Regge cuts in QCD amplitudes

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Introduction

One of the remarkable properties of QCD is the Reggeization of all elementary particles in perturbation theory, which is very important for the theoretical description of high energy processes. The gluon Reggeization is especially important because it determines the high energy behaviour of non-decreasing with energy cross sections in perturbative QCD. In particular, it appears to be the basis of the famous BFKL (Balitskii-Fadin-Kuraev-Lipatov) equation, which was first derived in non-Abelian theories with spontaneously broken symmetry
V.S. F., E.A. Kuraev, and L.N. Lipatov, Phys. Lett. B **60** (1975) 50-52
and whose applicability in QCD was then shown
I.I. Balitsky, and L.N. Lipatov, Yad. Fiz. **28** (1978) 1597-1611 [Sov. J. Nucl. Phys. **28** (1978) 822-829].

In the BFKL approach the primary Reggeon is the Reggeized gluon. The Pomeron, which determines the high energy behaviour of cross sections, appears as a compound state of two Reggeized gluons, and the Odderon, responsible for the difference of particle and antiparticle cross sections, as a compound state of three Reggeized gluons.
The Reggeization allows to express an infinite number of amplitudes through several Reggeon vertices and the Reggeized gluon trajectory. It means definite form not only of elastic amplitudes, but of inelastic amplitudes in the multi-Regge kinematics (MRK) as well.
It is extremely important that elastic amplitudes and real parts of inelastic amplitudes in the MRK with negative signature (symmetry with respect to $s \leftrightarrow u \simeq -s$) in cross-channels are determined by the Regge pole contributions and have a simple factorized form not only in the leading logarithmic approximation (LLA),

when in each order of perturbation theory only terms with the highest powers of the logarithm of c.m.s. energy $\sqrt{s}$ are retained, but also in the next-to-leading logarithmic approximation (NLLA), where the eldest of non-leading terms are also retained. Due to this, the Reggeization provides a simple derivation of the BFKL equation in the LLA and in the NLLA.

Validity of the Regge form is proved now in all orders of perturbation theory in the coupling constant g both in the LLA and in the NLLA.

The pole Regge form is violated in the NNLLA. It is necessary to say that, in general, breaking the pole Regge form is not a surprise. It is well known that Regge poles in the complex angular momenta plane generate Regge cuts. Moreover, in amplitudes with positive signature the Regge cuts appear already in the LLA. In particular, the BFKL Pomeron is the two-Reggeon cut in the complex angular momenta plane. But in amplitudes with negative signature Regge cuts appear only in the NNLLA.

It was natural to expect that the violation of the pole Regge form, firstly noticed in the non-logarithmic two loops contributions to elastic parton scattering amplitudes by
V. Del Duca and E. W. N. Glover, JHEP **0110** (2001) 035 [hep-ph/0109028],
and then confirmed and generalized to the case of logarithmic terms in the three loop contributions
V. Del Duca, G. Falcioni, L. Magnea and L. Vernazza, Phys. Lett. B **732** (2014) 233,
V. Del Duca, G. Falcioni, L. Magnea and L. Vernazza, PoS RADCOR **2013** (2013) 046 [hep-ph/1312.5098],
V. Del Duca, G. Falcioni, L. Magnea and L. Vernazza, JHEP **1502** (2015) 029,
could be explained by their contributions. It is shown
V. S. Fadin, AIP Conf. Proc. **1819** (2017) no.1, 060003 [arXiv:1612.04481 [hep-ph]],
S. Caron-Huot, E. Gardi and L. Vernazza, JHEP **1706** (2017) 016 [arXiv:1701.05241 [hep-ph]]
this is actually so.

I will discuss contributions of the Regge cuts to parton (quark and gluon) elastic scattering amplitudes with the same (negative) signature as the Reggeized gluon and the influence of the Regge cuts on the derivation of the BFKL equation. Recall that leading terms in the negative signature are real, whereas in the positive signature they are imaginary due to cancellation of leading logarithmic terms.

In contrast to the Reggeon, the three-Reggeon cuts can contribute to amplitudes with various representations of the colour group in the t-channel. Possible representations for quark-quark and quark-gluon scattering are only singlet (**1**) and octet (**8**) whereas for the gluon-gluon scattering there are also **10**, $\mathbf{10}^*$ and **27**. Due to Bose statistic for gluons, symmetry of the representations **1** and **27**, antisymmetry **10** and $\mathbf{10}^*$ and existence both symmetric $\mathbf{8_s}$ and antisymmetric $\mathbf{8_a}$ representations, except the Reggeon channel, there are amplitudes with negative signature in the representations **1** for quark-quark-scattering and in the representation **10** and $\mathbf{10}^*$ for the gluon-gluon scattering.

Three-Reggeon cuts in the lowest order

Let us consider parton (quark and gluon) elastic scattering amplitudes with negative signature in the two-loop approximation, and try to find the contribution of the Regge cuts in them. Due to the signature conservation the cuts with negative signature has to be three-Reggeon ones. Since our Reggeon is the Reggeized gluon, the cuts start with the diagrams with three *t*-channels gluons. They are presented below, where particles A, A' and B, B' can be quarks or gluons.

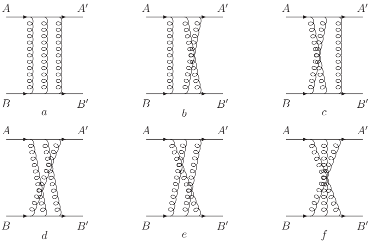

Figure 1: Three-gluon exchange diagrams

The colour factors of these diagrams are given by the convolution of the colour wave functions $\chi^{*\underline{a}'}_{A'}\chi^{\underline{a}}_{A}\chi^{*\underline{b}'}_{B'}\chi^{\underline{b}}_{B}$, where $\underline{a} = a$ for gluons and $\underline{a} = \alpha$ for quarks with $(\mathcal{T}^a_i\mathcal{T}^b_i\mathcal{T}^c_i)_{\underline{a}'\underline{a}}(\mathcal{T}^{\sigma a}_j\mathcal{T}^{\sigma b}_j\mathcal{T}^{\sigma c}_j)_{\underline{b}'\underline{b}}$, where σ means the permutation of the indices, $i = g(j = g)$ if the particles $A, A'(B, B')$ are gluons and $i = q(j = q)$ if the particles $A, A'(B, B')$ are quarks; $\mathcal{T}^a_i$ are the colour group generators in the corresponding representations, $[\mathcal{T}^a, \mathcal{T}^b] = if_{abc}\mathcal{T}^c$; $\mathcal{T}^a_{bc} = T^a_{bc} = -if_{abc}$ for gluons and $\mathcal{T}^a_{bc} = -t^a_{bc}$ for quarks.
These colour structures must be decomposed into irreducible representations R of the colour group in the t-channel. It can be done with the help of the projection operators $\hat{\mathcal{P}}^{(\mathcal{R})}$:

$$\chi^{*\underline{a}'}_{A'}\chi^{\underline{a}}_{A}\chi^{*\underline{b}'}_{B'}\chi^{\underline{b}}_{B}(\mathcal{T}^a_i\mathcal{T}^b_i\mathcal{T}^c_i)_{\underline{a}'\underline{a}}(\mathcal{T}^{\sigma a}_j\mathcal{T}^{\sigma b}_j\mathcal{T}^{\sigma c}_j)_{\underline{b}'\underline{b}} =$$

$$\sum_{\mathcal{R}}\chi^{*\underline{a}'}_{A'}\chi^{\underline{a}}_{A}\chi^{*\underline{b}'}_{B'}\chi^{\underline{b}}_{B}\langle\underline{a}'\underline{a}|\hat{\mathcal{P}}^{(\mathcal{R})}|\underline{b}'\underline{b}\rangle C^{(\mathcal{R})\sigma}_{ij}$$

where

$$C_{ij}^{(\mathcal{R})\sigma} = \frac{1}{n_{\mathcal{R}}} \langle \underline{a}' \underline{a} | \hat{\mathcal{P}}^{(\mathcal{R})} | \underline{b}' \underline{b} \rangle (\mathcal{T}_i^a \mathcal{T}_i^b \mathcal{T}_i^c)_{\underline{a}' \underline{a}} (\mathcal{T}_j^{\sigma a} \mathcal{T}_j^{\sigma b} \mathcal{T}_j^{\sigma c})_{\underline{b}' \underline{b}} \, ,$$

where $n_{\mathcal{R}}$ is the number of states in the representation $\mathcal{R}$. The projection operators, which we heed, are

$$\langle a' a | \hat{\mathcal{P}}^{(8_a)} | b' b \rangle = \frac{1}{N_c} f^{bb'c} f^{aa'c}$$

and

$$\langle aa' | \hat{\mathcal{P}}(10) + \hat{\mathcal{P}}^{(10^*)} | bb' \rangle = \frac{1}{2} (\delta^{ba} \delta^{b'a'} - \delta^{ba'} \delta^{b'a}) - \frac{1}{N_c} f^{bb'c} f^{aa'c}$$

$$= \frac{1}{2} (\delta^{ba} \delta^{b'a'} - \delta^{ba'} \delta^{b'a}) - \langle a' a | \hat{\mathcal{P}}^{(8_a)} | b' b \rangle$$

for gluon-gluon scattering,

$$\langle \alpha \alpha' | \hat{\mathcal{P}}^{(8)} | \beta \beta' \rangle = 2 t^a_{\alpha' \alpha} t^a_{\beta' \beta}$$

and

$$\langle \alpha \alpha' | \hat{\mathcal{P}}^{(1)} | \beta \beta' \rangle = \frac{1}{N_c} \delta_{\alpha' \alpha} \delta_{\beta' \beta}$$

for quark-quark scattering, and

$$\langle aa'|\hat{\mathcal{P}}^{(8)}|\beta\beta'\rangle = -\sqrt{\frac{2}{N_c}}if^{ca'a}t^c_{\beta'\beta}$$

for gluon-quark scattering.
Only octet representation gives contributions to amplitudes of gluon-gluon, gluon-quark and quark-quark scattering. In this case

$$C_{ij}^{(\mathcal{R})\sigma} = \mathrm{Tr}(\mathcal{T}^a\mathcal{T}^b\mathcal{T}^c\mathcal{T}^d)\mathrm{Tr}(\mathcal{T}^{\sigma a}\mathcal{T}^{\sigma b}\mathcal{T}^{\sigma c}\mathcal{T}^d)$$

Using the equalities

$$\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}\ , \quad t^a t^b = \frac{1}{2N_c}\delta^{ab}I + \frac{1}{2}(d^{abc} + if^{abc})t^c\ ,$$

one easily finds

$$\mathrm{Tr}(t^a t^b t^c t^d) = \frac{1}{N_c}\delta^{ad}\delta^{bc} + \frac{1}{8}(d^{adi}d^{bci} + f^{adi}f^{bci} + id^{adi}f^{bci} - if^{adi}d^{bci})\ .$$

The tensor $\mathrm{Tr}(T^a T^b T^c T^d)$ can be written as

$$\mathrm{Tr}(T^a T^b T^c T^d) = 4\mathrm{Tr}([t^a, t^i][t^j, t^b])\mathrm{Tr}([t^c, t^j][t^i, t^d])\ .$$

Using this representation, the completeness condition for the matrices t^a and the identity matrix I in the form

$$\mathrm{Tr}(t^i A)\mathrm{Tr}(t^i B) = -\frac{1}{2N_c}\mathrm{Tr}(A)\mathrm{Tr}(B) + \frac{1}{2}\mathrm{Tr}(AB) \ ,$$

and the relations (which are easily obtained from the completeness condition)

$$t^a t^a = C_F I, \quad t^a t^b t^a = \left(C_F - \frac{C_A}{2}\right) t^b,$$

$$t^a t^b t^c t^a = \frac{1}{4}\delta^{bc} I + \left(C_F - \frac{C_A}{2}\right) t^b t^c \ ,$$

where C_F and C_A are the values of the Casimir operators in the fundamental and

$$C_F = \frac{N_c^2 - 1}{2N_c} \ , \qquad C_A = N_c \ ,$$

we obtain

$$\mathrm{Tr}(T^aT^bT^cT^d) = \delta^{ad}\delta^{bc} + \frac{1}{2}(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd}) + \frac{N_c}{4}(f^{adi}f^{bci} + d^{adi}d^{bci}) \, .$$

The convolutions can be performed with the help of the relations

$$\mathrm{Tr}\left(T^aD^b\right) = 0\,, \quad \mathrm{Tr}\left(T^aT^b\right) = N_c\delta^{ab}, \quad \mathrm{Tr}\left(D^aD^b\right) = \frac{N_c^2-4}{N_c}\delta^{ab}\,,$$

$$\mathrm{Tr}\left(T^aT^bT^c\right) = i\frac{N_c}{2}f^{abc}, \quad \mathrm{Tr}\left(T^aT^bD^c\right) = \frac{N_c}{2}d^{abc}\,,$$

$$\mathrm{Tr}\left(D^aD^bT^c\right) = i\frac{N_c^2-4}{2N_c}f^{abc}, \quad \mathrm{Tr}\left(D^aD^bD^c\right) = \frac{N_c^2-12}{2N_c}d^{abc},$$

where $D^a_{bc} = d_{abc}$.

As the result, one obtains

$$C_{gg}^{(8_a)a} = \frac{3}{2} + \frac{N_c^2}{8}\,, \; C_{gq}^{(8_a)a} = \frac{1}{4} + \frac{N_c^2}{8}\,, \; C_{qq}^{(8)a} = \frac{1}{4}\left(1 + \frac{3}{N_c^2}\right)\,,$$

$$C_{gg}^{(8_a)b} = C_{gg}^{(8_a)c} = C_{gg}^{(8_a)d} = C_{gg}^{(8_a)e} = C_{gg}^{(8_a)} = \frac{3}{2}\,,$$

$$C_{gq}^{(8_a)b} = C_{gq}^{(8_a)c} = C_{gq}^{(8_a)d} = C_{gq}^{(8_a)e} = C_{gq}^{(8_a)} = \frac{1}{4}\,,$$

$$C_{qq}^{(8)b} = C_{qq}^{(8)c} = C_{qq}^{(8)d} = C_{qq}^{(8)e} = C_{qq}^{(8)} = \frac{1}{4}\left(-1 + \frac{3}{N_c^2}\right)\,,$$

$$C_{gg}^{(8_a)f} = \frac{3}{2} + \frac{N_c^2}{8}\,, \; C_{gq}^{(8_a)f} = \frac{1}{4} + \frac{N_c^2}{8}\,, \; C_{qq}^{(8)f} = \frac{1}{4}\left(N_c^2 - 3 + \frac{3}{N_c^2}\right)\,,$$

The representations (**10**) and (**10***) there are only in the gluon-gluon scattering. Since

$$\langle aa'|\hat{\mathcal{P}}^{(10)} + \hat{\mathcal{P}}^{(10^*)}|bb'\rangle = \frac{1}{2}(\delta^{ba}\delta^{b'a'} - \delta^{ba'}\delta^{b'a}) - \langle a'a|\hat{\mathcal{P}}^{(8_a)}|b'b\rangle\,,$$

for calculation of $C_{gg}^{(10+10^*)\sigma}$ we can use the results for $C_{gg}^{(8_a)\sigma}$.
The new thing which we need is

$$\mathrm{Tr}(T^a T^b T^c T^{\sigma a} T^{\sigma b} T^{\sigma c}) \, .$$

It's easy to find it using the equalities given above. The result is

$$C_{gg}^{(10+10^*)a} = C_{gg}^{(10+10^*)b} = C_{gg}^{(10+10^*)c}$$

$$= C_{gg}^{(10+10^*)d} = C_{gg}^{(10+10^*)e} = C_{gg}^{(10+10^*)f} = -\frac{3}{4} N_c \, .$$

The representation (**1**) in the quark-quark scattering is even more simple. The results are the following.
The octet part of the contribution of the diagrams with three-gluon exchanges can be written as

$$A_{ij}^{(8)} = \langle A' | \mathcal{T}^a | A \rangle \langle B' | \mathcal{T}^a | B \rangle \left[C_{ij} A^{eik} + \frac{N_c^2}{8} \left(A_{ij}^a + A_{ij}^f \right) \right.$$

$$\left. + \delta_{i,q} \delta_{j,q} \frac{4 - N_c^2}{8} \left(A_{ij}^a - A_{ij}^f \right) \right] \, ,$$

where A^{α}_{ij} is the contribution of the diagram α with omitted colour factors and $A^{eik}_{ij} = \sum_{\alpha} A^{\alpha}_{ij}$. Note that A^{eik} is gauge invariant.

$$A^{(10+10^*)}_{gg} = \varepsilon^{a}_{A}\varepsilon^{*a'}_{A}\varepsilon^{b}_{B}\varepsilon^{*b'}_{B}\langle aa'|\hat{\mathcal{P}}_{10} + \hat{\mathcal{P}}_{10^*}|bb'\rangle \frac{-3}{4} N_c A^{(eik)} \ , \quad (1)$$

where ε^{a}_{A} is the gluon A colour polarization vector, $\hat{\mathcal{P}}_R$ is the projection operator on the representation R,

$$\langle aa'|\hat{\mathcal{P}}_{10} + \hat{\mathcal{P}}_{10^*}|bb'\rangle = \frac{1}{2}(\delta^{ba}\delta^{b'a'} - \delta^{ba'}\delta^{b'a}) - \frac{1}{N_c} f^{bb'c} f^{aa'c},$$

and

$$A^{(1)}_{qq} = \chi^{*}_{A'\alpha'}\chi^{\alpha}_{A}\chi^{\beta}_{B}\chi^{*}_{B'\beta'}\delta^{\alpha'}_{\alpha}\delta^{\beta'}_{\beta}\frac{(N_c^2-4)(N_c^2-1)}{16N_c^3}A^{(eik)} \ ,$$

where χ^{α}_{A} is the quark A colour spinor.

The eikonal amplitude can be easily found:

$$A^{eik} = g^2 \frac{s}{t} \left(\frac{-4\pi^2}{3} \right) g^4 \, \vec{q}^{\,2} \, A_{\perp}^{(3)} \; ,$$

where $A_{\perp}^{(3)}$ the diagram

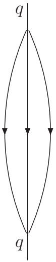

Figure 1: Diagrammatic representation of $A_{\perp}^{(3)}$ in the transverse momentum space

and is given by the integral

$$A_{\perp}^{(3)} = \int \frac{d^{2+2\epsilon} l_1 \, d^{2+2\epsilon} l_2}{(2\pi)^{2(3+2\epsilon)} \vec{l}_1^{\,2} \vec{l}_2^{\,2} (\vec{q} - \vec{l}_1 - \vec{l}_2)^2} =$$

$$= 3C_{\Gamma}^2 \frac{4}{\epsilon^2} \frac{(\vec{q}^{\,2})^{2\epsilon}}{\vec{q}^{\,2}} \frac{\Gamma^2(1+2\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)\Gamma(1+3\epsilon)} \, ,$$

where

$$C_{\Gamma} = \frac{\Gamma(1-\epsilon)\Gamma^2(1+\epsilon)}{(4\pi)^{2+\epsilon}\Gamma(1+2\epsilon)}$$

$$\frac{\Gamma^2(1+2\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)\Gamma(1+3\epsilon)} .$$

Note that we use the "infrared" ϵ, $\epsilon = (D-4)/2$, D is the space-time dimension.

The last term in $A_{ij}^{(8)}$ is not relevant; it is the contribution of positive signature in the quark-quark scattering. The second term does not violate the pole factorization and can be assigned to the Reggeized gluon contribution. This is not true for the first term, because

$$2C_{gq} \neq C_{qq} + C_{gg} \ , \ \ 2C_{gq} - C_{qq} - C_{gg} = -\frac{1}{4}\left(1 + \frac{1}{N_c^2}\right) \ ,$$

which means violation of the pole factorization. It is not difficult to see that the nonvanishing in the limit $\epsilon \to 0$ part of the amplitudes $C_{ij}A^{(eik)}$ coincides with $g^2(s/t)(\alpha_s/\pi)^2\mathcal{R}_i j^{(2),0,[8]}$ of the paper
V. Del Duca, G. Falcioni, L. Magnea and L. Vernazza, JHEP **1502** (2015) 029.
The values $\mathcal{R}_i j^{(2),0,[8]}$ are given there in Eq. (4.35), $(\alpha_s/\pi)^2\mathcal{R}_i j^{(2),0,[8]}$ is the first (two-loop, non-logarithmic) contributions to the "non-factorizing remainder function" $\mathcal{R}_{ij}^{[8]}$ introduced in Eq. (3.1) of this paper.

This means that the violation of the pole factorization is due to the eikonal part of the contribution of the diagrams with three-gluon exchange.
However, one can not affirm that this part is given entire by the three-Reggeon cut. Indeed, it can contain also the Reggeized gluon contribution. In fact, a non-factorizing remainder function is not uniquely defined.

Radiative corrections

The problem of separation of the pole and cut contributions can be solved by consideration of logarithmic radiative corrections to them. In the case of the Reggeized gluon contribution the correction comes solely from the Regge factor, so that the first order correction (more strictly, its relative value; this is assumed also in the following) is $\omega(t)\ln s$, where $\omega(t)$ is the gluon trajectory,

$$\omega(t) = -g^2 N_c \vec{q}^{\,2} \int \frac{d^{2+2\epsilon} l}{2(2\pi)^{(3+2\epsilon)} \vec{l}^{\,2} (\vec{q} - \vec{l})^2} = -g^2 N_c C_\Gamma \frac{2}{\epsilon} (\vec{q}^{\,2})^\epsilon \, .$$

In the case of the three-Reggeon cut, one has to take into account the Reggeization of each of three gluons and the interaction between them. The Reggeization gives $\ln s$ with the coefficient $3C_R$, where $A_{\perp}^{(3)} C_R$ is represented by the diagram

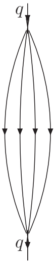

Figure 1: Diagrammatic representation of $A_{\perp}^{(3)} C_R$

in the transverse momentum space, and is given by the integral

$$A_\perp^{(3)} C_R = -g^2 N_c C_\Gamma \frac{2}{\epsilon} \int \frac{d^{2+2\epsilon} l_1 \, d^{2+2\epsilon} l_2}{(2\pi)^{2(3+2\epsilon)} \vec{l}_1^{\,2} \vec{l}_2^{\,2} (\vec{q} - \vec{l}_1 - \vec{l}_2)^{1-\epsilon}}$$

$$= -g^2 N_c C_\Gamma \frac{4}{3\epsilon} (\vec{q}^{\,2})^\epsilon \frac{\Gamma(1-3\epsilon)\Gamma(1+2\epsilon)\Gamma(1+3\epsilon)}{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)\Gamma(1+\epsilon)\Gamma(1+4\epsilon)} A_\perp^{(3)} \; .$$

Interaction between two Reggeons with transverse momenta $\vec{l}_1$ and $\vec{l}_2$ and colour indices c_1 and c_1 is defined by the real part of the BFKL kernel

$$\left[\mathcal{K}_r(\vec{q}_1, \vec{q}_2; \vec{k})\right]_{c_1 c_2}^{c_1' c_2'} = T^a_{c_1 c_1'} T^a_{c_2 c_2'} \frac{g^2}{(2\pi)^{D-1}} \left[\frac{\vec{q}_1^{\,2} \vec{q}_2^{\,\prime\, 2} + \vec{q}_2^{\,2} \vec{q}_1^{\,\prime\, 2}}{\vec{k}^{\,2}} - \vec{q}^{\,2} \right],$$

where $\vec{k}$ is the momentum transferred from one Reggeon to another in the interaction, $\vec{q}_1{}'$ and $\vec{q}_2{}'$ (c_1' and c_2') are the Reggeon momenta (colour indices) after the interaction, $\vec{q}_1{}' = \vec{q}_1 - \vec{k}$, $\vec{q}_2{}' = \vec{q}_2 + \vec{k}$, and $\vec{q} = \vec{q}_1 + \vec{q}_2 = \vec{q}_1{}' + \vec{q}_2{}'$.

Account of the interactions between all pairs of Reggeons leads in the sum to the colour coefficients which differ from the coefficients $C_{ij}^{(\mathcal{R})}$ only by the common factor $2N_c - C_{\mathcal{R}}$, where $C_{\mathcal{R}}$ is the value of the Casimir operator for the representation $\mathcal{R}$. It the octet case it can be easily obtained using its invariance of $\mathrm{Tr}(\mathcal{T}^{a_1}\mathcal{T}^{a_2}\mathcal{T}^{a_3}\mathcal{T}^{a_4})$ under the colour group transformation

$$\mathcal{T}^{a_i} \to e^{i\theta^c \mathcal{T}^c}\mathcal{T}^{a_i}e^{-i\theta^c \mathcal{T}^c} ,$$

which takes the form

$$\mathcal{T}^{a_i} \to \mathcal{T}^{a_i} - i\theta^c \hat{T}^c(i)\mathcal{T}^{a_i} , \quad \hat{T}^c(i)\mathcal{T}^{a_i} = T^c_{a_i a_i'}\mathcal{T}^{a_i'}$$

at small θ^c. It means that we can put

$$\hat{R}^c = \sum_i \hat{T}^c(i) = 0 ,$$

if $\hat{R}^c$ acts on $\mathrm{Tr}(\mathcal{T}^{a_1}\mathcal{T}^{a_2}\mathcal{T}^{a_3}\mathcal{T}^{a_4})$.

It gives

$$\sum_{i>j=2}^{4} \hat{T}^c(i)\hat{T}^c(j) = \frac{1}{2}\left(\sum_{i=2}^{4} \hat{T}^c(i)\hat{T}^c(i) - \hat{T}^c(1)\hat{T}^c(1)\right) .$$

Here on the left side we have the sum of the colours factors of the BFKL kernel for interactions between all pairs of Reggeons, and the right side is equal N_c.
Now about the kinematic part of the kernel. The first two terms in it correspond to the same diagrams, as for account of the Reggeization, and their total contribution to the coefficient of ln s in the first order correction is $-4C_R$. The last term corresponds to the diagram

Figure 1: Diagrammatic representation of $A_\perp^{(3)} C_3$

and its contribution to the coefficient of $\ln s$ in the first order correction is $-C_3$, where

$$C_3 = g^2 N_c C_\Gamma \frac{4}{\epsilon} \int \frac{d^{2+2\epsilon} l_1 \, d^{2+2\epsilon} l_2 (l_1 + l_2)^{2\epsilon}}{(2\pi)^{2(3+2\epsilon)} \vec{l}_1^{\,2} \vec{l}_2^{\,2} (\vec{q} - \vec{l}_1 - \vec{l}_2)^{1-2\epsilon}} \left(A_\perp^{(3)}\right)^{-1}$$

$$= g^2 N_c C_\Gamma \frac{32}{9\epsilon} (\vec{q}^{\,2})^\epsilon \frac{\Gamma(1-3\epsilon)\Gamma(1-\epsilon)\Gamma^2(1+3\epsilon)}{\Gamma^2(1-2\epsilon)\Gamma(1+2\epsilon)\Gamma(1+4\epsilon)} \, .$$

Therefore, the first order correction in the case of Reggeized gluon is $\omega(t) \ln s$, and in the case of the three-Reggeon cut is $(-C_R - C_3) \ln s$. If to present the coefficients C_{ij} as the sum

$$C_{ij} = C_{ij}^R + C_{ij}^C \; ,$$

where C_{ij}^R correspond to the pole, so that

$$2C_{gq}^R = C_{qq}^R + C_{gg}^R \; ,$$

and C_{ij}^C correspond to the cut, we obtain that with the logarithmic accuracy the total three-loop contributions to the coefficient of $\ln s$ are

Consistency with the infrared factorization

$$A^{eik}\left(C_{ij}^{R}\omega(t)-C_{ij}^{C}(C_R+C_3)\right)\ln s\ .$$

The infrared divergent part of these contributions must be compared with the functions $g^2(s/t)\mathcal{R}_{ij}^{(3),1,[8]}\ln s$ of the paper V. Del Duca, G. Falcioni, L. Magnea and L. Vernazza, JHEP **1502** (2015) 029.
The values $\mathcal{R}_ij^{(3),1,[8]}$ are given there in Eq. (4.59), $\mathcal{R}_ij^{(3),1,[8]}\ln s$ are the three-loop logarithmic contributions to the non-factorizing remainder function $\mathcal{R}_{ij}^{[8]}$. It is not difficult to see that with the accuracy with which the values $\mathcal{R}_ij^{(2),1,[8]}$ are known the equality

$$g^2(s/t)\mathcal{R}_{ij}^{(3),1,[8]}=A^{eik}\left(C_{ij}^{R}\omega(t)-C_{ij}^{C}(C_R+C_3)\right)$$

can be fulfilled if

$$C^C_{gg} = -\frac{3}{2}\,, \quad C^C_{gq} = -\frac{3}{2}\,, \quad C^C_{qq} = \frac{3(1-N_c^2)}{4N_c^2}\,,$$

and

$$aC^R_{gg} = 3\,, \quad C^R_{gq} = \frac{7}{4}\,, \quad C^R_{qq} = \frac{1}{2}\,.$$

It means that the restrictions imposed by the infrared factorization on the parton scattering amplitudes with the adjoint representation of the colour group in the t-channel and negative signature can be fulfilled in the NNLLA at two and three loops if besides the Regge pole contribution there is the Regge cut contribution

$$A^{eik} C^C_{ij} \left(1 - (C_R + C_3) \ln s\right)\,.$$

The BFKL equation was derived for summation of radiative corrections in the LLA to amplitudes of elastic scattering processes. These amplitudes were calculated using the s-channel unitarity and analyticity. The unitarity was used for calculation of discontinuities of elastic amplitudes, and analyticity for their full restoration. Use of the s-channel unitarity requires knowledge of multiple production amplitudes in the MRK. The assumption was made that all amplitudes in the unitarity relations for elastic amplitudes, both elastic and inelastic, are determined by the Regge pole contributions. With this assumption, the s-channel discontinuities of the elastic amplitudes can be presented as the convolution in the transverse momentum space of energy independent impact factors of colliding particles, describing their interaction with Reggeons, and the Green's function G for two interacting Reggeons, which is universal (process independent). The BFKL equation looks as

$$\frac{d\,G}{d\ln s} = \hat{\mathcal{K}}\,G\,,$$

where $\hat{\mathcal{K}}$ is the kernel of the BFKL equation. It consists of virtual and real parts; the first of them is expressed through the Regge trajectories and the second through effective vertices for s-channel production of particles in Reggeon interaction.
The assumption that all amplitudes in the unitarity relations are determined by the Regge pole contributions is very strong, and it should have been proven. The first check of this assumption was made already in 1975. Here it should be recalled that the BFKL equation is written for all t-channel colour states, which a system of two Reggeons can have, in particular, for the colour octet. Therefore, there is the bootstrap requirement: solution of the BFKL equation for the colour octet in the t-channel and negative signature must reproduce the pole Regge form, which was assumed in its derivation. It was shown that this requirement is satisfied.

Of course, it was not a proof of the assumption, but only verification in a very particular case. Later it was realized that it is possible to formulate the bootstrap conditions for amplitudes of multiple production in the MRK and to give a complete proof of the hypothesis on their basis. A similar (although much more complicated) proof for elastic amplitudes and for real parts of inelastic ones was carried out in the NLLA.

It turns out that all amplitudes in the unitarity relations are determined by the Regge pole contributions also in the NNLLA. The reason is that in this approximation one of two amplitudes in the unitarity relations can lose $\ln s$, while the second one must be taken in the LLA. The LLA amplitudes are real, so that only real parts of the NLLA amplitudes are important in the unitarity relations. Since they have a simple pole Regge form, the scheme of deviation of the BFKL equation in the NLLA remains unchanged.

The only difference is that we have to know the Reggeon trajectory and Reggeon-Reggeon-gluon production vertex with higher accuracy and to know also effective Reggeon-Reggeon$\rightarrow$ gluon-gluon and Reggeon-Reggeon$\rightarrow$ quark-antiquark vertices.

Unfortunately, this scheme is violated in the NNLLA. In this approximation two powers of $\ln s$ can be lost compared with the LLA in the product of two amplitudes in the unitarity relations. It can be done losing either one $\ln s$ in each of the amplitudes or $\ln^2 s$ in one of them. In the first case, discontinuities receive contributions from products of real parts of amplitudes with negative signature in the NLLA, products of imaginary parts of amplitudes with negative signature in the LLA, and products of amplitudes with positive signature in the LLA. Of course, account of these contributions greatly complicates derivation of the BFKL equation.

In particular, since for amplitudes with positive signature there are different colour group representations in the t-channel for quark-quark, quark-gluon and gluon-gluon scattering, their account violates unity of consideration.
However, these complications do not seem to be as great as in the second case, when $\ln^2 s$ is lost in one of the amplitudes in the unitarity relations. In this case one of the amplitudes must be taken in the NNLLA and the other in the LLA. Since the amplitudes in the LLA are real, only real parts of the NNLLA amplitudes are important. But even for these parts the pole Regge form becomes inapplicable because of the contributions of the three-Reggeon cuts which appear in this approximation. Note that account of these contributions also violates unity of consideration of quark-quark, quark-gluon and gluon-gluon scattering because the cuts give contributions to amplitudes with different representations of the colour group in the t-channel for these processes.

But even worse is that we actually do not know the contributions of the cuts.

We showed that the non-factorizing terms in the parton's elastic scattering amplitudes with the colour octet in the *t*-channel and negative signature calculated using the infrared factorization in two and three loops (non-logarithmic terms nonvanishing at $\epsilon \to 0$ in two loops and one-logarithmic terms singular at $\epsilon \to 0$ in three loops) can be explained by the contribution of the three-Reggeon cut. But these terms may have other explanations. In particular, in for their explanation, along with the three-Reggeon cut, mixing of the pole and the cut is used. In these paper, the coupling of the cut with partons and the Reggeon-cut mixing were introduced using effective theory of Wilson lines.

If we compare our current understanding of the cuts with the history of investigation of the gluon Reggeization, it seems that we are even in the worse position than after

L. N. Lipatov, Sov. J. Nucl. Phys. **23** (1976) 338 [Yad. Fiz. **23** (1976) 642], because the Regge form of elastic amplitudes was confirmed there by direct calculation in two loops with power accuracy. Remind that to prove this form in all orders of perturbation theory it was firstly generalized for multiple production amplitudes in the multi-Regge kinematics, then the bootstrap conditions for elastic and inelastic amplitudes were derived, after which it was proved that their fulfilment is sufficient for justification of the form, and finally it was shown that they are fulfilled.

Summary

- The gluon Reggeization is one of the remarkable features of QCD.
- In the LLA and in the NLLA the Reggeization provides a simple factorized form of QCD amplitudes.
- This form is violated in the NNLLA by the 3-Reggeon cut
- The cuts strongly complicate derivation of the BFKL equation.
- Investigation of the cuts is only in the beginning.