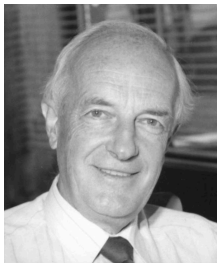
Gravity in the High Energy Limit

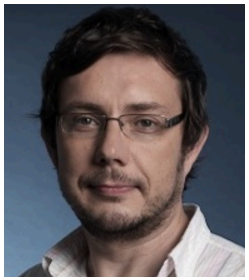
Agustín Sabio Vera

Universidad Autónoma de Madrid, Instituto de Física Teórica UAM/CSIC

Seminar at "New Frontiers in Physics", Crete, July 7th 2018



In 1996 I heard about pomerons for the first time, from Donnachie.
He told me about the work of a Russian scientist on the field.



A year later I studied BFKL with Forshaw, in Manchester.
He used to say that to read Lev's papers was like
"hitting your head against the wall"

I first met Lev at a Low x Physics Workshop in 2001, in Krakow.
He stood up after my talk and just stated
"What you are saying is wrong", with a very strong Russian accent.
It felt like a big hole appeared under me and I was going down through it.

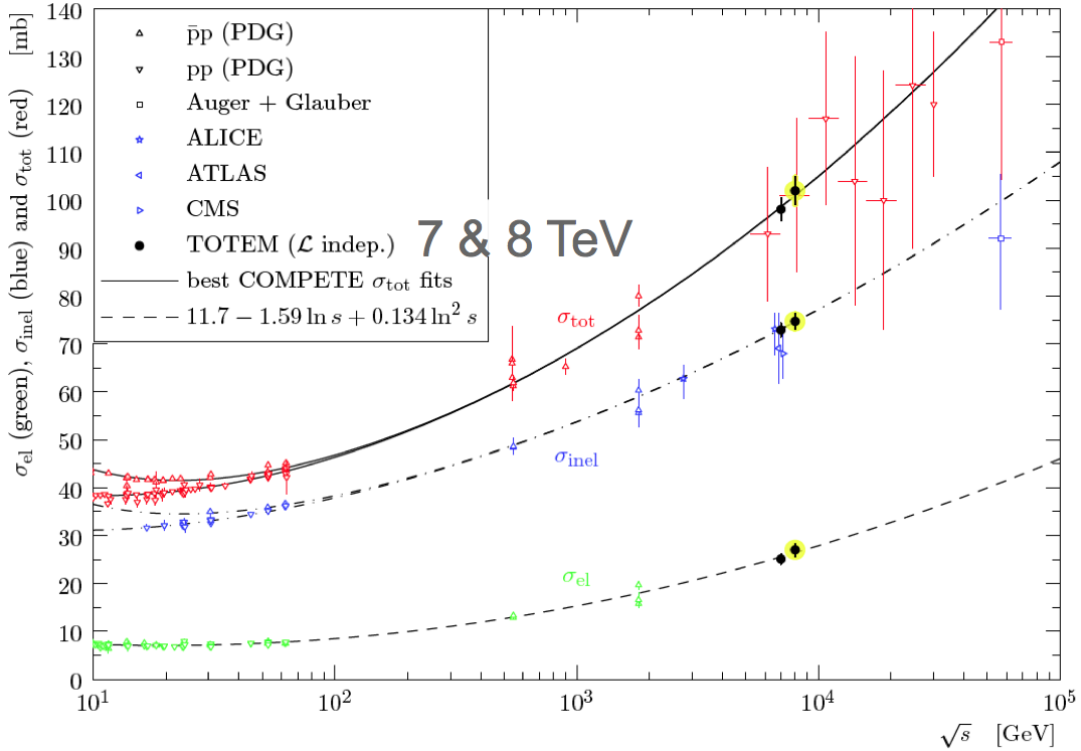
Since then we did not stop discussing about physics and everything else until August last year when we worked for a week in Cambridge. He had been in Madrid for a month in January.

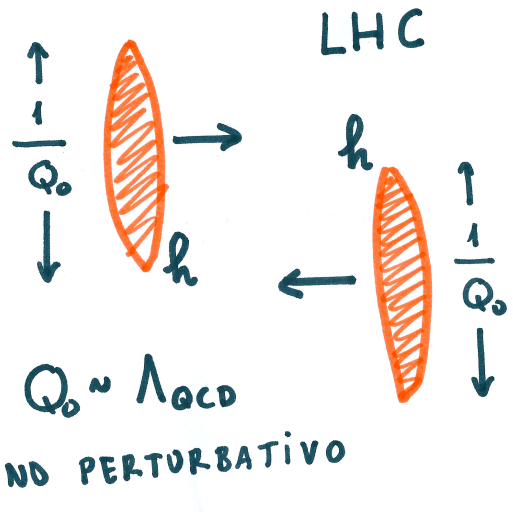
We wrote four papers together. Funny enough, none of them on QCD:

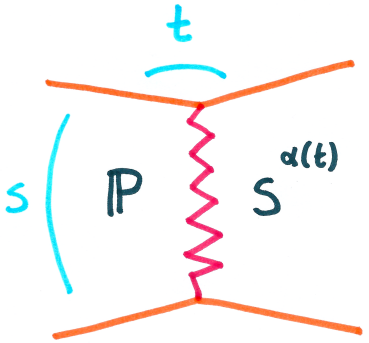
2005 Reflexive Numbers & Berger Graphs from Calabi-Yau Spaces.
with Velizhanin and Volkov
2008 BFKL Pomeron, Reggeized gluons & Bern-Dixon-Smirnov amplitudes
2008 N=4 SUSY amplitudes at high energies: the Regge cut piece
2012 Double-logarithms in Einstein-Hilbert gravity & supergravity.
these last three with Bartels

Working with Lev you always had the feeling that something special was about to happen, that a great discovery was around the corner ...

Hadron-hadron total cross section grows with energy:







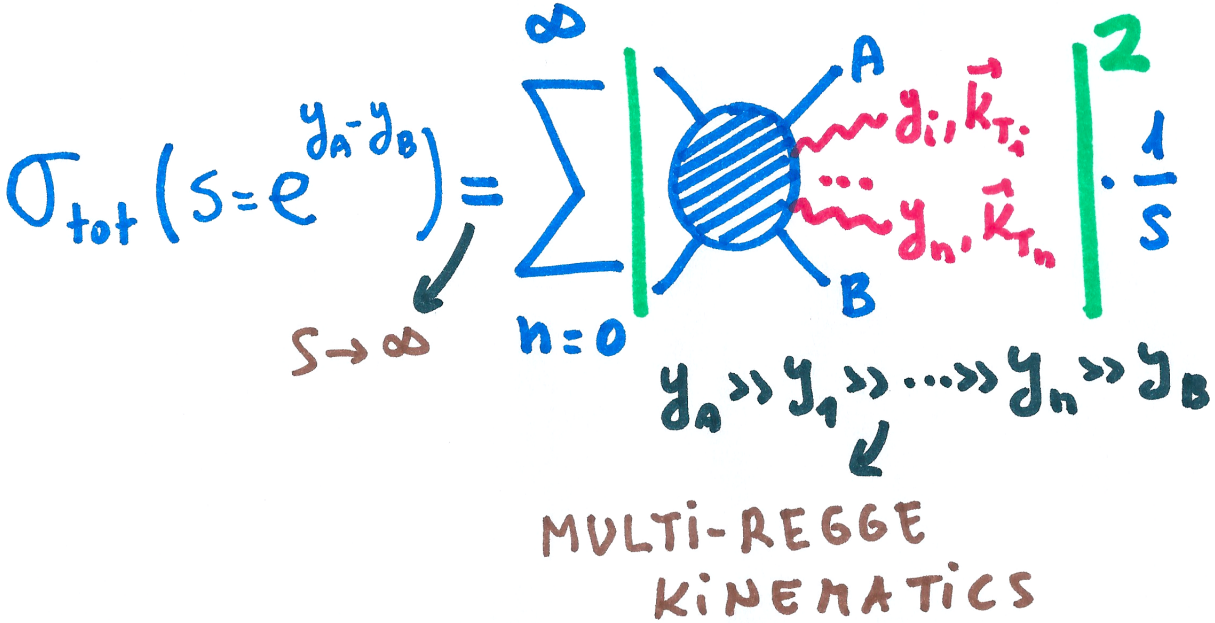
Consistent with Regge theory (Pomeron exchange) valid when $s \gg t$:

$$\sigma_{tot} \sim s^{\alpha(0)-1} = s^{0.1}$$ (Donnachie-Landshoff)

Regge theory preludes QCD. Pomeron in terms of quarks & gluons?

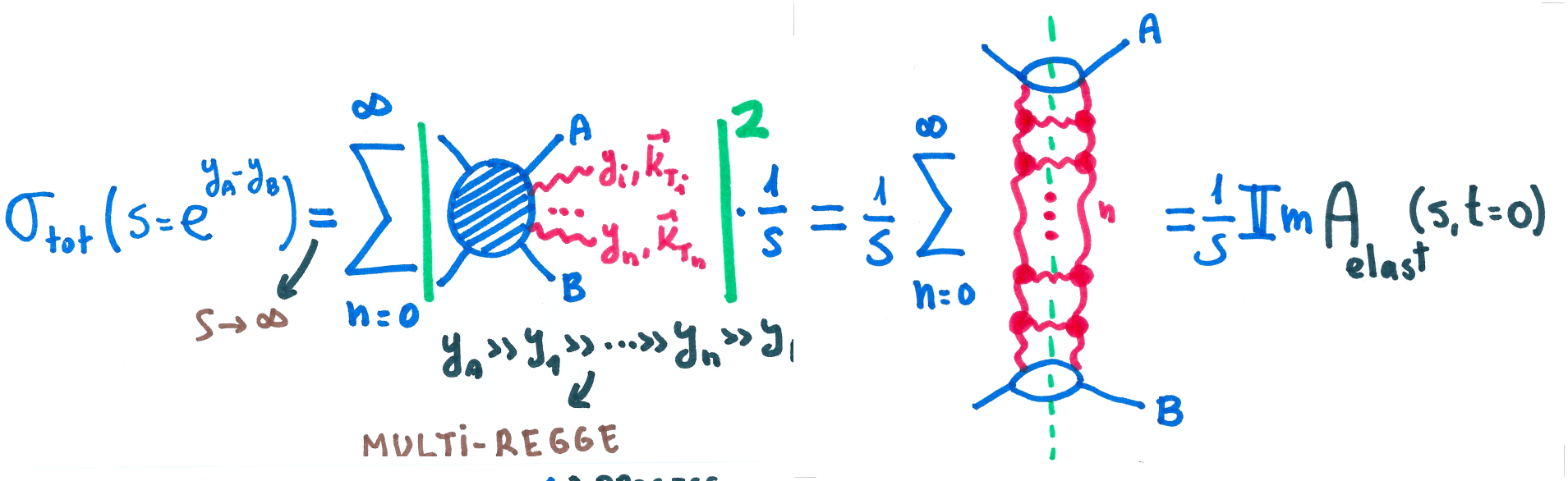
Perturbation theory with large scale $Q > \Lambda_{\rm QCD} \to \alpha_s(Q) \ll 1$.

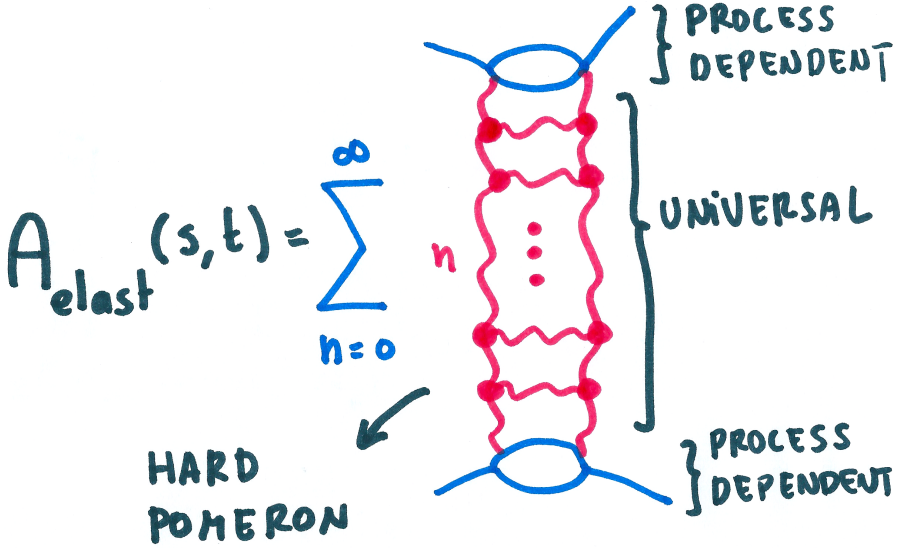
$s \gg t, Q^2 \to \alpha_s(Q) \log\left(\frac{s}{t}\right) \sim \mathcal{O}(1)$. Resummation needed.



$$\sigma_{\rm tot}^{\rm LL} = \sum_{n=0}^{\infty} \mathcal{C}_n^{\rm LL} \alpha_s^n \int_{y_B}^{y_A} dy_1 \int_{y_B}^{y_1} dy_2 \dots \int_{y_B}^{y_{n-1}} dy_n = \sum_{n=0}^{\infty} \frac{\mathcal{C}_n^{\rm LL}}{n!} \underbrace{\alpha_s^n (y_A - y_B)^n}_{\rm LL}$$

Multi-Regge linked to elastic amplitudes via optical theorem:



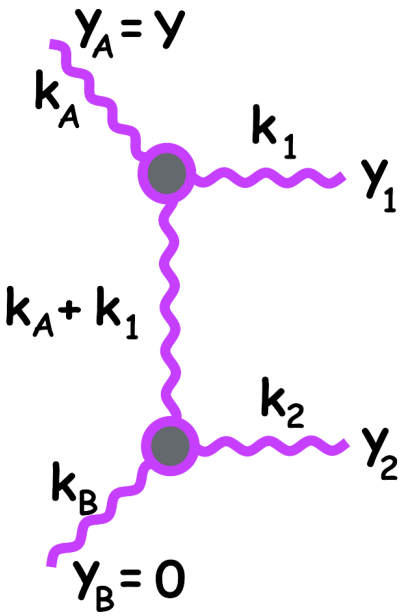


New degree of freedom $= g_R$
("Reggeized" gluon)

Pomeron= Bound state of 2 g_R

2-dimensional interaction Hamiltonian

Effective Feynman rules:



Gluon Regge trajectory: $\omega\left(\vec{q}\right) = -\frac{\alpha_s N_c}{\pi} \log \frac{q^2}{\lambda^2}$

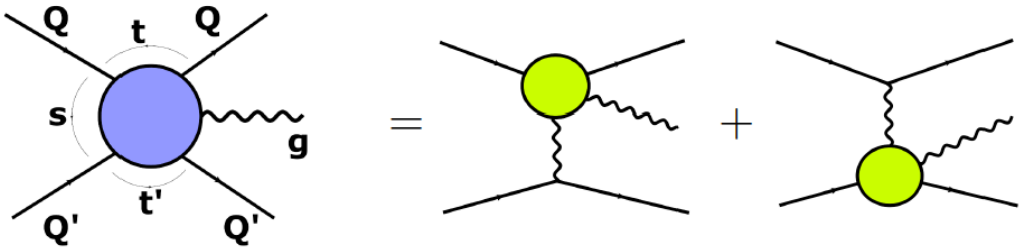
Modified propagators in the t-channel:

$$\left(\frac{s_i}{s_0}\right)^{\omega(t_i)} = e^{\omega(t_i)(y_i - y_{i+1})}$$

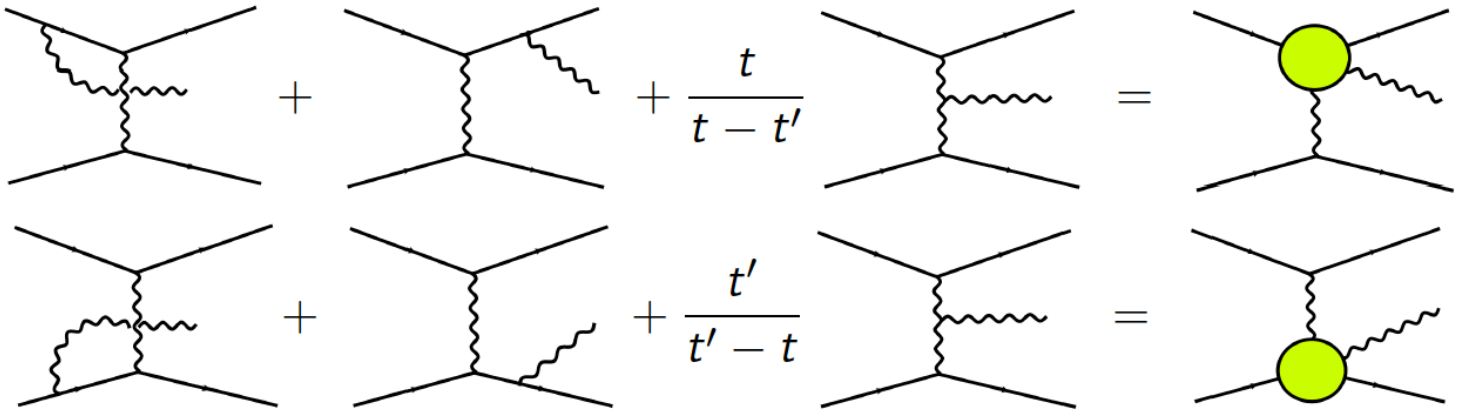
$$\left(\frac{\alpha_s N_c}{\pi}\right)^2 \int d^2\vec{k}_1 \frac{\theta\left(k_1^2 - \lambda^2\right)}{\pi k_1^2} \int d^2\vec{k}_2 \frac{\theta\left(k_2^2 - \lambda^2\right)}{\pi k_2^2} \delta^{(2)}\left(\vec{k}_A + \vec{k}_1 + \vec{k}_2 - \vec{k}_B\right)$$
$$\times \int_0^Y dy_1 \int_0^{y_1} dy_2\, e^{\omega\left(\vec{k}_A\right)(Y - y_1)} e^{\omega\left(\vec{k}_A + \vec{k}_1\right)(y_1 - y_2)} e^{\omega\left(\vec{k}_A + \vec{k}_1 + \vec{k}_2\right) y_2}$$

Is there a similar picture in gravity?

We reviewed Lev's calculation (SV-Serna-VazquezMozo)[2012]



where the amplitude is the sum of two gauge invariant parts

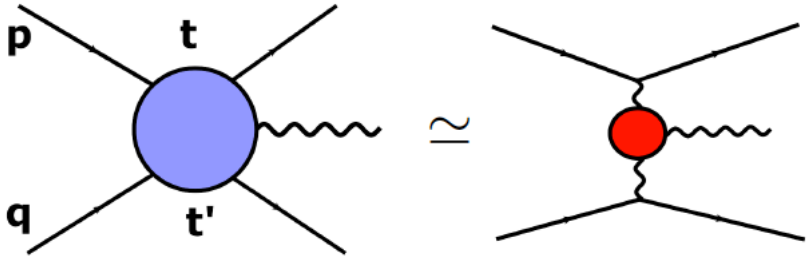


$t = k_1^2, t' = k_2^2$ Sudakov expansion

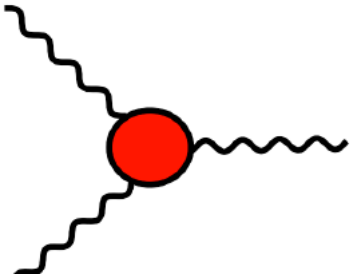
$$k_1 = \alpha_1 p + \beta_1 q + k_1^\perp \qquad k_2 = \alpha_2 p + \beta_2 q + k_2^\perp .$$

Multi-Regge kinematics (MRK):

$$1 \gg |\alpha_1| \gg |\alpha_2| = -\frac{t'}{s} \qquad 1 \gg |\beta_2| \gg |\beta_1| = -\frac{t}{s}$$

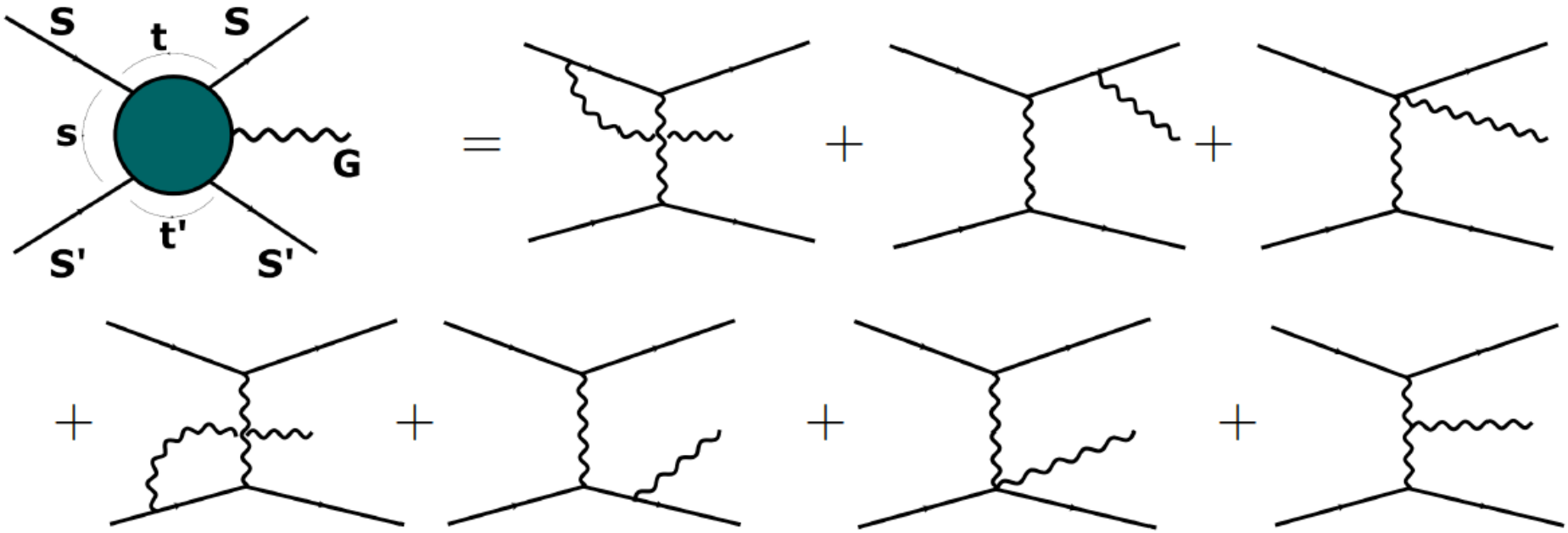


Universal Reggeized g - Reggeized g - g Effective Vertex (Lipatov) in MRK:

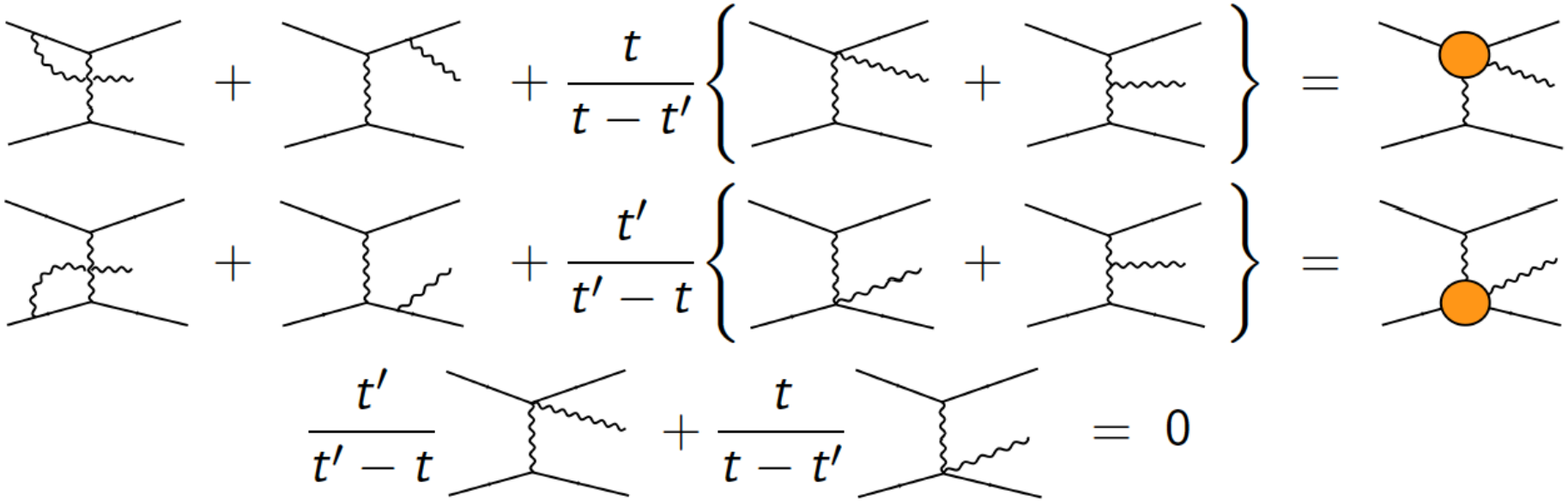


$$= ig\eta_{\mu\sigma}\left\{\left(\alpha_1 - \frac{2t}{s\beta_2}\right)p^\nu + \left(\beta_2 - \frac{2t'}{s\alpha_1}\right)q^\nu - \left(k_1^\perp + k_2^\perp\right)^2\right\}$$

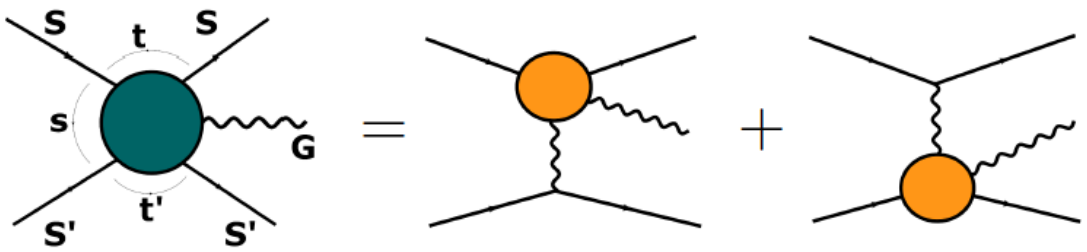
The Closest Calculation in Einstein-Hilbert Gravity:



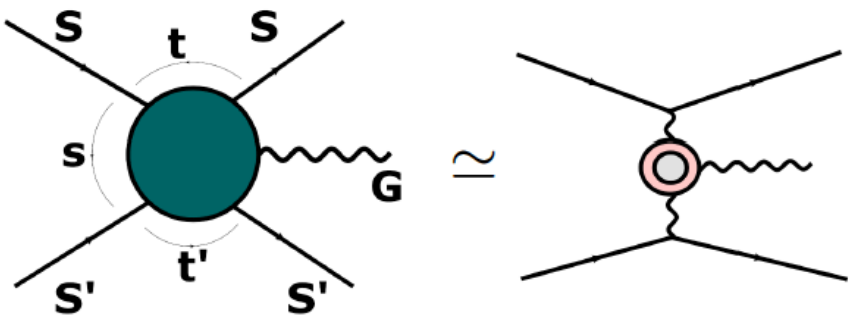
Nice Trick is Still Nice but More Tricky:



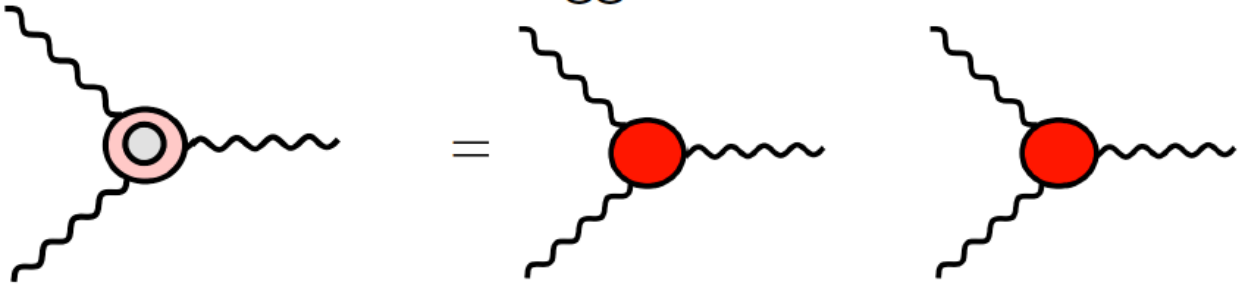
Exact Amplitude is the Sum of Two Gauge Invariant Sub-Amplitudes:



Using the same Sudakov expansion and Multi-Regge kinematics:



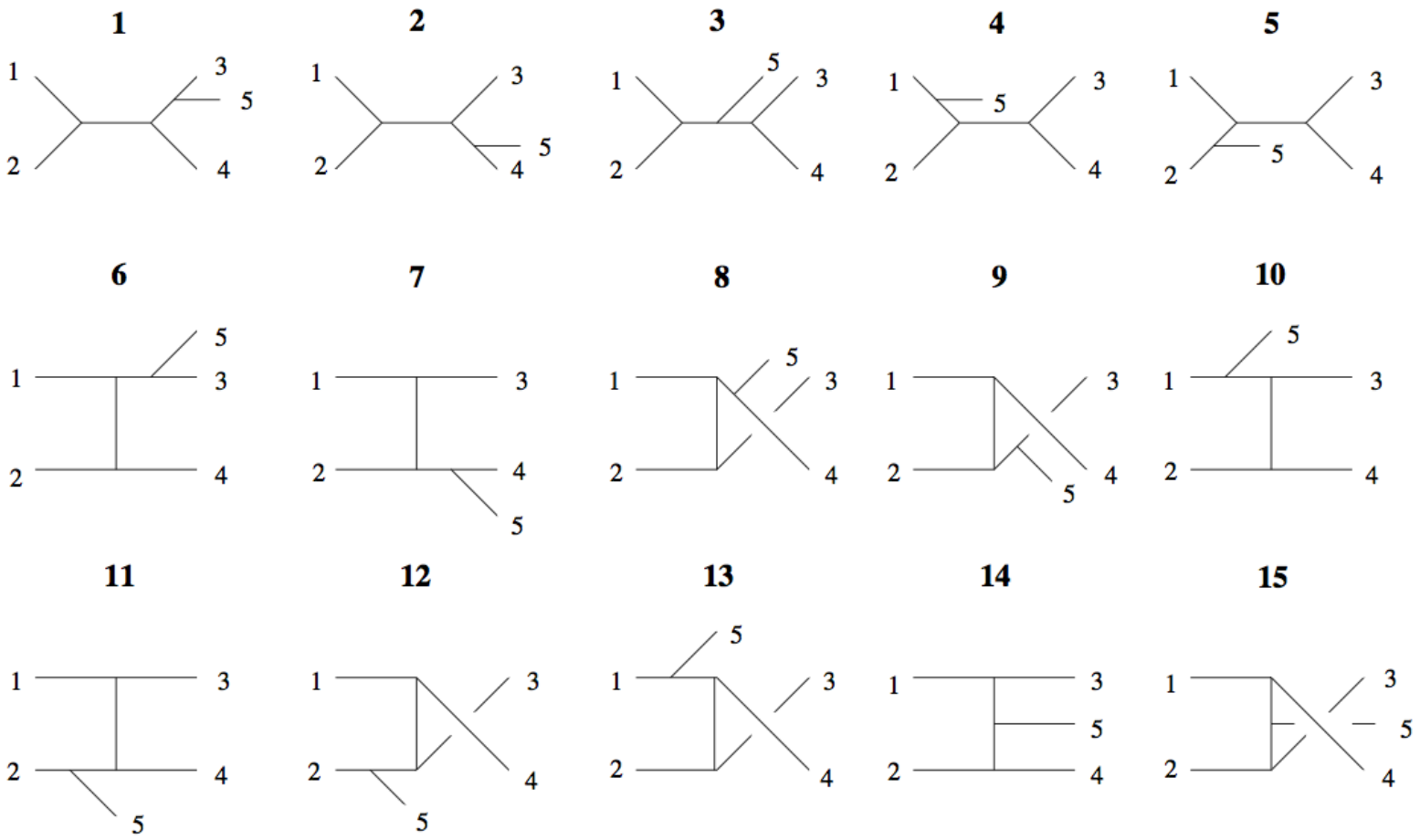
Universal Reggeized G - Reggeized G - G Effective Vertex (Lipatov):



$$+ \quad 4\beta_1\alpha_2\left\{\frac{p^\mu p^\nu}{\beta_2^2}+\frac{q^\mu q^\nu}{\alpha_1^2}+\frac{p^\mu q^\nu+q^\mu p^\nu}{\alpha_1\beta_2}\right\}$$

Subtraction Term to Fullfil Steinman Relations (no simultaneous singularities in overlapping channels).

This can be calculated also using the Color-Kinematics Duality (Johansson-SV-Serna-VazquezMozo)[2013].
With scalars and one gluon in general we have



$$\mathcal{A}_5 = g^3 \sum_{i=1}^{15} \frac{c_i n_i}{d_i}$$

c_i are color factors $(c_1 = f^{a_5 a_3 b} f^{b a_4 c} f^{c a_2 a_1})$
$d_i = \prod_{\alpha_i} s_{\alpha_i}$ are products of kinematical invariants
n_i are functions of momenta and polarizations from Feynman rules
c_i satisfy Jacobi identities $(j_1 \equiv c_{12} - c_9 + c_{15} = 0)$
These relations are not satisfied if $c_i \to n_i$
Perform the transformation $\mathcal{A}_5 = \sum_{i=1}^{15} \frac{c_i n_i}{d_i} + \sum_{\alpha=1}^{9} \gamma_\alpha j_\alpha = \sum_{i=1}^{15} \frac{c_i n_i'}{d_i}$,
such that the Jacobi identities with $c_i \to n_i'$ are satisfied.
The gravitational amplitude from BCJ double-copy:

$$-i\mathcal{M} = \left(\frac{\kappa}{2}\right)^3 \sum_{i=1}^{15} \frac{n_i' \widetilde{n}_i'}{d_i}$$

has the correct Lipatov's Regge limit. Test of double copy prescription.

Recent idea for amplitudes without Feynman diags. (Cachazo-He-Yuan):

$$\mathcal{A}_n = i\, g^{n-2} \int \frac{d^n\sigma}{\mathrm{Vol}[\mathrm{SL}(2,\mathbb{C})]} \sigma_{kl}\sigma_{lm}\sigma_{mk} \prod_{i\neq k,l,m} \delta\left(\sum_{j\neq i}^{n} \frac{2\, p_i\cdot p_j}{\sigma_{ij}}\right) I_L I_R$$

I_L carries the color traces $I_L = \sum_{\beta\in S_n/\mathbb{Z}_n} \frac{\mathrm{Tr}\left(T^{a_{\beta(1)}} T^{a_{\beta(2)}} \dots T^{a_{\beta(n)}}\right)}{\sigma_{\beta(1)\beta(2)}\sigma_{\beta(2)\beta(3)}\cdots\sigma_{\beta(n)\beta(1)}}$.

$I_R = \mathrm{Pf}' M_n$ is the reduced Pfaffian of $M_n = \begin{pmatrix} M_A & -M_C^T \\ M_C & M_B \end{pmatrix}$ where

$$M_A^{ij} = \begin{cases} \frac{p_i\cdot p_j}{\sigma_{ij}} \\ 0 \end{cases}, \quad M_B^{ij} = \begin{cases} \frac{\epsilon_i\cdot\epsilon_j}{\sigma_{ij}} \\ 0 \end{cases}, \quad M_C^{ij} = \begin{cases} \frac{\epsilon_i\cdot p_j}{\sigma_{ij}} & \text{for} \quad i\neq j \\ -\sum_{k\neq i} \frac{\epsilon_i\cdot p_k}{\sigma_{ik}} & \text{for} \quad i=j \end{cases},$$

ϵ_i: i-th gauge boson polarization. $(Pf\ A)^2 = \det(A)$.
$\mathrm{Pf}' M_n$ removes k-th row, ℓ-th column, times $(-1)^{k+\ell}\sigma_{k\ell}^{-1}$.

Gravity: $I_L = \mathrm{Pf}' M_n,\ I_R = \mathrm{Pf}' M_n$.

$s_{ij} = (p_i + p_j)^2 = 2p_i \cdot p_j, \qquad \sigma_{ij} = \sigma_i - \sigma_j$

$$\mathcal{A}_n = i\, g^{n-2} \int \frac{d^n\sigma}{\mathrm{Vol}[\mathrm{SL}(2,\mathbb{C})]} \sigma_{kl}\sigma_{lm}\sigma_{mk} \prod_{i\neq k,l,m} \delta\left(\mathcal{S}_i(\sigma)\right) I_L I_R$$

Support of the integral on $\boxed{\text{Scattering Equations}}$: $\mathcal{S}_i(\sigma) \equiv \sum_{j\neq i}^{n} \frac{s_{ij}}{\sigma_{ij}} = 0$

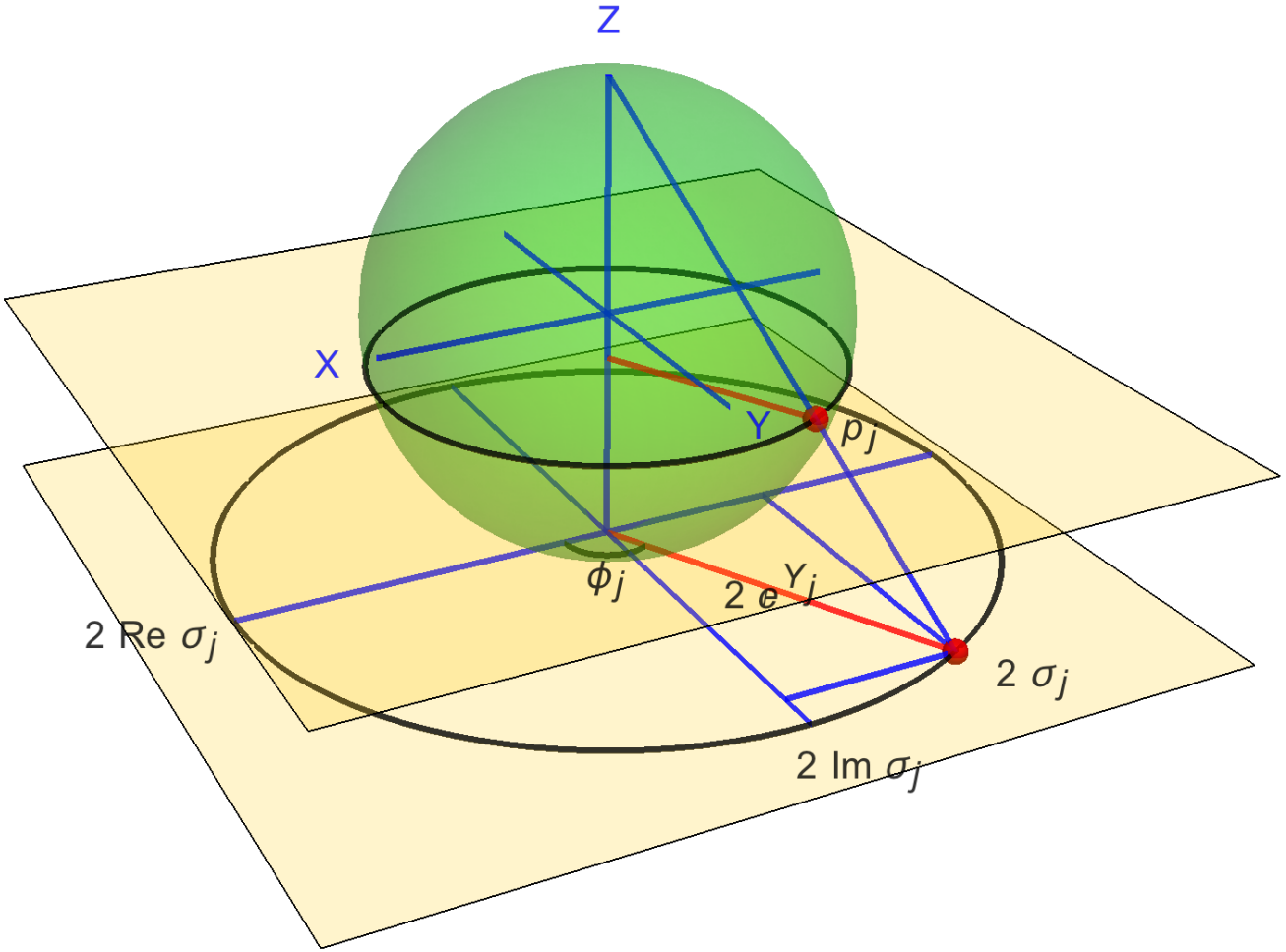
$(n-3)!$ solutions $(\sigma_1^{(i)}, \ldots, \sigma_n^{(i)})$ as n-punctured spheres.
To find all of them is a very complicated algebraic problem.

$\boxed{\text{Sudakov variables}}$ simplify the finding of solutions greatly.
(Chachamis-Medrano-SV-VázquezMozo)[2018]

Parametrize on-shell momenta with energy & $\mathbb{S}^2$ stereographic coordinates:

$$p_j = \omega_j \left(1, \frac{\zeta_j + \bar{\zeta}_j}{1+\zeta_j\bar{\zeta}_j}, i\frac{\bar{\zeta}_j - \zeta_j}{1+\zeta_j\bar{\zeta}_j}, \frac{\zeta_j\bar{\zeta}_j - 1}{1+\zeta_j\bar{\zeta}_j}\right)$$

In dimension four the solution $\sigma_j = \zeta_j = e^{Y_j + i\phi_j}$ by Fairlie always exists.



Geometric interpretation rapidity Y_j & azimuthal angle ϕ_j

$\sigma_j = \zeta_j = e^{Y_j + i\phi_j}$ on punctured sphere.

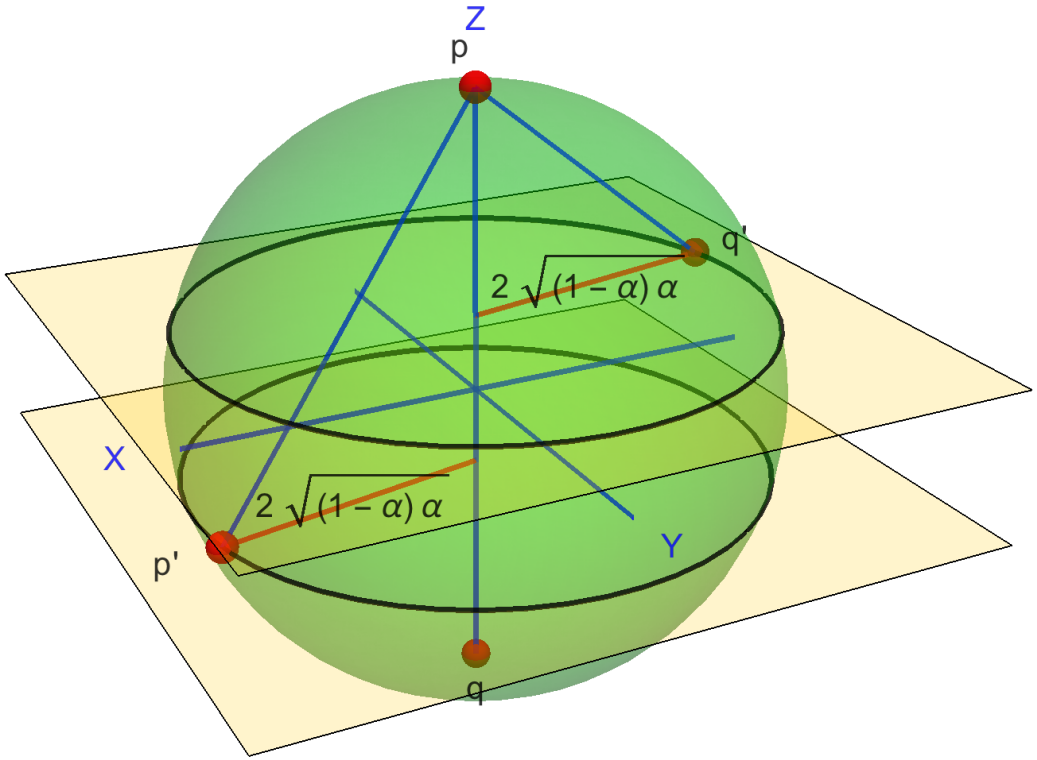
Take incoming momenta as $p = \frac{\sqrt{s}}{2}(1,0,0,1)$, $q = \frac{\sqrt{s}}{2}(1,0,0,-1)$

Then $\sigma_p = \frac{e^{i\phi}}{\epsilon} \longrightarrow \infty$, $\sigma_q = -e^{Y_q + i\phi} = -\epsilon\, e^{i\phi} \longrightarrow 0$

$\boxed{\text{Four-point}}$ amplitude with $p + q \to p' + q'$.

Introduce Sudakov representation

$$q_1 \equiv p - p' = \alpha\,(p - q) + \mathbf{q}_1, \quad \mathbf{q}_1 = q_1^{\perp}\,(0, \cos\theta_1, \sin\theta_1, 0)\,.$$



There is only one, (n-3)!, solution to the Scattering Equations:

$$\sigma_p = \infty, \;\; \sigma_q = 0, \;\; \sigma_{p'} = -\frac{Q_1}{\alpha}, \;\; \sigma_{q'} = \frac{Q_1}{1-\alpha}$$

$$Q_j = \frac{q_j^{\perp}}{\sqrt{s}} e^{i\theta_j}, \;\; |Q_1|^2 = \alpha(1-\alpha)$$

The four-point scalar amplitude is easy to calculate:

$$\mathcal{A}_4^{\varphi^3} = \int dz_{p'} \frac{z_{pq}^2 z_{qq'}^2 z_{q'p}^2}{\left(z_{pq} z_{qq'} z_{q'p'} z_{p'p}\right)^2} \delta\left(\mathcal{S}_{p'}\right) = \int \frac{dz_{p'}}{(z_{p'} - \sigma_{q'})^2} \delta\left(\frac{s_{p'q}}{z_{p'}} - \frac{s_{p'q'}}{z_{p'} - \sigma_{q'}}\right)$$

The Jacobian is very important:

$$\mathcal{A}_4^{\varphi^3} = \int dz_{p'} \left[z_{p'} - \frac{Q_1}{1-\alpha}\right]^{-2} \frac{Q_1^2}{s\alpha^3(\alpha-1)} \delta\left(z_{p'} + \frac{Q_1}{\alpha}\right)$$

$$= \left[\frac{\alpha^2(1-\alpha)^2}{Q_1^2}\right] \left[\frac{Q_1^2}{s\alpha^3(\alpha-1)}\right] = \frac{(\alpha-1)}{s\alpha} = \frac{1}{s} + \frac{1}{t}$$

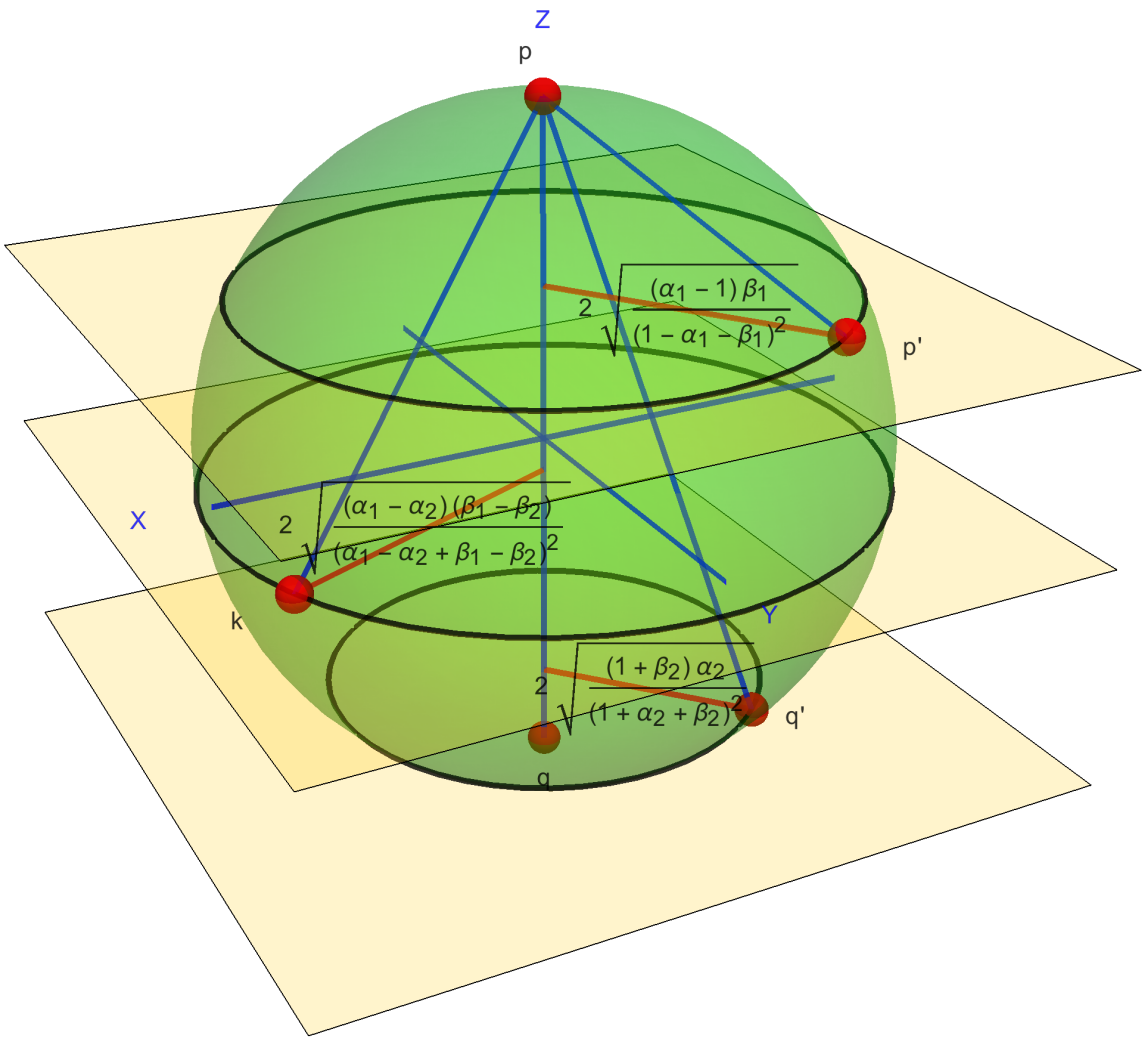
$\boxed{\text{Five-point}}$ amplitude $p + q \to p' + k + q'$. Sudakov representation:

$$
\begin{aligned}
q_1 &= p - p' = \alpha_1 p + \beta_1 q + \mathbf{q}_1, \\
q_2 &= q' - q = \alpha_2 p + \beta_2 q + \mathbf{q}_2, \\
k &= q_1 - q_2 = (\alpha_1 - \alpha_2)\, p + (\beta_1 - \beta_2)\, q + \mathbf{q}_1 - \mathbf{q}_2,
\end{aligned}
$$

Scattering Equations with (n-3)!=2 complex conjugated solutions:

$$
\begin{aligned}
\sigma_{p'}^{(+)} &= \sigma_{p'}^{(-)*} = \frac{Q_1 e^{-i\theta_2}}{\beta_1}, \\
\sigma_{q'}^{(+)} &= \sigma_{q'}^{(-)*} = \frac{Q_2 e^{-i\theta_2}}{1 + \beta_2}, \\
\sigma_{k}^{(+)} &= \sigma_{k}^{(-)*} = \frac{(Q_1 - Q_2) e^{-i\theta_2}}{\beta_1 - \beta_2}.
\end{aligned}
$$

In Sudakov space the CHY approach is much simpler.



Punctures on the Riemann sphere for the five-particle amplitude

$$\mathcal{A}_5^{\varphi^3} = \int dz_{p'} dz_{q'}\, \delta\left(\mathcal{S}_{p'}\right) \delta\left(\mathcal{S}_{q'}\right) \frac{z_{pq}^2 z_{qk}^2 z_{kp}^2}{\left(z_{pq} z_{qq'} z_{q'k} z_{kp'} z_{p'p}\right)^2}$$

$$= \int dz_{p'} dz_{q'}\, \mathcal{J}^{-1} \delta\left(z_{p'} - \sigma_{p'}\right) \delta\left(z_{q'} - \sigma_{q'}\right) \frac{z_k^2}{z_{q'}^2 z_{q'k}^2 z_{kp'}^2} + \text{c.c.}$$

$$= \frac{2}{s^2} \mathrm{Re}\left[\left(\frac{\sigma_{p'}}{\sigma_{q'}}\right) \frac{1}{L\widetilde{L} - R\widetilde{R}}\right]$$

$$= \frac{1}{s^2}\left[\frac{1}{\alpha_1+\beta_1} - \frac{1}{\alpha_2+\beta_2} + \frac{1}{(\alpha_1+\beta_1)\beta_1} - \frac{1}{\beta_1\alpha_2} + \frac{1}{\alpha_2(\alpha_2+\beta_2)}\right]$$

where

$$L = \frac{\sigma_{p'k}}{\sigma_{p'q'}}\left[(\alpha_1 - 1)\frac{\sigma_{q'}}{\sigma_{p'}} + \beta_1 \frac{\sigma_{p'}}{\sigma_{q'}}\right], \quad R = \left(\frac{\sigma_{p'}\sigma_{p'k}}{\sigma_{p'q'}}\right) \frac{(1-\alpha_1+\alpha_2-\beta_1+\beta_2)(\alpha_1+\beta_1)}{(\alpha_1-\alpha_2+\beta_1)\sigma_{p'} - (1+\beta_2)\sigma_{q'}}$$

$$\widetilde{\mathcal{O}}\Big(\alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1 - \theta_2\Big) = \mathcal{O}\Big(1-\alpha_2, 1-\alpha_1, -1-\beta_2, -1-\beta_1, \theta_2 - \theta_1\Big)$$

Factorization channels are encoded in the Jacobian ...

Geometrical meaning of subtraction to double copy found by Lev?

Four-graviton scattering in N-SUGRA $\boxed{\mathcal{A}_{4,(N)} = \mathcal{A}_4^{\rm Born}\mathcal{M}_{4,(N)}}$

At 1-loop there are three contributions ($\alpha = \text{Newton's constant}/\pi$):

$$\begin{aligned}
\mathcal{M}^{(1)}_{4,(N=8)} &= \underbrace{\alpha\, t \ln\left(\frac{-s}{-t}\right)\ln\left(\frac{-u}{-t}\right)}_{\text{Double Logs}} \\
&+ \underbrace{\alpha\,\frac{t}{2}\ln\left(\frac{-t}{\lambda^2}\right)\left(\ln\left(\frac{-s}{-t}\right)+\ln\left(\frac{-u}{-t}\right)\right)}_{\text{Trajectory}} \\
&- \underbrace{\alpha\frac{(s-u)}{2}\ln\left(\frac{-t}{\lambda^2}\right)\ln\left(\frac{-s}{-u}\right)}_{\text{Eikonal}}
\end{aligned}$$

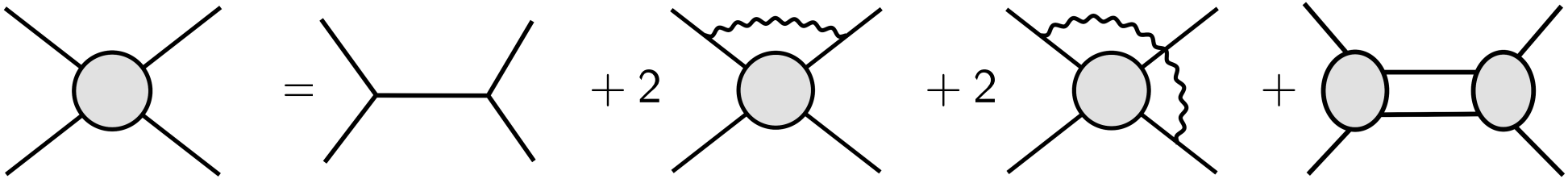
In the Regge limit $u \simeq -s$

$$\begin{aligned}\mathcal{M}^{(1)}_{4,(N=8)} &\simeq \underbrace{(\alpha\, t)\ln^2\left(\frac{s}{-t}\right)}_{\text{Double Logs}} \\ &+ \underbrace{(\alpha\, t)\ln\left(\frac{-t}{\lambda^2}\right)\ln\left(\frac{s}{-t}\right)}_{\text{Trajectory}} \\ &+ i\,\pi\,\underbrace{(\alpha\, s)\,\ln\left(\frac{-t}{\lambda^2}\right)}_{\text{Eikonal}}\end{aligned}$$

Can we calculate the Double Logs to all orders? We can use

$$\mathcal{A}_{4,(N)} = \mathcal{A}_4^{\text{Born}}\left(\frac{s}{-t}\right)^{\alpha\, t\,\ln\left(\frac{-t}{\lambda^2}\right)}\int_{\delta-i\infty}^{\delta+i\infty}\frac{d\,\omega}{2\pi i}\left(\frac{s}{-t}\right)^{\omega}\frac{f_\omega^{(N)}}{\omega}$$

The diagrammatic origin of the Double Logs is



with (1) virtual gravitons with lowest energy
(2) a pair of t-channel gravitons/gravitinos with lowest energy

The corresponding equation is

$$f_\omega^{(N)} = 1 - (\alpha\, t)\frac{d}{d\,\omega}\left(\frac{f_\omega^{(N)}}{\omega}\right) + (\alpha\, t)\left(\frac{N-6}{2}\right)\left(\frac{f_\omega^{(N)}}{\omega}\right)^2$$

with perturbative solution (N = number of gravitinos):

$$\begin{aligned} f_\omega^{(N)} &= 1 + (\alpha\, t)\frac{(N-4)}{2\omega^2} + (\alpha\, t)^2\frac{(N-4)(N-3)}{2\omega^4} \\ &- (\alpha\, t)^3\frac{(N-4)\left(5N^2-26N+36\right)}{8\omega^6} + \ldots \end{aligned}$$

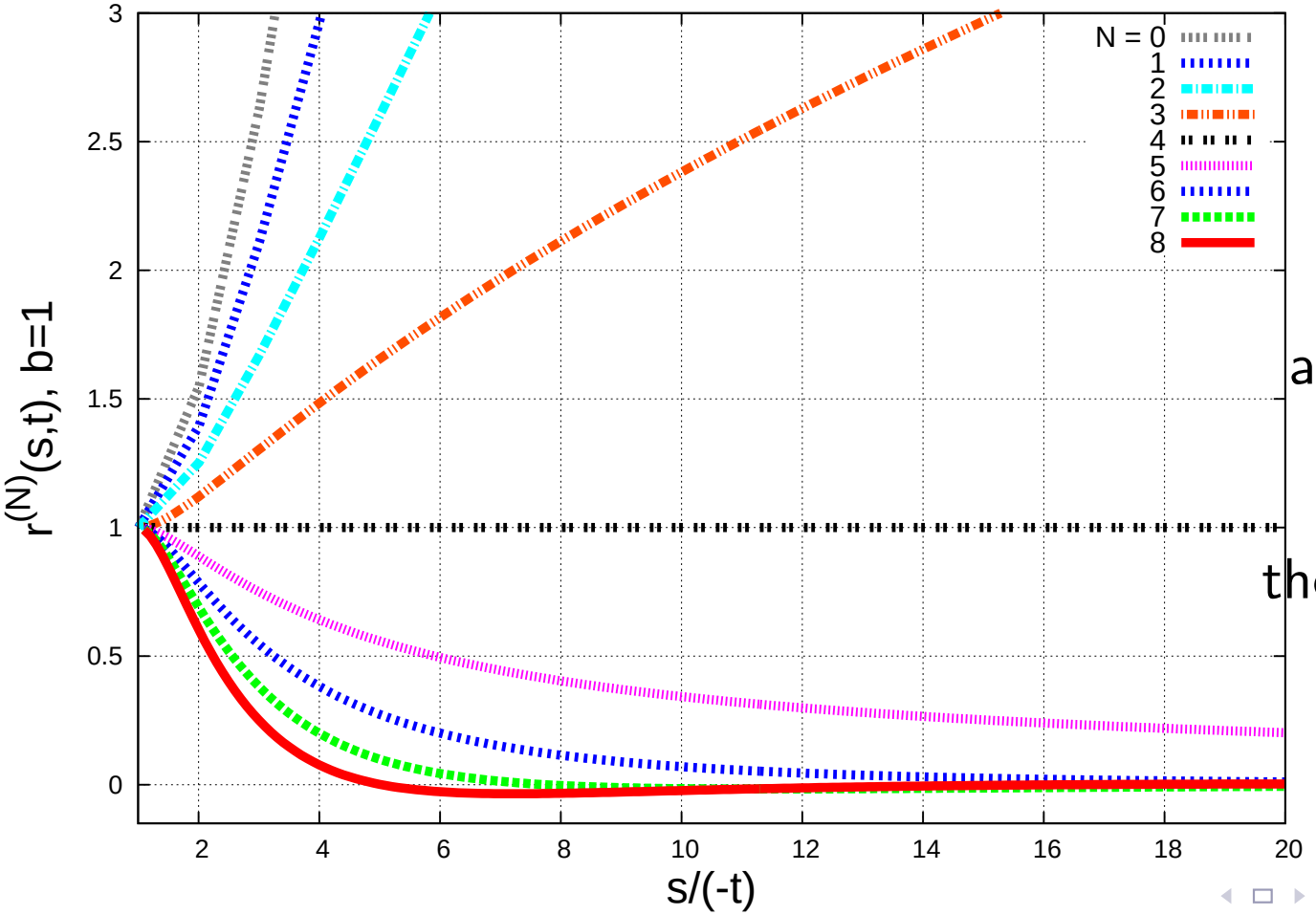
In agreement with the two-loop results for $N = 4, \ldots, 8$ SUGRA obtained using the conjecture that gravity is a double copy of gauge theories (Dixon, Bern, Carrasco, Johansson).

We have predictions to all orders *e.g.* $N = 8$:

$$\begin{aligned}
\mathcal{A}_{4,(N=8)} &= \mathcal{A}_4^{\text{Born}} \left(\frac{-t}{\lambda^2}\right)^{\alpha t\left(\ln\left(\frac{s}{-t}\right)+i\pi\left(\frac{s}{t}\right)\right)} \\
&\times \left\{ 1 + 2\left(\frac{\alpha\, t}{2}\right) \ln^2\left(\frac{s}{-t}\right) + \frac{5}{3}\left(\frac{\alpha\, t}{2}\right)^2 \ln^4\left(\frac{s}{-t}\right) \right. \\
&+ \frac{37}{45}\left(\frac{\alpha\, t}{2}\right)^3 \ln^6\left(\frac{s}{-t}\right) + \frac{353}{1260}\left(\frac{\alpha\, t}{2}\right)^4 \ln^8\left(\frac{s}{-t}\right) \\
&\left. + \frac{583}{8100}\left(\frac{\alpha\, t}{2}\right)^5 \ln^{10}\left(\frac{s}{-t}\right) + \ldots \right\}
\end{aligned}$$

Which the BCJ conjecture should reproduce for different N.

We can resum all terms (Bartels-Lipatov-ASV):
Solution of a Schrödinger equation in terms of parabolic cylinder functions



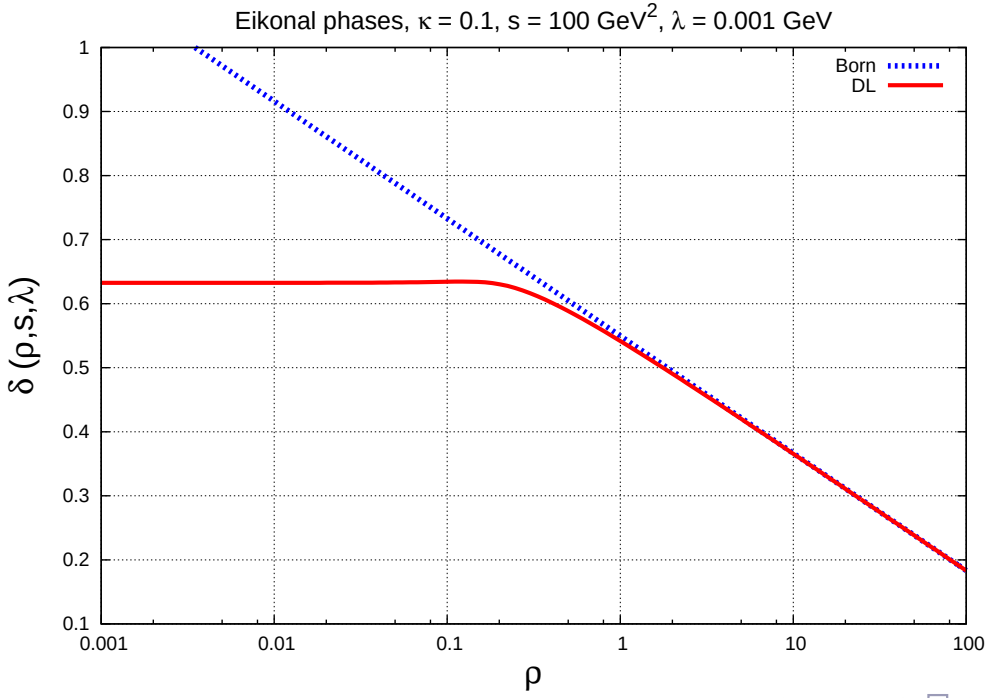
$N = 4$ SUGRA is a critical theory among
Finite ($N > 4$)
Non-finite ($N < 4$)
theories at high energies

Eikonal phase is modified. Example, in $N = 8$ SUGRA:

$$\delta^{(8)}_{\rm DL}(\rho, s, \lambda) = -\frac{s\kappa^2}{4\pi}\ln(\lambda\rho) - \frac{s\kappa^2}{8\pi}\frac{\beta_1}{\sqrt{\beta_1^2+\beta_2^2}}\frac{\rho_c(s)}{\rho}$$

$$-\frac{s\kappa^2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)(2(n-1))!}{(2n+1)n!(n-1)!}\cos\left((2n+1)\arctan\frac{\beta_2}{\beta_1}\right)\left(\frac{\rho_c(s)}{4\rho}\right)^{2n+1}$$

Eikonal phases in Born and DL approximations as a function of ρ

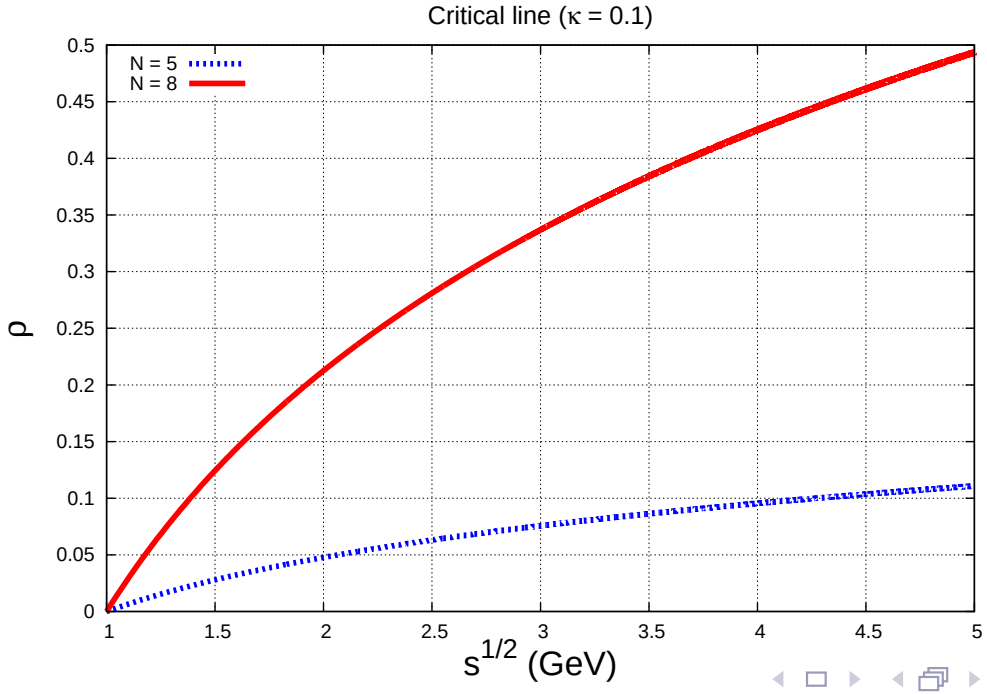


The associated Einstein deflection angle is

$$\frac{\partial \delta_{\rm DL}^{(8)}}{\partial \rho} = -\frac{s\kappa^2}{4\pi\rho} + \frac{s\kappa^2}{8\pi}\frac{\beta_1}{\sqrt{\beta_1^2+\beta_2^2}}\frac{\rho_c(s)}{\rho^2}$$

$$+\frac{4s\kappa^2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)(2(n-1))!}{n!(n-1)!\rho_c(s)}\cos\left((2n+1)\arctan\frac{\beta_2}{\beta_1}\right)\left(\frac{\rho_c(s)}{4\rho}\right)^{2n+2}$$

Critical lines in the impact parameter vs s plot: $\rho_c(s) = 2\sqrt{\beta_1^2+\beta_2^2}\,\kappa \ln s$



1. Working with Lev
2. Multi-Regge limit
3. Lipatov vertex in gravity
   1. From Feynman rules in Sudakov variables
   2. From Bern-Carrasco-Johansson (BCJ) Color-Kinematics duality
   3. From Cachazo-He-Yuan (CHY) in Sudakov representation
4. Double logarithms in gravity and supergravity

Let us see how far we are able to push the limits of MRK, now without the help of a friend and a true giant in physics ...

Lev N. Lipatov, Southampton, August 2017