Relativistic Gravitational Instability: the Weight of Heat

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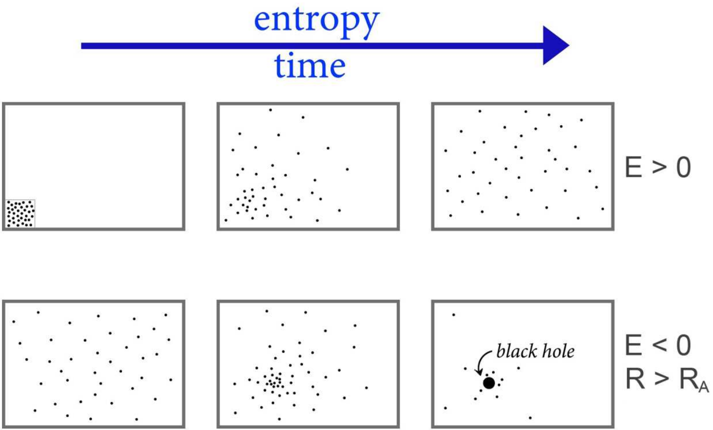
ROUPAS, Classical and Quantum Gravity 32 135023 (2015)
ROUPAS, Classical and Quantum Gravity (2013) and 32, 119501 (2015)
ROUPAS, Physical Review D, 91, 023001 (2015)

Layout


entropy
time
E > 0
black hole
E < 0
R > R_A

Newtonian Gravitational Instabilities

Extremization of Boltzmann Entropy & Poisson eq. give Emden.

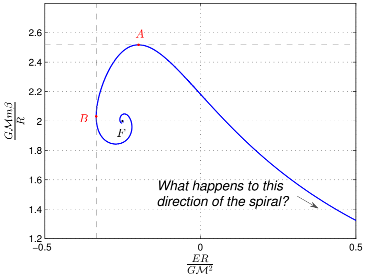
$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\phi(r)\right) = 4\pi G\rho_0 e^{-m\beta(\phi(r)-\phi(0))}$$

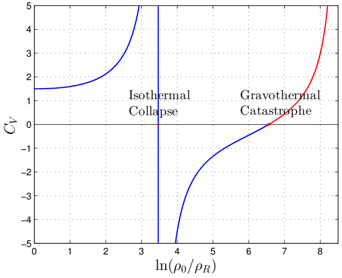
$\Leftrightarrow$

Hydrostatic equilibrium & ideal equation of state $P = \rho/m\beta$.

$$\frac{dP(r)}{dr} = -\rho(r)\frac{GM(r)}{r^2}$$

The solutions of Emden eq. are called isothermal spheres.



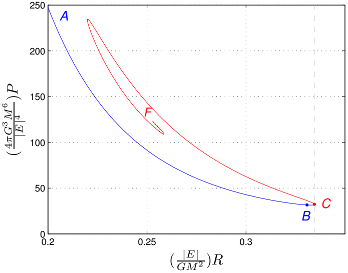


At point A: Canonical instability *"Isothermal Collapse"*.

At point B: Microcanonical instability *"Gravothermal Catastrophe"*.

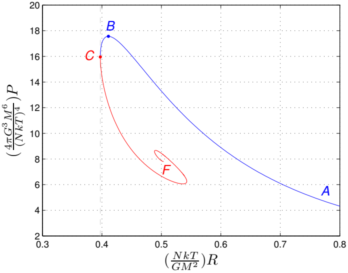
Constant Energy & Volume:

Instability at big radii.



Constant Temperature & Volume:

Instability at small radii.

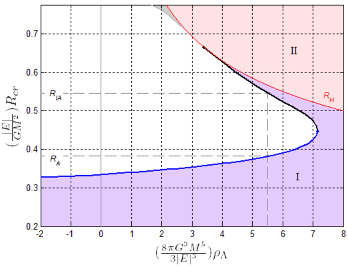


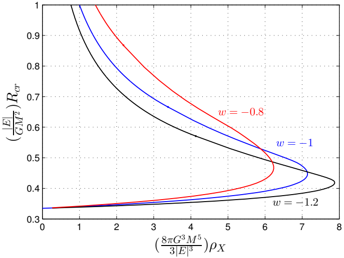
The Newtonian Gravitational Instability applies to big, low-energy systems.

Counter-intuitive to the relativistic paradigm!

Gravothermal Instability with Dark Energy [1, 2, 3]

The reentrant radius defines the maximum size of a perturbation that can lead to structure formation, i.e. the maximum turnaround radius. Quintessence increases the reentrant radius, while phantom dark energy decreases it. Therefore, a quintessence universe is expected to present richer large-scale structures, with more and larger bounded systems, than a phantom universe.





[1] ROUPAS, Axenides, Georgiou, Saridakis, PRD 89 (2014)
[2] Axenides, Georgiou, ROUPAS, Nucl Phys B, 871 (2013)
[3] Axenides, Georgiou, ROUPAS, PRD, 86 (2012)

Layout

TOV equation from Maximum Entropy

TOV equation (hydrostatic equilibrium) is derived in G.R. by Einstein's equations. I have derived it from a maximum entropy principle as follows.

Let us preassume only the expression for the proper volume

$$d^3x = 4\pi r^2 {g_{rr}}^{\frac{1}{2}} dr \ , \ g_{rr} = \left(1 - \frac{2G\hat{M}}{rc^2}\right)^{-1}$$

and not the whole set of Einstein's equations.

For constant mass-energy M and number of particles N, the entropy

$$S = \int_0^R s(r) \left(1 - \frac{2G\hat{M}(r)}{rc^2}\right)^{-\frac{1}{2}} 4\pi r^2 dr, \ ds = \frac{c^2}{T} d\rho - \frac{\mu}{T} dn, \tag{1}$$

attains an extremum if

$$\delta S - \tilde{\beta} c^2 \delta M + \alpha \delta N = 0. \tag{2}$$

This condition gives for the Lagrange multipliers

$$\alpha = \frac{\mu(r)}{T(r)} = const. \tag{3}$$

$$\tilde{\beta} = \frac{1}{T}(g_{rr})^{\frac{1}{2}} + \frac{4\pi G}{c^4} \int_r^R \frac{\rho(\bar{r})c^2 + P(\bar{r})}{T(\bar{r})} (g_{rr}(\bar{r}))^{\frac{3}{2}} \bar{r} d\bar{r} = const. \tag{4}$$

and by use of standard thermodynamic relations and after some algebra we get TOV equation and g_{tt}.

Theorem[a]

[a] ROUPAS, Classical and Quantum Gravity (2013) and 32, 119501 (2015)

For static spherically symmetric perfect fluids in General Relativity, thermal equilibrium requires these conditions to hold:

1. TOV equation:

$$\frac{dP}{dr} = -(P/c^2 + \rho)\left(\frac{GM(r)}{r^2} + 4\pi G P r/c^2\right)\left(1 - \frac{2GM(r)}{rc^2}\right)^{-1}$$

2. Tolman's & Klein's relations:

$$T(r)\sqrt{g_{tt}} = \tilde{T} \equiv const. \quad \text{and} \quad \mu(r)\sqrt{g_{tt}} = \tilde{\mu} \equiv const.$$

3. The function g_{tt} equals:

$$g_{tt} = e^{-2\int_r^\infty dr\left(\frac{G\hat{M}}{\bar{r}^2} + 4\pi G\frac{P}{c^2}\bar{r}\right)\left(1 - \frac{2G\hat{M}}{\bar{r}c^2}\right)^{-1}}.$$

Thus, I was able to derive from entropy maximum, the equation of hydrostatic equilibrium (TOV), Tolman's effect and the time-time component of the metric with no use of Einstein's equations, besides the proper volume.

Tolman-Ehrenfest effect: the 'weight of heat'

In General Relativity, *thermal energy rearranges itself to counterbalance its own gravitational attraction*. Hence, at thermal equilibrium, not only local density, but also *local temperature, are inhomogeneous*. This is expressed by Tolman relation $T(r)\sqrt{g_{tt}} = const.$, which may be expressed differentially in our case

$$\frac{dT(r)}{dr} = -T(r)\left(\frac{GM(r)}{r^2} + 4\pi GPr/c^2\right)\left(1 - \frac{2GM(r)}{rc^2}\right)^{-1}$$

In the Newtonian limit this equation reads

$$\frac{\nabla T}{T} = \frac{\vec{g}}{c^2}, \tag{5}$$

Let us derive it from the *maximum entropy principle*. Assume that a quantity of heat $|dE_1|$ flows from one subsystem to another at a gravitational potential lower by $\Delta\phi$. The energy received by the second subsystem equals to $dE_2 = -(dE_1 + m\Delta\phi)$ with $m = -dE_1/c^2$ the gravitational mass corresponding to the transferred heat. Now, assuming that the two systems achieve equilibrium, the entropy is $dS = dS_1 + dS_2 = 0$ which after differentianting by dE_2 and using $1/T = dS/dE$ gives

$$\frac{dS_1}{dE_1} - \frac{dS_2}{dE_2} = -\frac{dS_2}{dE_2}\frac{\Delta\phi}{c^2} \Rightarrow \frac{\Delta T}{T} = -\frac{\Delta\phi}{c^2}, \tag{6}$$

which expresses equation (5). Evidently, the temperature gradient is a result of the "mass of heat" $m = |dE_1|/c^2$.

Relativistic Equation of State

Using the relativistic one-particle energy:

$$\epsilon = mc^2 \left\{ \left(1 + \frac{p^2}{m^2c^2}\right)^{\frac{1}{2}} - 1 \right\}$$

the Boltzmann distribution $f = A(\vec{r})e^{-\beta\varepsilon}$ leads to the equation of state:

The Relativistic Ideal Gas

$$P(r) = mc^2 \frac{n(r)}{b(r)} \Leftrightarrow P = \frac{1}{b(1+\mathcal{F}(b))}\rho c^2$$

- Newtonian limit $b \to \infty$: $\mathcal{F} \simeq \frac{3}{2b}$, $\rho \simeq mn + \frac{3m}{2b}$, $P \simeq \frac{\rho}{\beta}$
- Ultra-relativistic limit $b \to 0$: $\mathcal{F} \simeq \frac{3}{b}$, $P \simeq \frac{1}{3}\rho c^2$

where $b = mc^2/kT$ and:

$$\mathcal{F}(b) = \frac{K_1(b)}{K_2(b)} + \frac{3}{b} - 1\ ,\ K_n(b) = \int_0^\infty e^{-b\cosh\zeta}\cosh(n\zeta)d\zeta$$

For a generally covariant proof see [W. Israel, J. Math. Phys. 1963]

Solutions of the following system are in dynamical and thermal equilibrium in General Relativity:

$$\frac{dP}{dr} = -(\frac{P}{c^2} + \rho)\left(\frac{G\hat{M}}{r^2} + 4\pi G\frac{P}{c^2}r\right)\left(1 - \frac{2G\hat{M}}{rc^2}\right)^{-1}$$

$$\frac{d\hat{M}}{dr} = 4\pi\rho r^2$$

$$\frac{db}{dr} = -\frac{b}{P + \rho c^2}\frac{dP}{dr}$$

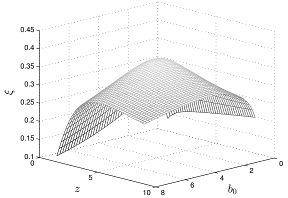
$$P = \frac{1}{b(1 + \mathcal{F}(b))}\rho c^2$$

with initial conditions: $\rho(0) = \rho_0,\ b(0) = b_0,\ \hat{M}(0) = 0$ for $r \in [0, R]$.
We will also impose the constraint of fixed total rest mass

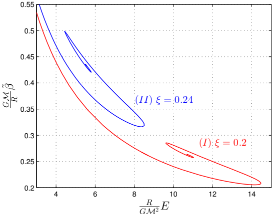
$$\mathcal{M} \equiv mN = \int_0^R m\, n(r)\left(1 - \frac{2G\hat{M}}{rc^2}\right)^{-\frac{1}{2}} 4\pi r^2 dr.$$

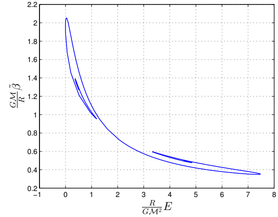
Relativistic Gravothermal Instability [4]

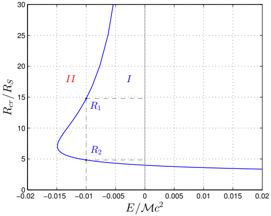
Ultra max rest mass $= 0.35Rc^2/2G$



Relativistic Spiral





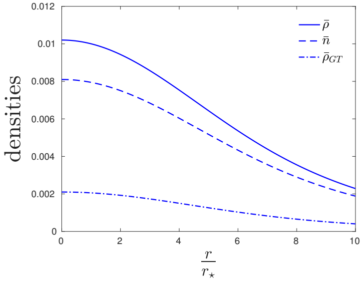


where $\xi = \frac{2G}{Rc^2}\mathcal{M}_{rest}$ and $E = M - \mathcal{M}_{rest}$ = thermal + gravitational energy

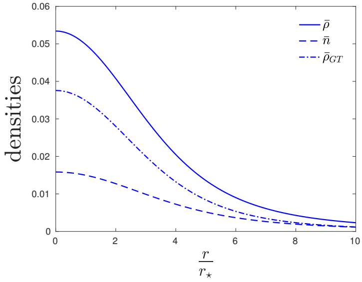
[4] ROUPAS, Classical and Quantum Gravity 32 135023 (2015)

Gravitational Collapse due to the Weight of Heat

A stable equilibrium at lower temperature.



Relativistic Gravothermal Instability.



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Relativistic Fermi Gas

The equations of thermal equilibrium, normalized to:

$$\rho_* = \frac{8\pi m^4 c^3}{h^3} \ , \ P_* = \rho_* c^2 \ , \ r_* = \left(\frac{4\pi G}{c^2}\rho_*\right)^{-\frac{1}{2}} \ , \ M_* = r_* \frac{c^2}{G}$$

become for the relativistic Fermi gas

$$P = \frac{1}{3}\int_0^\infty \frac{\sinh^4\theta d\theta}{e^{-\alpha + b\cosh\theta} + 1} \ , \ \frac{db}{dr} = b\left(\frac{\hat{M}}{r^2} + Pr\right)\left(1 - \frac{2\hat{M}}{r}\right)^{-1}$$

$$\rho = \int_0^\infty \frac{\sinh^2\theta \cosh^2\theta d\theta}{e^{-\alpha + b\cosh\theta} + 1} \ , \ \frac{d\hat{M}}{dr} = \rho r^2$$

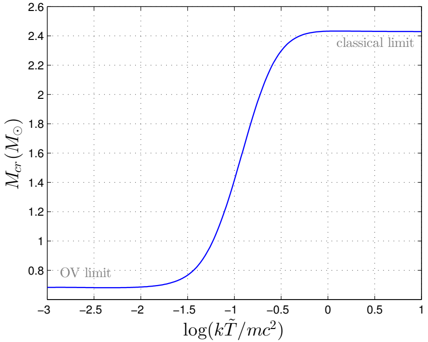
This is the system to be solved with initial conditions:

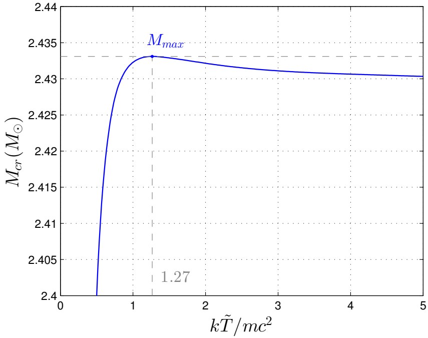
$$b(0) = b_0 \ , \ \hat{M}(0) = 0$$

and boundary condition:

$$\rho_R = 0.5\rho_N$$

Thermal Mass Limit of Neutron Cores [5]





- $M_{max} = 2.43 M_\odot$ at $\tilde{T} = 1190 MeV$ with $R = 15.2 km$
- For $M_{obs} = 2 M_\odot$ it is $\tilde{T} = 180 MeV$ with $R = 14.4 km$

Relevant to hot proto-neutron stars and core-collapse supernovae.

[5] ROUPAS, Physical Review D, 91, 023001 (2015)

Conclusions

- Gravitation is characterized by the presence of a **double spiral** in the energy-temperature space. Thermal equilibria of a self-gravitating gas are subject to a Newtonian gravitational instability at ‘low’ energies and ‘big’ radii and a **relativistic gravitational instability** at ‘high’ energies and ‘small’ radii.
- This relativistic instability of a hot gas is due to the **weight of heat**, namely it occurs when the gravity of heat overcomes its stabilizing effect against gravitational collapse.
- There is an **upper limit of the amount of rest mass** that an ideal self-gravitating gas can attain under any conditions, namely $M_{rest} < 0.35Rc^2/2G$.
- TOV equation, Tolman’s relation and the time component of the metric may be derived by maximum entropy principle and the proper volume with no use of Einstein’s equations.
- Hot protoneutron stars are subject to the relativistic gravitational instability with $M_{max} = 2M_{\odot}$ at $R = 14.4km$.