Next-to-leading order npi calculations

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Outline

Introduction

strong coupling $\rightarrow$ can't use perturbation theory

different approaches (for example):

• lattice calculations

$\rightarrow$ *continuum and infinite volume limits*

• continuum methods

- Schwinger-Dyson equations
- renormalization group (RG)
- *n*-particle irreducible (*n*pi) effective theories

Introduction to npi

2pi for scalar theories:

generating functional with local and bi-local sources

$$Z[J,B] = e^{iW[J,B]} = \int \mathcal{D}\varphi e^{i(S[\varphi]+J_i\varphi_i+\frac{1}{2}\varphi_i B_{ij}\varphi_j)}$$

short-hand notation:

$$\int dx \int dy\; \varphi(x)B(x,y)\varphi(y) \to \varphi_i B_{ij}\varphi_j \to B\varphi^2$$

Legendre transform:

$$\Gamma[\phi,G] = W[J,B] - J_i\phi_i - \frac{1}{2}B_{ij}\phi_i\phi_j$$

$$= S_{\rm cl}[\phi] + \frac{i}{2}\mathrm{Tr}\ln G^{-1} + \frac{i}{2}\mathrm{Tr}\, G_0^{-1}(G-G_0) + \Gamma_2[\phi,G]$$

$\Gamma[\phi,G]$ is a functional of the 1- and 2-point functions

ϕ and G determined self-consistently from equations of motion

variational principle (in the absence of sources)

$$\frac{\delta\Gamma}{\delta\phi} = \frac{\delta\Gamma}{\delta G} = 0$$

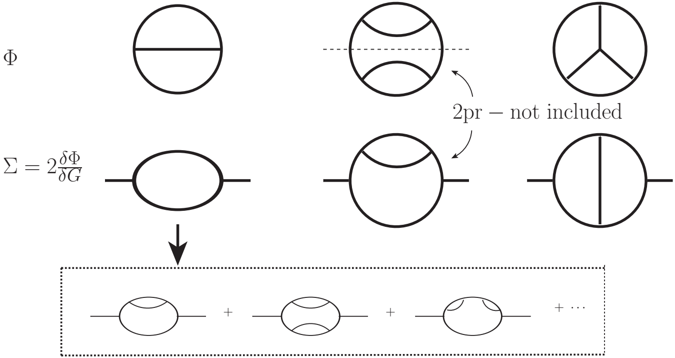
compare to $\Gamma[\phi]$ = 1pi effective action:

• $\Gamma[\phi, G]$ depends on the self consistent propagator

$\to$ truncated $\Gamma[\phi, G]$ includes an infinite resummation of diagrams

$\to$ non-perturbative

• $\Gamma[\phi, G]$ is 2pi - no double counting



npi effective action

npi Γ is a functional of n-point functions

3pi $\Gamma[\phi, G, U]$, 4pi $\Gamma[\phi, G, U, V] \cdots$

n-point functions determined self-consistently from the eom's

$\Rightarrow$ hierarchy of coupled equations

- no exact solution method is available
- use approximation techniques: truncate the effective action

Key features:

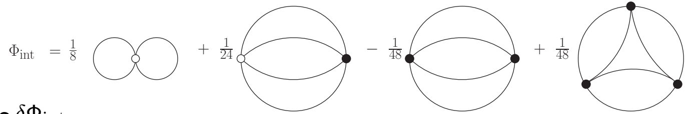
- non-perturbative
  infinite resummations of selected classes of diagrams
- action based approximation
  $\rightarrow$ symmetries of original theory
- renormalizable ?
  counterterm renormalization at 2pi level is understood

  - can't apply same method to higher order approximations

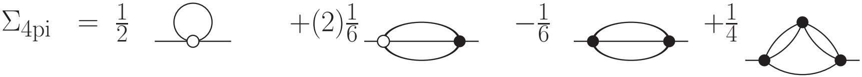
  $\rightarrow$ new method based on renormalization group (RG)

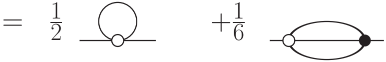
examples of variational eom's

$\Phi_{\rm int} = i\Gamma_{\rm int}$ 4-loop 4pi (symmetric)



$\Sigma = 2\frac{\delta\Phi_{\rm int}}{\delta G}$



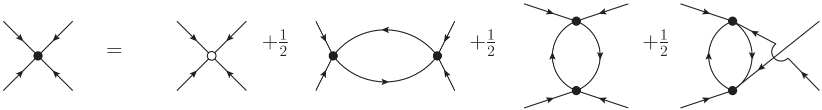


MEC and Yun Guo, PRD 83, 016006 (2010); PRD 85, 076008 (2012).

npi 4-vertices

Variational 4-vertex (4-loop 4pi)

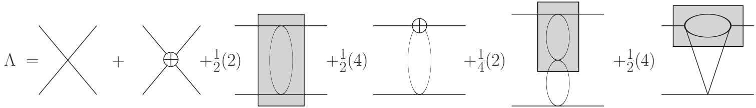
$\frac{\delta \Phi_{\text{int}}}{\delta V} = 0$



2pi also provides a non-perturbative 4-vertex $\cdots$

Npi renormalization – 4 dimensions

4-loop 2pi 4-kernel $\Lambda = 4\frac{\delta^2\Phi_{\text{int}}}{\delta G^2}$



counterterms appear in positions to cancel 1-loop divergences

- but there is no one $\delta\lambda_1$ that works

this is typical of npi theories - combinatorics are different

Resolution for 2pi

need 2 ct's . . .

1) they both come from the action

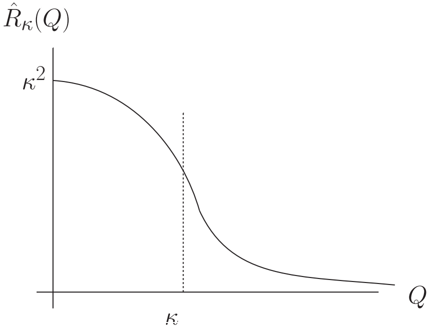
2) at $L \to \infty$ loops they are equal

H. van Hees, J. Knoll, Phys. Rev. **D65**, *025010 (2002);*
J-P Blaizot, E. Iancu, U. Reinosa, Nucl. Phys. **A736**, *149 (2004);*
J. Berges, Sz. Borsányi, U. Reinosa, J. Serreau, Annals Phys. **320**, *344 (2005).*

BUT unknown how to use counterterms beyond the 2pi level
must develop another method to renormalize

Renormalization group method

add to the action a non-local regulator term $\Delta S_\kappa[\varphi] = -\frac{1}{2} R_\kappa \varphi^2$



$$R_\kappa = \frac{Q^2}{e^{Q^2/\kappa^2} - 1}$$

$R_\kappa(Q) \sim \kappa^2$ for $Q \ll \kappa$

fluctuations $Q \ll \kappa$ suppressed

$R_\kappa(Q) \to 0$ for $Q \geq \kappa$

fluctuations $Q \gg \kappa$ unaffected

family of theories indexed by the continuous parameter κ

fluctuations are smoothly taken into account as κ is lowered to zero

$\kappa \to \infty$ regulated action $\to$ classical action

$\kappa \to 0$ regulated action $\to$ full quantum action

J.-P. Blaizot, A. Ipp, N. Wschebor, Nucl. Phys. A **849**, *165 (2011)*
J.-P. Blaizot, J.M. Pawlowski and U. Reinosa, Phys. Lett. B **696**, *523 (2011)*

will see in a minute how to use this $\cdots$

generating functionals defined in the usual way

$$Z_\kappa[J,B] = \int [d\varphi] \exp\left\{ i\Big(S[\varphi] - \frac{1}{2}\hat{R}_\kappa\varphi^2 + J\varphi + \frac{1}{2}B\varphi^2 + \cdots \Big)\right\}$$

calculate 1pi, 2pi, $\cdots$ effective action

action depends on κ: Φ_κ

action flow eqn: $$\partial_\kappa \Phi_\kappa = \frac{1}{2}\partial_\kappa R_\kappa\, G$$

C. Wetterich, Phys. Lett., B 301, 90 (1993).

Hierarchies of flow equations

definitions of kernels: $\Phi^{(nm)}_{\mathrm{int}\cdot\kappa} = 4!^n 2^m G^{-4n} \frac{\delta^{n+m}\Phi_{\mathrm{int}}}{\delta V^n \delta G^m}\Big|_{\substack{V=V_\kappa \\ G=G_\kappa \\ \phi=o}}$

example: $n=0$ and $m=1$ $\rightarrow$ $\Phi^{(01)}_{\mathrm{int}\cdot\kappa} = \Sigma_\kappa$

functional derivatives of action flow eqn:

$$\partial_\kappa \Phi^{(nm)}_{\mathrm{int}\cdot\kappa}\Big|_{\substack{G=G_\kappa \\ \phi=o}} = \frac{1}{2}\int dQ\, \partial_\kappa\,(R_\kappa + \Sigma_\kappa)\, G^2_\kappa(Q)\, \Phi^{(n,m+1)}_{\mathrm{int}\cdot\kappa}(Q,\quad) + \frac{1}{4!}\int dQ_i\, \partial_\kappa\, V_\kappa\, G^4_\kappa(Q_i)\, \Phi^{(n+1,m)}_{\mathrm{int}\cdot\kappa}(Q_i,\quad)$$

$\Rightarrow$ infinite hierarchy of coupled flow eqns for the n-point kernels

TRUNCATION: flow equations truncate when action is truncated
more in a minute $\cdots$

Method

so far:

we have a hierarchy of differential flow eqns for κ dependent n-point fcns

role of kappa:

$\kappa \to \infty$ regulated action $\to$ classical action

$\kappa \to 0$ regulated action $\to$ full quantum action

$\to$ method to solve flow equations:

1. choose an uv scale $\kappa = \mu$ (defn of bare parameters)

theory is classical at this scale (all fluctuations suppressed)

$\to$ n-point functions are known functions of the bare parameters

2. solve differential flow equations starting from bc's at $\kappa = \mu$

$\to$ obtain the n-point fcns at $\kappa = 0$ (the quantum solutions)

Technicalities

KEY:

bc's chosen at $\kappa = \mu$ $\leftarrow$ classical scale where theory is simple

rc's are imposed at $\kappa = 0$ $\leftarrow$ this is the full quantum theory

3 Issues:

1. Tuning: definition of physical parameters ($\kappa = 0$)
$\rightsquigarrow$ constrains initial conditions on the flow equations ($\kappa = \mu$)

2. Consistency: can we satisfy both the bc's and the rc's?

flow equations $\rightarrow$ vertex functions up to κ independent constant

$\rightarrow$ can always satisfy bc with an appropriate choice of this constant

the rc's are satisfied if

$$\lim_{P_i \to 0} \left[\Lambda_0(P_1, P_2 \cdots P_m) - \Lambda_0(0, 0 \cdots 0) \right] = \text{constant}$$

looks obvious . . .

key: sub-divergences could give something ill defined like $\infty \times 0$

3. Truncation:

it is obvious that hierarchy of flow eqns truncates with action

but actually: can truncate as soon as we find a kernel that satisfies

$\lim_{P_i \to 0} \left[\Lambda_0(P_1, P_2 \cdots P_m) - \Lambda_0(0, 0 \cdots 0)\right] = \mathrm{constant}$

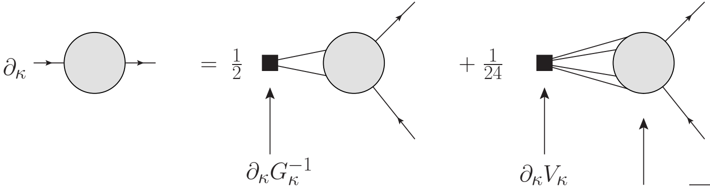
$\Rightarrow$ *no sub-divergence in quantum n-point function*

KEY to truncation:

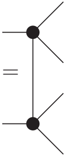
kernel with a sub-divergence must be obtained from its flow eqn

kernel without a sub-divergence doesn't have to be flowed
- substitute directly into previous flow equation

example flow equation



$\Phi^{11} =$

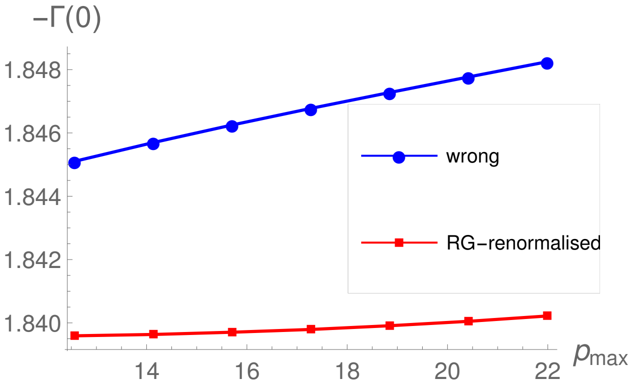


summary:

4 loop 2pi $\rightarrow$ 2 coupled flow eqns for $\Phi^{01} = \Sigma$, $\Phi^{02} = \Lambda$

4 loop 4pi $\rightarrow$ 3 coupled flow eqns for $\Phi^{01} = \Sigma$, $\Phi^{02} = \Lambda$, $\Phi^{10} = V$

Results – 4-loop 2pi arXiv:1711.09135



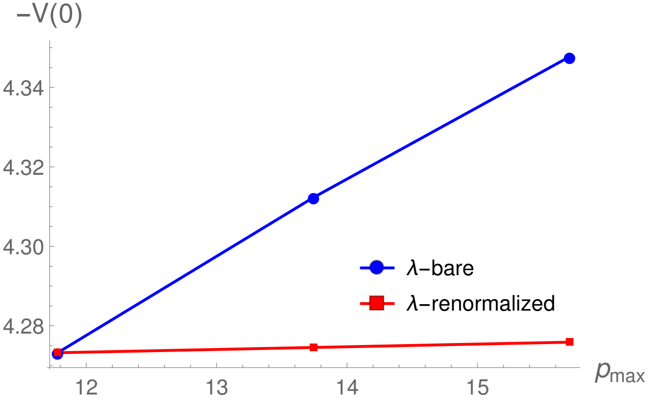
$\lambda = T = 2$

Results – 4-loop 4pi preliminary

the 4pi calculation is technically much more difficult

1. memory constraints $\rightarrow$ spherical coordinates
   must calculate 13 loops (some angles are "free")
   4 matsubara frequencies
   5 angles (very weak dependence)
   4 momentum magnitudes
2. use symmetries (for example under leg permutations)
3. must store a 9 dimensional array for the variational 4 vertex

preliminary results



$\lambda = 4, \; T = 2$ (bare data shifted)

Conclusions

- 2pi can be renormalized with counterterms

at ≥ 4 loop require two counterterms: $\delta\lambda$ and $\delta\lambda'$
can't be generalized to higher order theories

- functional renormalization group regulator $\Rightarrow$ λ_b

all divergences are absorbed into one bare coupling which is introduced at the level of the lagrangian
agrees with counterterm renormalization for the 2pi calculation
method generalizes to higher order nPI

further 4pi numerical calculations are in progress