# Subleading corrections to the BFKL equation: singlet versus color adjoint state

Alex Prygarin
Ariel University, Israel

July 2018, ICNFP, Crete

Dedicated to the memory of Lev Lipatov
1940-2017

Photo courtesy of M. Shifman

# Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation

Historic remark: I was one-year-old when the BFKL equation was derived.

Alphabetic order: Would be "BKLF" if written in Russian and Greek.

Would be "BLKF" if written in Hebrew.

All letters commute, BUT the letter "L" commutes the most.

$$\frac{\partial}{\partial \log s}\Psi = H \bigotimes \Psi$$

# Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation

Some "good" facts about BFKL

- Bound state of two or more reggeized gluons
- Originally derived for representing Pomeron in QCD
- Derived from first principles: analyticity, unitarity and crossing symmetry
- Complies with experiment
- Introduces integrability to high energy physics
- Arbitrary color configuration

Some "not-so-good" facts about BFKL

- Unitarity issues
- Very complicated beyond leading order
- Internal structure not clear (Schroedinger, Bethe-Salpeter or any other)
- Only few people really understand what's going on

# Perturbation Theory

BFKL re-sums

powers of $\alpha_s^n \log^n s$ in leading order (LO)
powers of $\alpha_s^n \log^{n-1} s$ in next-to-leading order (NLO)
...
powers of $\alpha_s^n \log^{n-k} s$ in $(\text{next})^k$-to-leading order ($\text{N}^k$ LO)

This multiplies some function of transverse momenta, which gives "BFKL Hamiltonian"

$$\frac{\partial}{\partial \log s}\Phi = H \bigotimes \Phi$$

BFKL is essentially the eigenvalue and eigenfunction problem

$$H \bigotimes \Phi = \omega \Phi$$

# Conformal properties at LO (singlet)

*Eigenvalue* and *Eigenfunction* problem

$$H \bigotimes \Phi = \omega \Phi$$

In the LO color singlet case $H$ is invariant under conformal transformation in coordinate space $\vec{r} = (x, y)$

$$z = x + iy, \qquad z \to \frac{az + b}{cz + d}, \qquad ad - bc \neq 0$$

Eigenfunction

$$E_{n,\nu}(\vec{r}_{10}, \vec{r}_{20}) = (-1)^n \left( \frac{z_{12}}{z_{10} z_{20}} \right)^{\frac{1}{2} + i\nu - \frac{n}{2}} \left( \frac{z_{12}^*}{z_{10}^* z_{20}^*} \right)^{\frac{1}{2} + i\nu + \frac{n}{2}}$$

Eigenvalue

$$\omega(n, \nu) = 4a \left[ 2\Psi(1) - \Psi\left( \frac{1}{2} + i\nu + \frac{|n|}{2} \right) - \Psi\left( \frac{1}{2} - i\nu + \frac{|n|}{2} \right) \right]$$

For zero momentum transfer the eigenfunction is much simpler, but the eigenvalue is the same.

# Conformal properties at LO (adjoint)

*Eigenvalue* and *Eigenfunction* problem

$$H_{adj} \bigotimes \Phi_{adj} = \omega_{adj} \Phi_{adj}$$

In the LO color adjoint case $H$ is invariant under conformal transformation in momentum space $\vec{k} = (k_x, k_y)$

$$z = k_x + ik_y, \quad z \to \frac{az+b}{cz+d}, \quad ad - bc \neq 0$$

LO eigenfunction

$$f_{n,\nu}(\vec{k}_1, \vec{k}_2, \vec{q}_1, \vec{q}_3) = (-1)^n \left( \frac{z_{02} z_{0'1}}{z_{0'2} z_{01}} \right)^{i\nu - \frac{n}{2}} \left( \frac{z^*_{02} z^*_{0'1}}{z^*_{0'2} z^*_{01}} \right)^{i\nu + \frac{n}{2}}$$

Dual coordinates

$$q_1 \to z_{01}, \; k_1 \to z_{0'1}, q_3 \to z_{02}, k_2 \to z_{20'}$$

LO eigenvalue

$$\omega(n,\nu)_{adj} = 4a\mathcal{R}e \left[ 2\Psi(1) - \Psi\left(1 + i\nu + \frac{|n|}{2}\right) - \Psi\left(1 - i\nu + \frac{|n|}{2}\right) \right]$$

# Singlet vs Adjoint BFKL

- Kernel is IR finite (singlet) and IR divergent (adjoint)
- Corresponds to closed spin chain (singlet) and open spin chain (adjoint)
- Hamiltonian has conformal symmetry in coordinate space (singlet) and momentum space (adjoint)
- Different normalization of eigenfunctions in (singlet) and (adjoint) cases
- Different argument of functions building eigenvalue $\frac{1}{2} + i\nu + \frac{n}{2}$ (singlet) and $1 + i\nu + \frac{n}{2}$ (adjoint)
- Different space of functions building eigenvalue (see on the next slide)

# Space of functions

- At LO level same function for singlet and adjoint, i.e. digamma $\psi(z) = \frac{d \log \Gamma(z)}{dz}$
- At all orders of adjoint BFKL eigenvalue in planar SYM $N = 4$ enter only polygamma (derivatives of digamma)
- Already at NLO singlet BFKL eigenvalue enter more complicated functions, double sign alternating sums
- At NNLO singlet BFKL eigenvalue is calculated only for definite conformal spins $n = 0, 1, 2, ..$
  No closed expression is found, the space of function is not known.

# Singlet vs Adjoint BFKL

- Kernel is IR finite (singlet) and IR divergent (adjoint)
- Corresponds to closed spin chain (singlet) and open spin chain (adjoint)
- Hamiltonian has conformal symmetry in coordinate space (singlet) and momentum space (adjoint)
- Different normalization of eigenfunctions in (singlet) and (adjoint) cases
- Different argument of functions building eigenvalue $\frac{1}{2} + i\nu + \frac{n}{2}$ (singlet) and $1 + i\nu + \frac{n}{2}$ (adjoint)
- Different space of functions building eigenvalue (see on the next slide)

# Harmonic sums

The principle of maximal transcendentality predicts transcendentality of functions at each order of the perturbations theory in SYM $N = 4$.

Nested harmonic sums

$$S_{a,b,c..}(n) = \sum_{k=1}^{n} \frac{sign(a)^k}{k^{|a|}} \sum_{l=1}^{k} \frac{sign(b)^l}{l^{|b|}} \sum_{j=1}^{l} \frac{sign(c)^j}{j^{|c|}} \ldots$$

are the natural candidate for a good basis.

Nested harmonic sums have two main problems

- they form overcompleted basis at each weight (i.e. transcendentality )
- they need an analytic continuation to the complex plane

Both of the problems are overcomed now using available Mathematica packages

# Holomorphic separability at LO

Both singlet and adjoint have similar structure at LO. Holomorphic

coordinates
singlet

$$z = \frac{1}{2} + i\nu + \frac{n}{2}, z^* = \frac{1}{2} - i\nu + \frac{n}{2}$$

adjoint

$$z = 1 + i\nu + \frac{n}{2}, z^* = 1 - i\nu + \frac{n}{2}$$

At LO both singlet and adjoint eigenvalue can be written as

$$\omega^{LO} = f(z) + f(z^*)$$

# Holomorphic separability at NLO

At NLO holomorphic separability has form of product of two holomorphic function.
At NLO both singlet and adjoint eigenvalue can be written as

$$\omega^{NLO} = \{f(z) + f(z^*)\}\{\rho(z) + \rho(z^*)\} + g(z) + g(z^*)$$

where $\omega = a\,\omega^{LO} + a^2\omega^{NLO} + \ldots$ Which suggests that the BFKL

equation is actually the Bethe-Salpeter equation

$$1 = \frac{a(\bar{f} + a\,\bar{g})}{\omega} + a\,\bar{\rho}$$

where $\bar{f} = f(z) + f(z^*),\ \bar{g} = g(z) + g(z^*),\ \bar{\rho} = \rho(z) + \rho(z^*)$

# Holomorphic separability at NNLO

Adjoint BFKL eigenvalue in planar SYM $N = 4$ satisfies the generalized Bethe-Salpeter form.

Singlet BFKL eigenvalue in planar SYM $N = 4$ known only for specific $n$ and presented in form of

$$\omega^{NNLO} = F(z) + F(z^*), \quad \text{for } z = \frac{1}{2} + i\nu + \frac{n}{2}, z^* = \frac{1}{2} - i\nu + \frac{n}{2}$$

Note: for $n = 0$ we have $z^* = 1 - z$ and then product of two holomorphic functions can be written as linear combination of more complicated holomorphic functions

$$S_1(z)S_1(1-z) = \frac{\pi^2}{3} + S_{1,1}(z) + S_{1,1}(1-z)$$

related to digamma after analytic continuation by

$$S_1(z) = \psi(1+z) - \psi(1)$$

# Interrelation between singlet and adjoint BFKL

Two major observations:

- In adjoint BFKL eigenvalue in SYM $N = 4$ only harmonic sums with positive index appear, like $S_{2,1}(z)$ but those are much easier reducible due to identities between harmonic sums.
- In singlet BFKL eigenvalue in SYM $N = 4$ only harmonic sums with at most one negative index appear, like $S_{-2,1}(z)$ and not $S_{-2,-1}(z)$

Which imply

- In adjoint BFKL eigenvalue in SYM $N = 4$ only simple functions appear due to multiple cancellations
- Only one negative index suggests that singlet and adjoint BFKL are built as a linear combination of the same operators (left-right movers) with positive (adjoint) and negative(singlet) relative sign between them.

# Interrelation between singlet and adjoint BFKL

Only one negative index suggests that singlet and adjoint BFKL are built as a linear combination of the same operators (left-right movers) with positive (adjoint) and negative(singlet) relative sign between them.

The arguments of functions corresponding to those operators are half of the conventional arguments with some shifts by $\frac{1}{2}$

Some examples

$$S_{+2}(z) = \frac{1}{4} S_2\left(\frac{z}{2} - \frac{1}{2}\right) + \frac{1}{4} S_2\left(\frac{z}{2}\right) + \text{some const}$$

$$S_{-2}(z) = \frac{1}{4} S_2\left(\frac{z}{2} - \frac{1}{2}\right) - \frac{1}{4} S_2\left(\frac{z}{2}\right) + \text{some const}$$

# Conclusions and Discussions

- Singlet and adjoint BFKL both correspond to closed and open spin chains
- Singlet and adjoint BFKL both have conformal symmetry in different spaces(coordinate and momentum)
- Eigenfunctions and eigenvalues are very similar
- Might be represented by plus/minus linear combination of the same operators
- Can be written in more transparent form of Bethe-Salpeter equation
- Eigenvalues can be written using only Nested Harmonic Sums, no need in more complicated functions

Dedicated to the memory of Lev Lipatov
1940-2017

Photo courtesy of M. Shifman