Subleading corrections to the BFKL equation: singlet versus color adjoint state

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Dedicated to the memory of Lev Lipatov
1940-2017

Photo courtesy of M. Shifman
Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation

Historic remark: I was one-year-old when the BFKL equation was derived.

Alphabetic order: Would be ”BKLF” if written in Russian and Greek.

Would be ”BLKF” if written in Hebrew.

All letters commute, BUT the letter ”L” commutes the most.

\[
\frac{\partial}{\partial \log s} \psi = H \otimes \psi
\]
Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation

Some ”good” facts about BFKL

- Bound state of two or more reggeized gluons
- Originally derived for representing Pomeron in QCD
- Derived from first principles: analyticity, unitarity and crossing symmetry
- Complies with experiment
- Introduces integrability to high energy physics
- Arbitrary color configuration

Some ”not-so-good” facts about BFKL

- Unitarity issues
- Very complicated beyond leading order
- Internal structure not clear (Schroedinger, Bethe-Salpeter or any other)
- Only few people really understand what’s going on
Perturbation Theory

BFKL re-sums
powers of $\alpha^s_n \log^n s$ in leading order (LO)
powers of $\alpha^s_n \log^{n-1} s$ in next-to-leading order (NLO)
...
powers of $\alpha^s_n \log^{n-k} s$ in $(next)^k$-to-leading order ($N^k$ LO)

This multiplies some function of transverse momenta, which gives "BFKL Hamiltonian"

\[ \frac{\partial}{\partial \log s} \Phi = H \bigotimes \Phi \]

BFKL is essentially the eigenvalue and eigenfunction problem

\[ H \bigotimes \Phi = \omega \Phi \]
Conformal properties at LO (singlet)

**Eigenvalue and Eigenfunction** problem

\[ H \otimes \Phi = \omega \Phi \]

In the LO color singlet case \( H \) is invariant under conformal transformation in coordinate space \( \vec{r} = (x, y) \)

\[ z = x + iy, \quad z \rightarrow \frac{az + b}{cz + d}, \quad ad - bc \neq 0 \]

**Eigenfunction**

\[ E_{n, \nu}(\vec{r}_1, \vec{r}_2) = (-1)^n \left( \frac{z_{12}}{z_{10}z_{20}} \right)^{\frac{1}{2} + i\nu - \frac{n}{2}} \left( \frac{z_{12}^*}{z_{10}^*z_{20}^*} \right)^{\frac{1}{2} + i\nu + \frac{n}{2}} \]

**Eigenvalue**

\[ \omega(n, \nu) = 4a \left[ 2\Psi(1) - \Psi \left( \frac{1}{2} + i\nu + \frac{|n|}{2} \right) - \Psi \left( \frac{1}{2} - i\nu + \frac{|n|}{2} \right) \right] \]

For zero momentum transfer the eigenfunction is much simpler, but the eigenvalue is the same.
Conformal properties at LO (adjoint)

Eigenvalue and Eigenfunction problem

\[ H_{\text{adj}} \bigotimes \Phi_{\text{adj}} = \omega_{\text{adj}} \Phi_{\text{adj}} \]

In the LO color adjoint case \( H \) is invariant under conformal transformation in momentum space \( \vec{k} = (k_x, k_y) \)

\[ z = k_x + i k_y, \quad z \rightarrow \frac{a z + b}{c z + d}, \quad ad - bc \neq 0 \]

LO eigenfunction

\[ f_{n,\nu}(\vec{k}_1, \vec{k}_2, \vec{q}_1, \vec{q}_3) = (-1)^n \left( \frac{Z_0 Z_0' 1}{Z_0' 2 Z_0 1} \right)^{i \nu - \frac{n}{2}} \left( \frac{Z_0^* Z_0'^* 1}{Z_0'^* 2 Z_0^* 1} \right)^{i \nu + \frac{n}{2}} \]

Dual coordinates

\[ q_1 \rightarrow z_0 1, \; k_1 \rightarrow z_0' 1, \; q_3 \rightarrow z_0 2, \; k_2 \rightarrow z_2 0' \]

LO eigenvalue

\[ \omega(n, \nu)_{\text{adj}} = 4 a \text{Re} \left[ 2 \Psi(1) - \Psi \left( 1 + i \nu + \frac{|n|}{2} \right) - \Psi \left( 1 - i \nu + \frac{|n|}{2} \right) \right] \]
Singlet vs Adjoint BFKL

- Kernel is IR finite (singlet) and IR divergent (adjoint)
- Corresponds to closed spin chain (singlet) and open spin chain (adjoint)
- Hamiltonian has conformal symmetry in coordinate space (singlet) and momentum space (adjoint)
- Different normalization of eigenfunctions in (singlet) and (adjoint) cases
- Different argument of functions building eigenvalue $\frac{1}{2} + i\nu + \frac{n}{2}$ (singlet) and $1 + i\nu + \frac{n}{2}$ (adjoint)
- Different space of functions building eigenvalue (see on the next slide)
Space of functions

- At LO level **same function** for singlet and adjoint, i.e. digamma $\psi(z) = \frac{d \log \Gamma(z)}{dz}$

- At all orders of **adjoint** BFKL eigenvalue in planar SYM $N = 4$ enter **only polygamma** (derivatives of digamma)

- Already at NLO **singlet** BFKL eigenvalue enter more **complicated functions**, double sign alternating sums

- At NNLO **singlet** BFKL eigenvalue is calculated only for **definite** conformal spins $n = 0, 1, 2, ..$
  No closed expression is found, the **space of function** is not known.
Singlet vs Adjoint BFKL

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Harmonic sums

The principle of maximal transcendentality predicts transcendentality of functions at each order of the perturbations theory in SYM $N = 4$.

Nested harmonic sums

$$S_{a,b,c...}(n) = \sum_{k=1}^{n} \frac{\text{sign}(a)^k}{k|a|} \sum_{l=1}^{k} \frac{\text{sign}(b)^l}{l|b|} \sum_{j=1}^{l} \frac{\text{sign}(c)^j}{j|c|} \ldots$$

are the natural candidate for a good basis.

Nested harmonic sums have two main problems

- they form overcompleted basis at each weight (i.e. transcendentality )
- they need an analytic continuation to the complex plane

Both of the problems are overcome now using available Mathematica packages
Holomorphic separability at LO

Both singlet and adjoint have similar structure at LO. Holomorphic coordinates

singlet

\[ z = \frac{1}{2} + i\nu + \frac{n}{2}, \quad z^* = \frac{1}{2} - i\nu + \frac{n}{2} \]

adjoint

\[ z = 1 + i\nu + \frac{n}{2}, \quad z^* = 1 - i\nu + \frac{n}{2} \]

At LO both singlet and adjoint eigenvalue can be written as

\[ \omega^{LO} = f(z) + f(z^*) \]
At NLO holomorphic separability has form of product of two holomorphic function.

At NLO both singlet and adjoint eigenvalue can be written as

$$\omega^{NLO} = \{ f(z) + f(z^*) \} \{ \rho(z) + \rho(z^*) \} + g(z) + g(z^*)$$

where

$$\omega = a \omega^{LO} + a^2 \omega^{NLO} + \ldots$$

Which suggests that the BFKL equation is actually the Bethe-Salpeter equation

$$1 = \frac{a(\bar{f} + a \bar{g})}{\omega} + a \bar{\rho}$$

where

$$\bar{f} = f(z) + f(z^*), \quad \bar{g} = g(z) + g(z^*), \quad \bar{\rho} = \rho(z) + \rho(z^*)$$
Holomorphic separability at NNLO

**Adjoint** BFKL eigenvalue in planar SYM $N = 4$ satisfies the generalized Bethe-Salpeter form.

**Singlet** BFKL eigenvalue in planar SYM $N = 4$ known only for specific $n$ and presented in form of

$$\omega^{NNLO} = F(z) + F(z^*), \quad \text{for } z = \frac{1}{2} + i\nu + \frac{n}{2}, z^* = \frac{1}{2} - i\nu + \frac{n}{2}$$

Note: for $n = 0$ we have $z^* = 1 - z$ and then product of two holomorphic functions can be written as linear combination of more complicated holomorphic functions

$$S_1(z)S_1(1 - z) = \frac{\pi^2}{3} + S_{1,1}(z) + S_{1,1}(1 - z)$$

related to digamma after analytic continuation by

$$S_1(z) = \psi(1 + z) - \psi(1)$$
Interrelation between singlet and adjoint BFKL

Two major observations:

- In adjoint BFKL eigenvalue in SYM $N = 4$ only harmonic sums with positive index appear, like $S_{2,1}(z)$ but those are much easier reducible due to identities between harmonic sums.

- In singlet BFKL eigenvalue in SYM $N = 4$ only harmonic sums with at most one negative index appear, like $S_{-2,1}(z)$ and not $S_{-2,-1}(z)$

Which imply

- In adjoint BFKL eigenvalue in SYM $N = 4$ only simple functions appear due to multiple cancellations.

- Only one negative index suggests that singlet and adjoint BFKL are built as a linear combination of the same operators (left-right movers) with positive (adjoint) and negative(singlet) relative sign between them.
Interrelation between singlet and adjoint BFKL

Only one negative index suggests that singlet and adjoint BFKL are built as a linear combination of the same operators (left-right movers) with positive (adjoint) and negative (singlet) relative sign between them.

The arguments of functions corresponding to those operators are half of the conventional arguments with some shifts by $\frac{1}{2}$.

Some examples

$$S_{+2}(z) = \frac{1}{4} S_2 \left( \frac{z}{2} - \frac{1}{2} \right) + \frac{1}{4} S_2 \left( \frac{z}{2} \right) + \text{some const}$$

$$S_{-2}(z) = \frac{1}{4} S_2 \left( \frac{z}{2} - \frac{1}{2} \right) - \frac{1}{4} S_2 \left( \frac{z}{2} \right) + \text{some const}$$

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Conclusions and Discussions

- Singlet and adjoint BFKL both correspond to closed and open spin chains.
- Singlet and adjoint BFKL both have conformal symmetry in different spaces (coordinate and momentum).
- Eigenfunctions and eigenvalues are very similar.
- Might be represented by plus/minus linear combination of the same operators.
- Can be written in more transparent form of Bethe-Salpeter equation.
- Eigenvalues can be written using only Nested Harmonic Sums, no need in more complicated functions.
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