Quantum Diffusion during Inflation and Primordial Black Holes

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Outline

• Quantum State of Cosmological Perturbations
• The Stochastic-δN Inflation Formalism
• Primordial Black Holes

Based on:
- VV and A. Starobinsky, 1506.04732 (EPC)
- H. Assadullahi, H. Firouzjahi, M. Noorbala, VV, D. Wands, 1604.04502 (JCAP)
- VV, H. Assadullahi, H. Firouzjahi, M. Noorbala, D. Wands, 1604.06017 (PRL)
- C. Pattison, VV, H. Assadullah, D. Wands, 1705.05746
Cosmological Perturbations in Inflation

- Inflation is a high energy phase of accelerated expansion in the early Universe

\[ ds^2 = -dt^2 + a^2(t) \, d\vec{x}^2 \quad \text{with} \quad \ddot{a} > 0 \]

- Quantum vacuum fluctuations are stretched to cosmological scales

**Quantum Mechanics on Cosmological Scales!**
Cosmological Perturbations in Inflation

- One scalar degree of freedom: \( v \propto \zeta \) (curvature perturbation) \( \propto \delta T/T \) (CMB T° fluctuation)

\[
|\Psi\rangle = \bigotimes_{\mathbf{k} \in \mathbb{R}^3} |\Psi_k\rangle \quad \text{with} \quad |\Psi_k\rangle = \frac{1}{\cosh r_k} \sum_{n=0}^{\infty} e^{2in\varphi_k} (-1)^n \tanh^n r_k |n_k, n_{-k}\rangle
\]

Two-mode squeezed state (Gaussian state)

- Wigner function

\[
W(v_k, p_k) = \int \frac{dx}{2\pi^2} \Psi^*(v_k - \frac{x}{2}) e^{-ip_k x} \Psi(v_k + \frac{x}{2})
\]

- Evolution Equation

\[
\frac{\partial}{\partial t} W(v, p, t) = - \{ W(v, p, t), H(v, p, t) \}_{\text{Poisson Bracket}}
\]

For quadratic Hamiltonians
Quantum State of Cosmological Perturbations

\[ \frac{\partial}{\partial t} W (v, p, t) = - \{ W (v, p, t) , H (v, p, t) \} \text{Poisson Bracket} \]
Cosmological Perturbations in Inflation

- One scalar degree of freedom: $v \propto \zeta$ (curvature perturbation) $\propto \delta T / T$ (CMB $T^\circ$ fluctuation)

$$|\Psi\rangle = \bigotimes_{k \in \mathbb{R}^3^+} |\Psi_k\rangle \quad \text{with} \quad |\Psi_k\rangle = \frac{1}{\cosh r_k} \sum_{n=0}^{\infty} e^{2in\varphi_k} (-1)^n \tanh^n r_k |n_k; n_{-k}\rangle$$

Two-mode squeezed state (Gaussian state)

- Wigner function:
  $$W(v_k, p_k) = \int \frac{dx}{2\pi^2} \Psi^*(v_k - \frac{x}{2}) e^{-ip_k x} \Psi(v_k + \frac{x}{2})$$

- Evolution Equation
  $$\frac{\partial}{\partial t} W(v, p, t) = -\{W(v, p, t), H(v, p, t)\}_{\text{Poisson Bracket}}$$

For quadratic Hamiltonians

- Quantum Mean Value and Stochastic Average
  $$\langle \hat{O}(\hat{v}, \hat{p}) \rangle_{\text{quant}} \simeq \int W(v, p) O(v, p) \, dv \, dp$$

Super-Hubble limit

Example: $\langle vp \rangle_{\text{squeezed}} \longrightarrow e^{\Delta N_*} + \frac{i}{2} \frac{\dot{N}}{N}$
Stochastic Formalism

At leading order in slow roll:

\[ \dot{\phi}_{\text{cg}} + \frac{V'(\phi_{\text{cg}})}{3H^2} = \xi_1 \]

with \( \xi_1 = -\int \frac{dk}{(2\pi)^{3/2}} \frac{\partial}{\partial t} \left[ \theta \left( \frac{k}{\sigma aH} \right) \right] \left[ \phi_k(t) e^{-ikx} \hat{a}_k + \text{h.c.} \right] \)

Modes smaller than the coarse-graining scale are constantly escaping the Hubble radius and **source the coarse-grained sector**.

**Large Squeezing Approximation:**

\[ \hat{\xi}_1 \rightarrow \xi_1 \]

quantum operator \hspace{1cm} stochastic variable
Primordial Black Holes from Inflation

- Primordial density perturbations when modes re-enter the Hubble radius after inflation
  \[
  \frac{\delta \rho}{\rho} \bigg|_{k = aH} \sim \zeta
  \]

- Rare fluctuations exceeding critical value $\zeta > \zeta_c \sim 1$ collapse to form black holes

- Mass fraction
  \[
  \beta(M) < 10^{-8}
  \]
  \[
  \beta(M) = \int_{\zeta_c}^{\infty} P(\zeta) \, d\zeta
  \]
The Stochastic-δN Formalism

The number of e-fold is a stochastic variable $\mathcal{N}(\phi)$

$$\zeta_{\text{coarse grained}} = \mathcal{N} - \langle \mathcal{N} \rangle$$

Moments obey an iterative relation \((\text{Vennin and Starobinsky 2015})\)

$$\langle \mathcal{N}^n \rangle'' - \frac{\nu'}{\nu^2} \langle \mathcal{N}^n \rangle' = -\frac{n}{\nu M_{P1}^2} \langle \mathcal{N}^{n-1} \rangle$$
Full PDF required for PBHs!

Define characteristic function (includes all the moments)
\[
\chi_N(t, \phi) = \langle e^{itN} (\phi) \rangle = \int e^{itN} (\phi) P(N, \phi) \, dN
\]

Obeys partial differential equation
\[
\left( \frac{\partial^2}{\partial \phi^2} - \frac{v'}{v^2} \frac{\partial}{\partial \phi} + \frac{it}{vM_{Pl}^2} \right) \chi_N(t, \phi) = 0
\]

Inverse Fourier transform gives full probability distribution
\[
P(N, \phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itN} \chi_N(t, \phi) \, dt
\]
Example: \( v(\phi) = v_0 \left[ 1 + \left( \frac{\phi}{\phi_0} \right)^p \right] \)

\( v^2 v'' \sim v'^2 \)

*not the “naïve” criterion!*

\[ P_\zeta \]

- **full result**
- **slow–roll classical limit**
- **classical limit**
- **stochastic limit**
Example: \[ v(\phi) = v_0 \left[ 1 + \left( \frac{\phi}{\phi_0} \right)^p \right] \]

"Classical" Regime

Is the Gaussian approximation sufficient?
Example: \( v(\phi) = v_0 \left[ 1 + \left( \frac{\phi}{\phi_0} \right)^p \right] \)

"Stochastic" Regime

\[ \mu^2 = \frac{\Delta \phi_{\text{well}}^2}{v_0 M_{\text{Pl}}^2} \]
Example: \[ v(\phi) = v_0 \left[ 1 + \left( \frac{\phi}{\phi_0} \right)^p \right] \]

“Stochastic” Regime

\[ \mu^2 = \frac{\Delta \phi_{\text{well}}^2}{v_0 M_{\text{Pl}}^2} \]
Example: $v(\phi) = v_0 \left[ 1 + \left( \frac{\phi}{\phi_0} \right)^p \right]$
Conclusions

• **Stochastic-δN** needed to calculate primordial density perturbations beyond perturbative approach

• In the classical regime, **Gaussian approximation may fail!**

• Primordial Black Hole bounds require **N<1** in quantum diffusion regime

• Extension to **multi-field**?

• Transient **slow-roll violation** (inflection point models)?