \int_0^x \frac{1}{p!(p+2)!} \left( \frac{d^p}{dt'^p} (S_k(s, t') + T_k(s, t')) \right) \times \\
\sum_{k=1}^{n-2} \sum_{p=0}^{2k-2} \frac{1}{(p+2)!} \left( S_{n-1-k}(s, t') + T_{n-1-k}(s, t') \right)|_{t'=-sx} (tsx(1-x))^p
\]

**Summation** \( \Sigma_3(s, t, z) = \Sigma_1(s, t, z) - S_2(s, t)z^2 + S_1(s, t)z, \quad \Sigma_2(s, t, z) = \Sigma_1(s, t, z) + S_1(s, t)z \)

**Diff eqn**

\[
\frac{d}{dz} \Sigma(s, t, z) = -\frac{1}{12} + 2s^2 \int_0^1 dx \int_0^x dy \, y(1-x) \left( \Sigma(s, t', z) + \Sigma(t', s, z) \right)|_{t'=tx+yu} \\
-\frac{1}{s^4} \int_0^1 \int_0^x \frac{1}{p!(p+2)!} \left( \frac{d^p}{dt'^p} (\Sigma(s, t', z) + \Sigma(t', s, z)) |_{t'=sx}^2 \right) (tsx(1-x))^p.
\]
All loop Solution (leading divs)

$D=8 \quad N=1$

\[ L(s, z) = \frac{p}{3} \frac{4 \tan(zs^2/(8\sqrt{15}))}{1 - \tan(zs^2/(8\sqrt{15}))\sqrt{5/3}} \]

PT and Padé versus ladder for $t=s$
Subleading divergences

\[ \Sigma_L(z) + \epsilon \Sigma_{NL}(z) + \epsilon^2 \Sigma_{NNL}(z) + \cdots \]

\[ \Sigma(z) = \sum_{n} z^n F_n \]

\[ \begin{array}{llll}
D = 4 & N = 4 & z = g^2/\epsilon \\
D = 6 & N = 2 & z = g^2s/\epsilon, z = g^2t/\epsilon \\
D = 8 & N = 1 & z = g^2s^2/\epsilon, z = g^2st/\epsilon, \ldots \\
D = 10 & N = 1 & z = g^2s^3/\epsilon, z = g^2s^2t/\epsilon, \ldots \\
\end{array} \]

D=8 N=1

sLadder case

\[ \Sigma_{NL} = s\Sigma_{sB}(z) + t\Sigma_{tB}(z) \]

\[ z = \frac{g^2s^2}{\epsilon} \]
Subleading divergences

\[ \Sigma_L(z) + \epsilon \Sigma_{NL}(z) + \epsilon^2 \Sigma_{NNL}(z) + \cdots \]

\[ \Sigma(z) = \sum_{n} z^n F_n \]

\(D = 4\) \(N = 4\) \(z = g^2/\epsilon\)

\(D = 6\) \(N = 2\) \(z = g^2 s/\epsilon, z = g^2 t/\epsilon\)

\(D = 8\) \(N = 1\) \(z = g^2 s^2/\epsilon, z = g^2 st/\epsilon, \ldots\)

\(D = 10\) \(N = 1\) \(z = g^2 s^3/\epsilon, z = g^2 s^2 t/\epsilon, \ldots\)

**D=8 N=1**

**sLadder case** \(\Sigma_{NL} = s \Sigma_{sB}(z) + t \Sigma_{tB}(z)\)

\[ z = \frac{g^2 s^2}{\epsilon} \]

\[ \Sigma'_{tB}(z) = \frac{5}{6} \left[ e^{z/60} (2 \cos(z/30) - \sin(z/30)) - 2 \right] \]

\[ \Sigma_{tB} = -\frac{1}{36} \left[ 60 + z + e^{z/60} (- (60 + z) \cos(z/30) - 2(-15 + z) \sin(z/30)) \right] \]
\[
\Sigma'_{sB} = \sum_{n=2}^{\infty} z^n B'_{s_n}
\]

\[
d\frac{d^2 \Sigma'_{sB}(z)}{dz^2} + f_1(z) \frac{d \Sigma'_{sB}(z)}{dz} + f_2(z) \Sigma'_{sB}(z) = f_3(z)
\]

**Diff eqn**

\[
f_1(z) = -\frac{1}{6} + \frac{\Sigma_A}{15},
\]

\[
f_2(z) = \frac{1}{80} - \frac{\Sigma_A}{360} + \frac{\Sigma_A^2}{600} + \frac{1}{15} \frac{d \Sigma_A}{dz},
\]

\[
f_3(z) = \frac{2321}{5!5!2} \Sigma_A + \frac{11}{1800} \Sigma'_{tB} - \frac{47}{5!45} \Sigma_A^2 - \frac{1}{5!72} \Sigma_A \Sigma'_{tB} + \frac{23}{6750} \Sigma_A^3 + \frac{1}{1200} \Sigma_A^2 \Sigma'_{tB}
\]

\[
- \frac{19}{36} \frac{d \Sigma_A}{dz} - \frac{1}{15} \frac{d \Sigma'_{tB}}{dz} + \frac{23}{225} \frac{d \Sigma_A^2}{dz} + \frac{1}{30} \frac{d(\Sigma_A \Sigma'_{tB})}{dz} - \frac{3}{32}
\]
Sum of Ladder diagrams (subleading divs)

\[ \Sigma'_{sB} = \sum_{n=2}^{\infty} z^n B'_{sn} \]

\[ \frac{d^2 \Sigma'_{sB}(z)}{dz^2} + f_1(z) \frac{d \Sigma'_{sB}(z)}{dz} + f_2(z) \Sigma'_{sB}(z) = f_3(z) \]

Diff eqn

\[ f_1(z) = -\frac{1}{6} + \frac{\Sigma_A}{15}, \]

\[ f_2(z) = \frac{1}{80} - \frac{\Sigma_A}{360} + \frac{\Sigma_A^2}{600} + \frac{1}{15} \frac{d \Sigma_A}{dz}, \]

\[ f_3(z) = \frac{2321}{5!5!} \Sigma_A + \frac{11}{1800} \Sigma'_{tB} - \frac{47}{5!45} \Sigma_A^2 - \frac{1}{5!72} \Sigma_A \Sigma'_{tB} + \frac{23}{6750} \Sigma_A^3 + \frac{1}{1200} \Sigma_A^2 \Sigma'_{tB} \]

\[ -\frac{19}{36} \frac{d \Sigma_A}{dz} - \frac{1}{15} \frac{d \Sigma'_{tB}}{dz} + \frac{23}{225} \frac{d \Sigma_A^2}{dz} + \frac{1}{30} \frac{d \Sigma_A}{dz} + \frac{3}{32} \]

Solution to Diff eqn

\[ \Sigma'_{sB}(z) = \frac{d \Sigma_A}{dz} u(z) \]

\[ u(z) = \int_{0}^{z} dy \int_{0}^{y} dx \frac{f_3(x)}{d \Sigma_A(x)/dx} \]

smooth monotonic function
Scheme dependence and arbitrariness of subtraction

**subleading case**

\[
A'_1 + B'_{s1} = \frac{1}{6\epsilon} (1 + c_1 \epsilon) \quad \Delta \Sigma'_{sB} = c_1 z \frac{d \Sigma'_{sA}}{dz}.
\]

\[ z \rightarrow z(1 + c_1 \epsilon). \]

**sub-subleading case**

\[
A'_2 + B'_{s2} = \frac{s}{3!4!\epsilon^2} \left(1 - \frac{5}{12} \epsilon + 2c_1 \epsilon + c_2 \epsilon^2\right) \quad \Delta \Sigma'_{sC} = c_2 z^2 \frac{d \Sigma'_{sA}}{dz}.
\]

\[ z \rightarrow z(1 + c_1 \epsilon) + z^2 c_2 \epsilon^2. \]

\[
\Delta \Sigma'_{sC} = -c_1^2 \frac{z}{4!} \left(\frac{d \Sigma'_{sA}}{dz} - 12\frac{d^2 \Sigma'_{sA}}{dz^2}\right) \quad z \rightarrow z(1 + c_1 \epsilon) + z^2 (c_2 + c_1^2 / 4!) \epsilon^2.
\]
Scheme dependence and arbitrariness of subtraction

subleading case

\[ A'_1 + B'_{s1} = \frac{1}{6\epsilon} (1 + c_1 \epsilon) \quad \Delta \Sigma'_{sB} = c_1 z \frac{d\Sigma'_A}{dz} \rightarrow z \rightarrow z(1 + c_1 \epsilon). \]

sub-subleading case

\[ A'_2 + B'_2 = \frac{s}{3! 4! \epsilon^2} \left( 1 - \frac{5}{12} \epsilon + 2c_1 \epsilon + c_2 \epsilon^2 \right) \quad \Delta \Sigma'_{sC} = c_2 \epsilon^2 \frac{d\Sigma'_A}{dz}. \]

\[ \Delta \Sigma'_{sC} = -\frac{c^2_1}{4!} \left( \frac{d\Sigma'_A}{dz} - 12 \frac{d^2 \Sigma'_A}{dz^2} \right) \rightarrow z \rightarrow z(1 + c_1 \epsilon) + z^2 (c_2 + \frac{c^2_1}{4!}) \epsilon^2. \]
Scheme dependence and arbitrariness of subtraction

**Sub-subleading case**

\[
A'_2 + B'_2 = \frac{s}{3!4!\epsilon^2} \left( 1 - \frac{5}{12} \epsilon + 2c_1\epsilon + c_2\epsilon^2 \right)
\]

New contribution from subleading term

\[
\Delta \Sigma'_{sC}(3 - \text{loop}) = -\frac{719c_1 s^2}{1036800\epsilon}
\]

\[
\Sigma'_{sB}(3 - \text{loop}) = -\frac{71 s^2}{345600\epsilon^2}
\]

The source of a problem

\[
\Delta \Sigma'_{sC}(3 - \text{loop}) = c_1z \frac{d\Sigma'_{sB}}{dz}(3 - \text{loop})
\]

\[
z \to z(1 + c_1\epsilon) + z^2(c_2 - c_1^2/4!)\epsilon^2 + z^3c_1^3/6!\epsilon^3 - z^4c_1^4/4!6!\epsilon^4 + \ldots.
\]
Scheme dependence and arbitrariness of subtraction

**sub-subleading case**

\[
A'_2 + B'_2 = \frac{s}{3!4!\epsilon^2} \left( 1 - \frac{5}{12} \epsilon + 2c_1\epsilon + c_2\epsilon^2 \right)
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\]
Kinematically dependent renormalization

- R-operation is equivalent to

renormalizable theories

\[ \bar{A}_4 = Z_4(g^2)\bar{A}_4^{bare} \mid g^2_{bare} \rightarrow g^2 Z_4 \]

\[ g^2_{bare} = \mu^\epsilon Z_4(g^2)g^2. \]

\[ Z = 1 - \sum_i K R' G_i \]

simple multiplication

nonrenormalizable theories

\[ Z = 1 + \frac{g^2}{\epsilon} + g^4\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\right) + \ldots \]

operator multiplication

\[ Z = 1 + \frac{g^2}{\epsilon}st + g^4 st\left(\frac{s^2 + t^2}{\epsilon^2} + \frac{s^2 + st + t^2}{\epsilon}\right) + \ldots \]

\[ \frac{g^2}{\epsilon}(D_\rho D_\sigma F_{\mu\nu})^2 \]
Kinematically dependent renormalization

**operator** kinematically dependent renormalization

at 2 loops

\[
\bar{A}_4 = 1 - \frac{g^2 s t}{3!\epsilon} - \frac{g^4 s t}{3!4!}\left(\frac{s^2 + t^2}{\epsilon^2} + \frac{27/4s^2 + 1/3st + 27/4t^2}{\epsilon}\right) + \ldots
\]

\[
\bar{A}_4 = Z_4(g^2) \bar{A}_4^{\text{bare}} |_{g_{\text{bare}}^2 \rightarrow g^2 Z_4}
\]

\[
Z_4 = 1 + \frac{g^2 s t}{3!\epsilon} + \frac{g^4 s t}{3!4!}\left(-\frac{s^2 + t^2}{\epsilon^2} + \frac{5/12s^2 + 1/3st + 5/12t^2}{\epsilon}\right)
\]

\[
g_B^2 = g^2 \left(1 + \frac{g^2}{3!\epsilon}\right)
\]

\[
g^2 s t \quad \square \quad \Rightarrow \quad g^2 \left(\frac{s}{\epsilon} \quad \triangle \quad + \quad \frac{t}{\epsilon} \quad \triangledown \right)
\]

this is operator action!

\[
R' \quad \square \quad \Rightarrow \quad \square \quad \triangledown
\]

compare with R-operation
Kinematically dependent renormalization

Two-loop box operator action

\[ g^4 st \left( \begin{array}{c|c} & \\ \hline & \end{array} + \begin{array}{c|c} & \\ \hline & \end{array} \right) \rightarrow g^4 \left( s + t + t + s \right) \]

Three-loop box counterterms

Tennis court counterterms

Z-operator reproduces R-operation like in renormalizable theories
Kinematically dependent renormalization

renormalizable theories

scheme dependence

\[ g^2 = zg'^2, \quad z = 1 + g'^2 c_1 + g'^4 c_2 + \ldots \]

infinite number of free parameters lead to a single multiplication constant -> redefinition of a single coupling

nonrenormalizable theories

scheme dependence

\[ g^2 = zg'^2, \quad z = 1 + g'^2 c_1 + g'^4 st(s^2 + t^2) c_2 + \ldots \]

infinite number of free parameters lead to a single multiplication constant acting as an operator -> redefinition of a series of couplings
Conclusions
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As a result, one can construct the higher derivative theory that gives the finite scattering amplitudes with a single arbitrary coupling $g$ defined in PT within the given renormalization scheme.
Conclusions

- The structure of UV divergences in non-renormalizable theories essentially copies that of renormalizable ones.

- The main difference is that the renormalization constant $Z$ depends on kinematics and acts like an operator rather than simple multiplication.

- As a result, one can construct the higher derivative theory that gives the finite scattering amplitudes with a single arbitrary coupling $g$ defined in PT within the given renormalization scheme.

- Transition to another scheme is performed by the action on the amplitude of a finite renormalization operator $z$ that depends on kinematics.
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The main difference is that the renormalization constant $Z$ depends on kinematics and acts like an operator rather than simple multiplication.

As a result, one can construct the higher derivative theory that gives the finite scattering amplitudes with a single arbitrary coupling $g$ defined in PT within the given renormalization scheme.

Transition to another scheme is performed by the action on the amplitude of a finite renormalization operator $z$ that depends on kinematics.

Assuming that one accepts these arguments, there is still a problem that at each order of PT the amplitude increases with energy, thus violating unitarity. However, apparently, this problem has to be addressed after summation of the whole PT series.