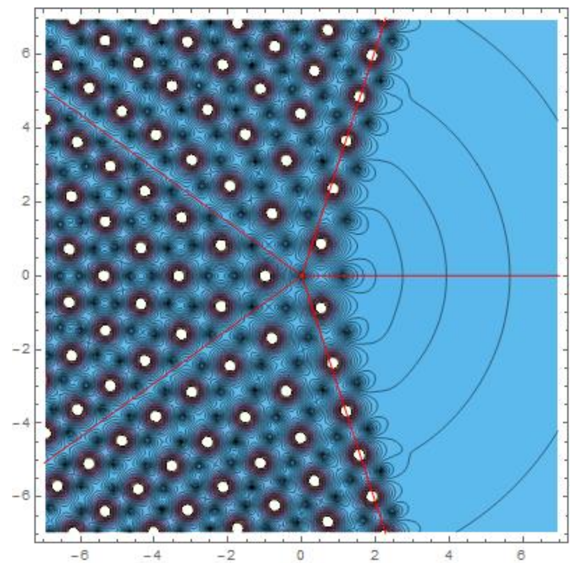
Analytic Transseries Summation for Painlevé I



Marcel Vonk (University of Amsterdam)

Workshop on resurgent asymptotics
in physics and matematics

ICNFP Crete, 10 July 2018

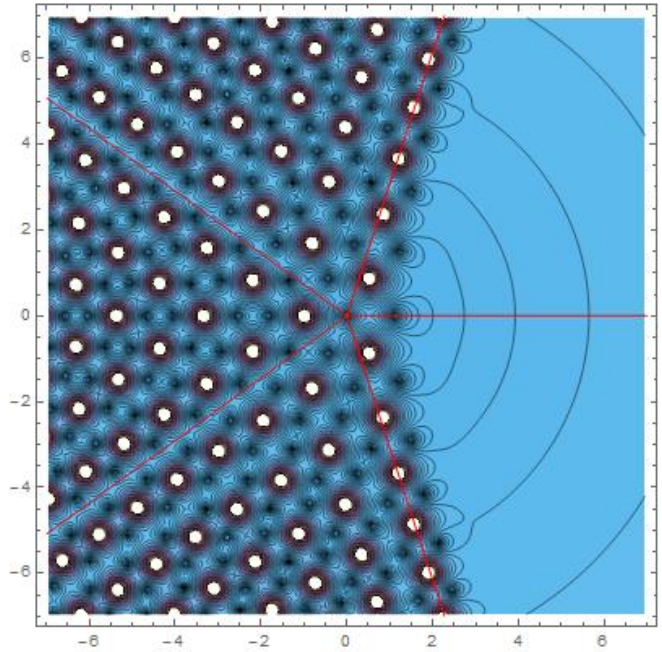
This talk describes part of a recent project with Inês Aniceto and Ricardo Schiappa.

<u>Goal:</u> in a tractable setting, get a **full** understanding of the physics and mathematics encoded in resurgent asymptotic (trans)series.

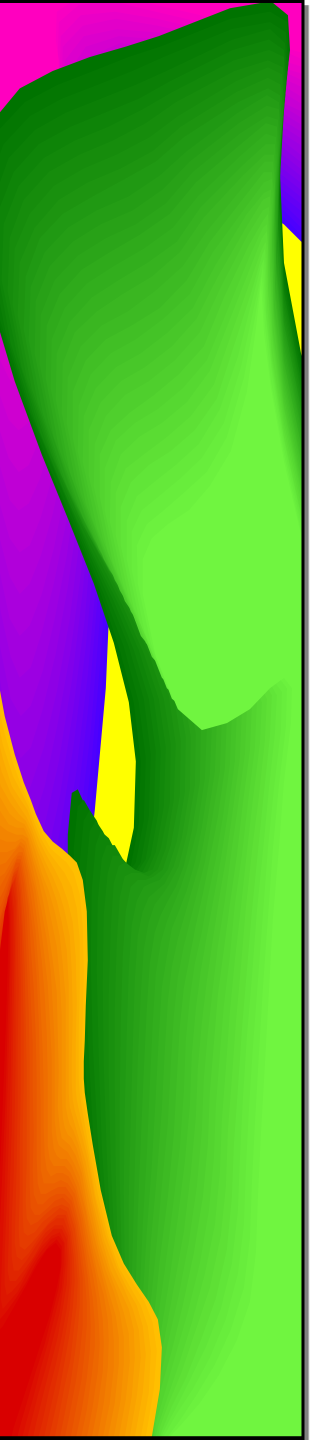
Aniceto, Schiappa, Vonk – to appear

Outline

1. Motivation: 2D quantum gravity and Painlevé I
2. Painlevé I: properties
3. Transseries solution
4. Analytic transseries summation: linear case
5. Analytic transseries summation: quadratic case
6. The second parameter
7. Conclusion and outlook



1. Motivation: 2D quantum gravity and Painlevé I

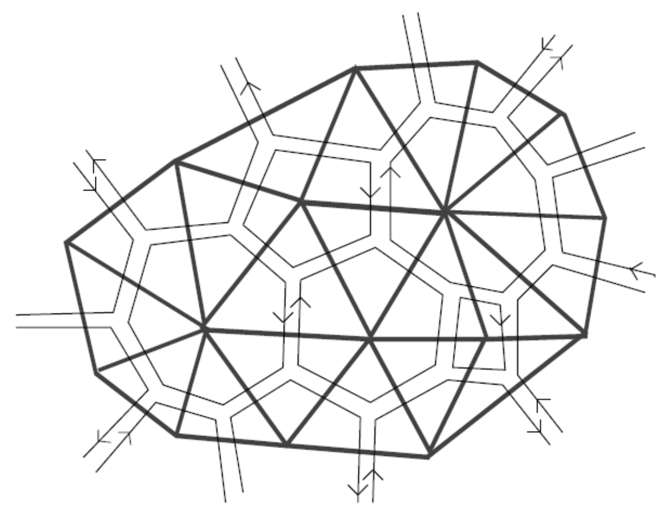


2D quantum gravity

Some physics background on the problem we study. A (more than) 25-year-old story!

- A good candidate to investigate quantum gravity is **string theory**.
- What is the simplest string theory one can study?
- Discretize world sheet: **matrix models**!

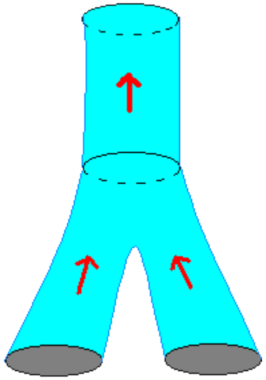
Douglas, Shenker – 1990
Brézin, Kazakov – 1990
Gross, Migdal – 1990



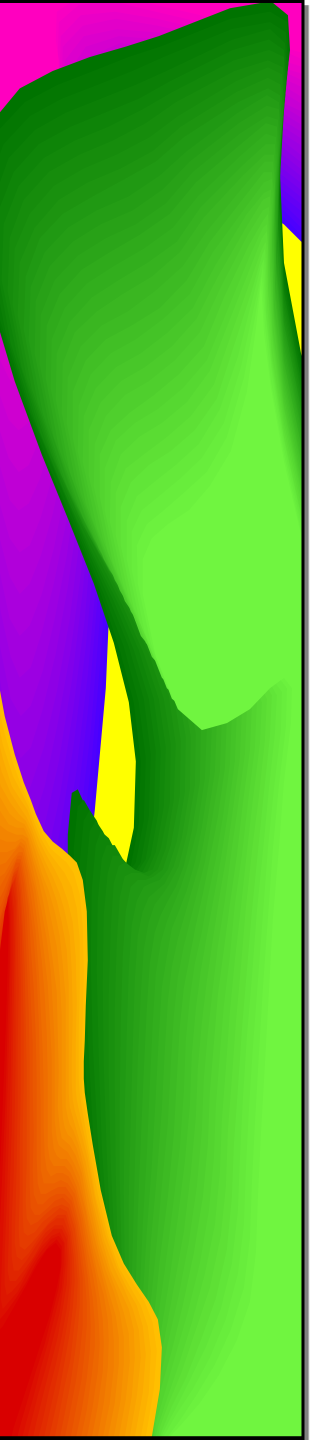
(Image: Ginsparg/Moore)

2D quantum gravity

- Choice and scaling determines which **target space** theory we study.
- 1/N (size of the matrix) is related to the **string coupling constant**. Interested in a large N expansion.

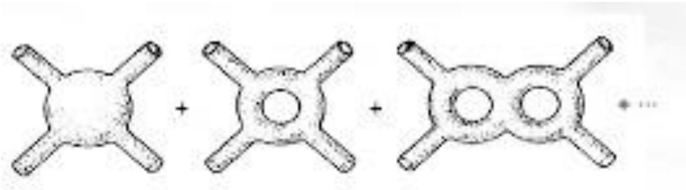


- Strict large N limit: **tree level strings** in 0D. But one can do more!
- Scale matrix couplings too: pure gravity coupled to minimal CFTs. Simplest nontrivial case: **(2,3) minimal model**.



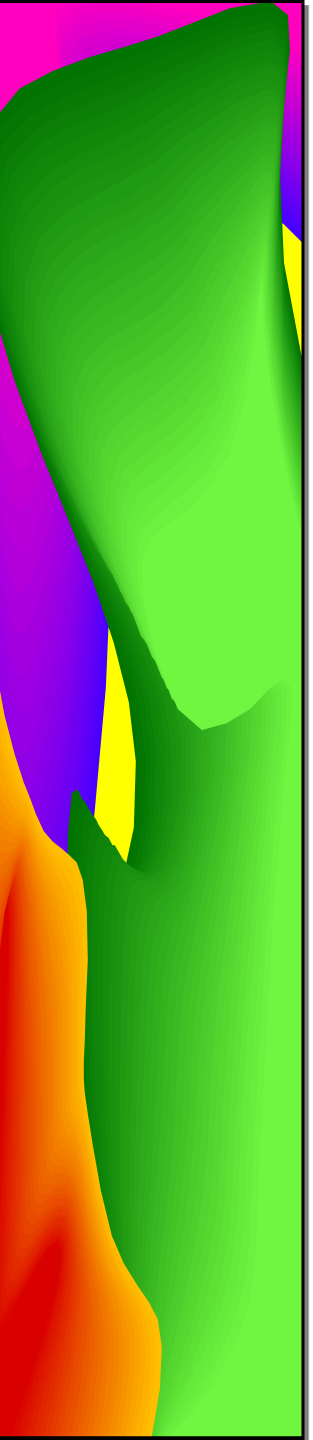
2D quantum gravity

- Partition function can be expressed in terms of a simple **function** u(z).
- The scaled version of the string coupling constant is $z^{-5/4}$: **large z** expansion.
- Asymptotic expansion, formally satisfies the **Painlevé I** ODE.



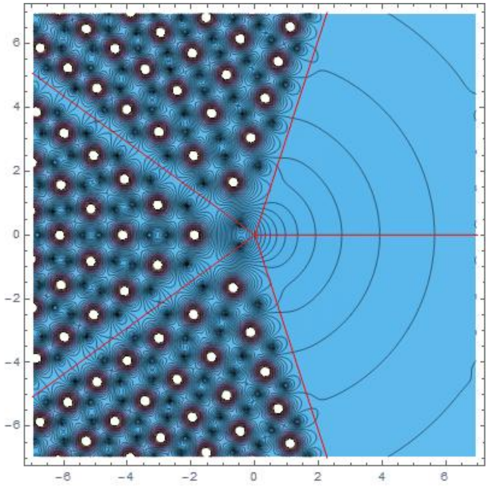
(Image: Green/Schwarz/Witten)

- Can one **sum this** into an actual function?
- We have studied the full matrix model, but in this talk: focus on Painlevé I.

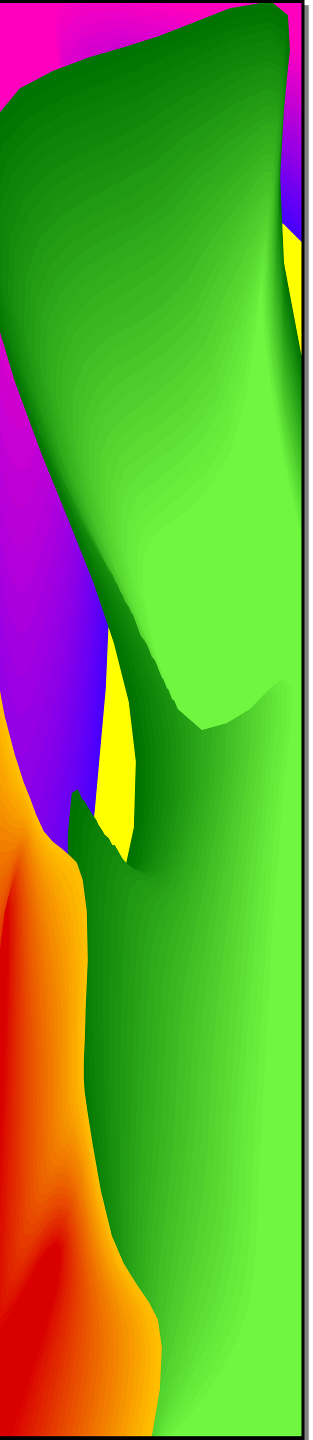


The Painlevé I equation

In mathematics, the story is even older: 100-year-old problem.



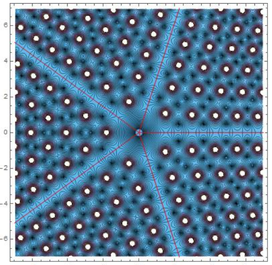
- Paul Painlevé (1863-1933) studied second order ODEs whose only moveable singu-larities are poles.
- 6 classes found: **Painlevé transcendants**. We are interested in Painlevé I.
- Boutroux classified its solutions in 1913.



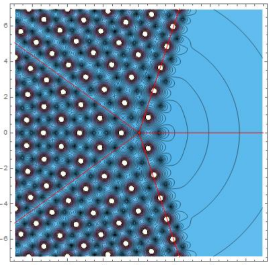
The Painlevé I equation

2^{nd} order ODE: 2 integration parameters.

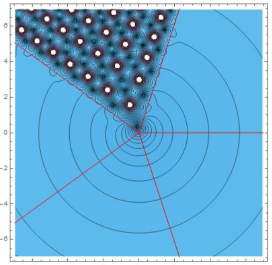
- A **generic solution** has poles throughout the complex plane.



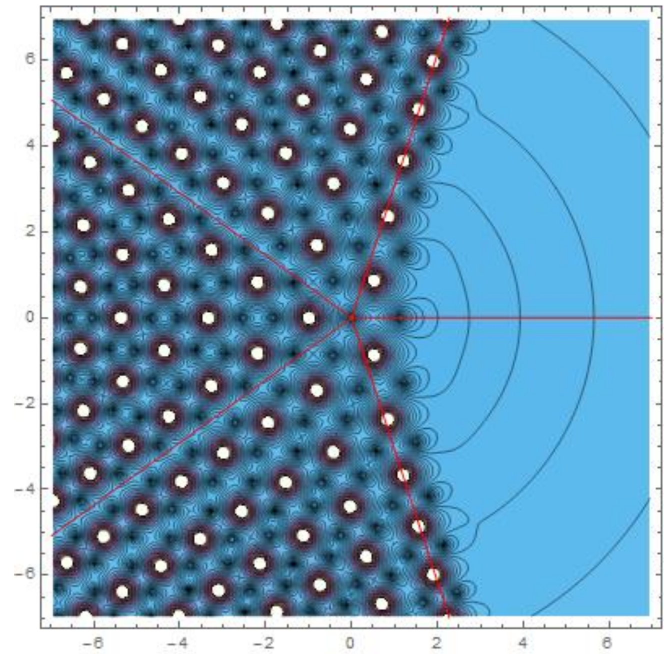
- The ***tronquées* solutions** (1 parameter) have two “empty quintants”.



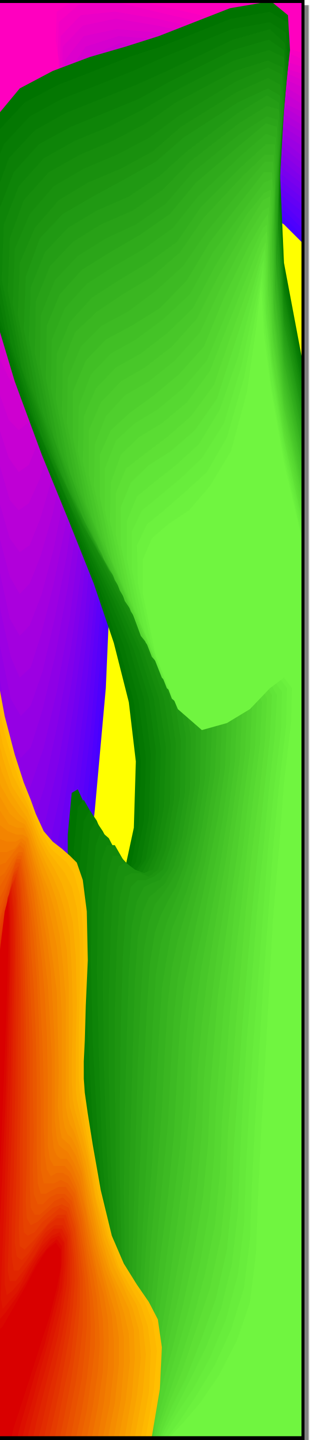
- The ***tritronquées* solutions** (discrete set) have four empty quintants.



How does this relate to formal solutions?



2. Painlevé I: properties



The Painlevé I equation

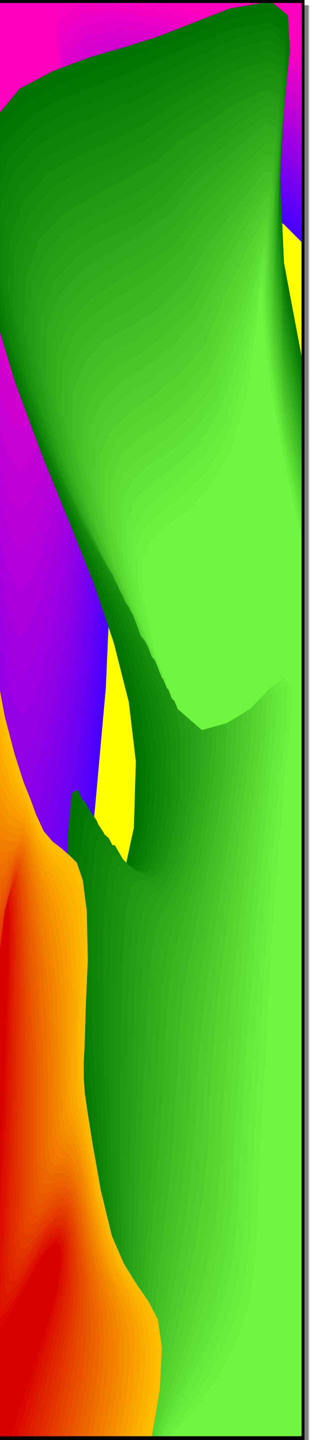
Painlevé I: $$u^2(z) - \frac{1}{6}u''(z) = z$$

Some **properties**:

1) The equation has the symmetry

$$z \to e^{2\pi i/5} z, \qquad u \to e^{-4\pi i/5} u$$

As a result, there is a $\mathbf{Z}_5$-action on the space of solutions. Moreover, the *z*-plane can be divided into **five sectors** where the solutions may have different asymptotics.



The Painlevé I equation

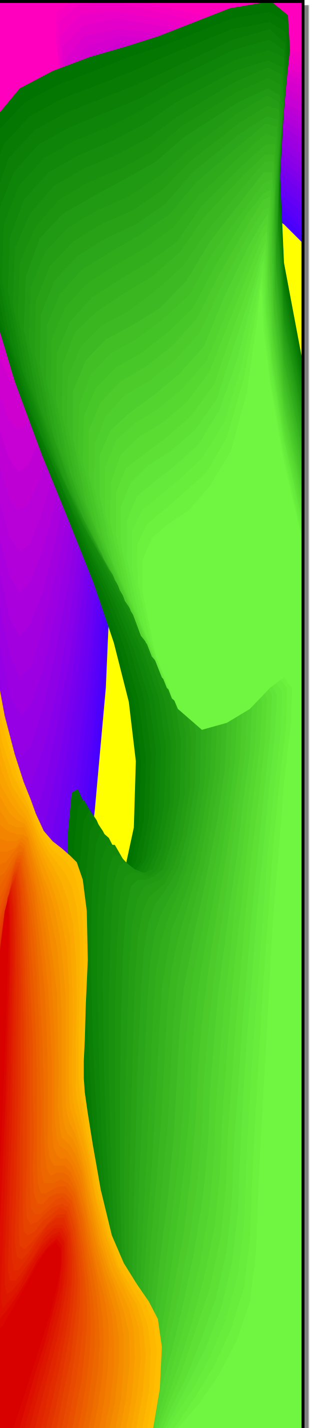
$$u^2(z) - \frac{1}{6}u''(z) = z$$

2) All poles are **double poles** with the same leading coefficient:

$$u(z) = \frac{1}{(z-z_0)^2} + \frac{3z_0}{5}(z-z_0)^2 + (z-z_0)^3 + h(z-z_0)^4 + \mathcal{O}\left((z-z_0)^5\right)$$

Note the second parameter, *h*.

Generic solution has **infinitely many poles** throughout the complex *z*-plane.



The Painlevé I equation

$$u(z) = \frac{1}{(z-z_0)^2} + \frac{3z_0}{5}(z-z_0)^2 + (z-z_0)^3 + h(z-z_0)^4 + \mathcal{O}\left((z-z_0)^5\right)$$

In physics, one is often interested in the associated **free energy** and **partition function**:

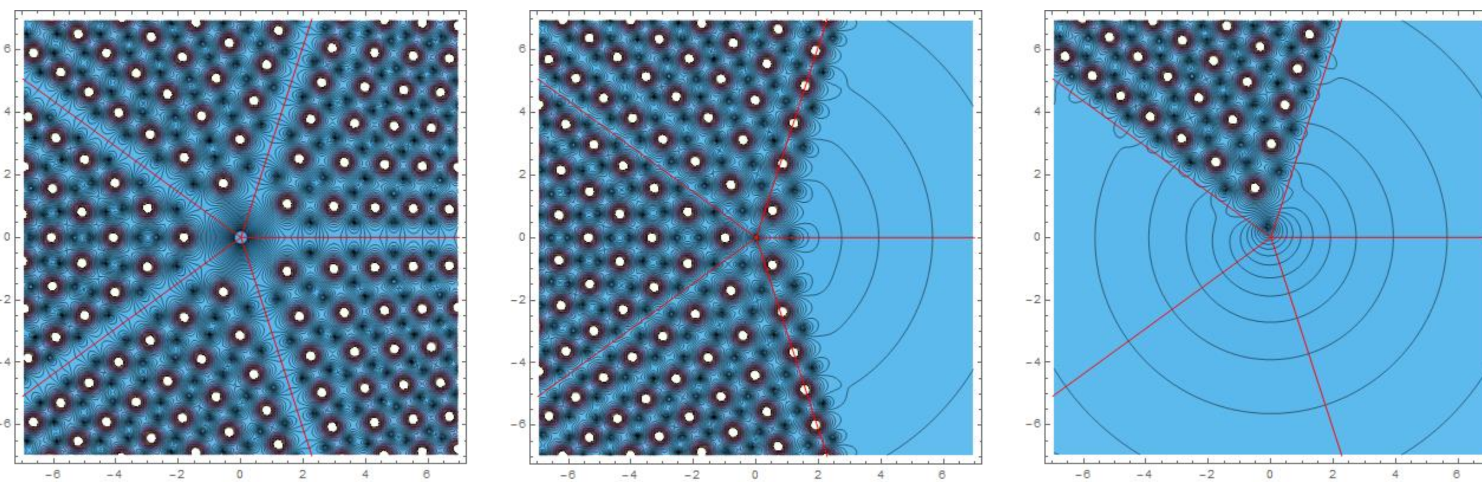
$$F''(z) = -u(z), \qquad Z(z) = e^{F(z)}$$

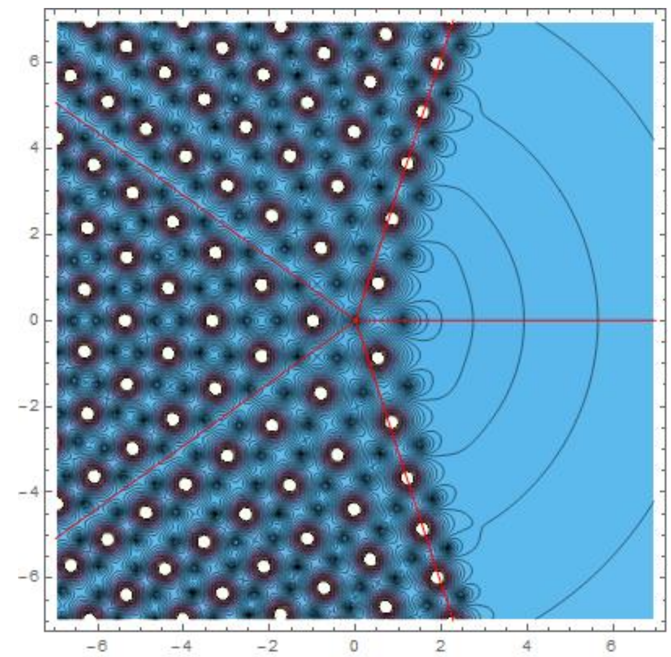
Note: double pole of u $\leftrightarrow$ single zero of Z.

The Painlevé I equation

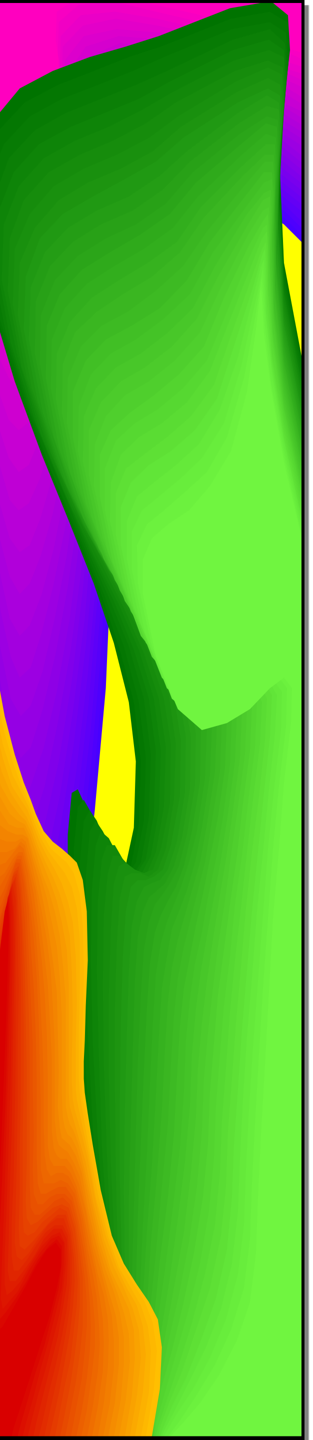
$$u^2(z) - \frac{1}{6}u''(z) = z$$

3) Special solutions: **tronquées** and **tritronquées**.





3. Transseries solution



Transseries solution

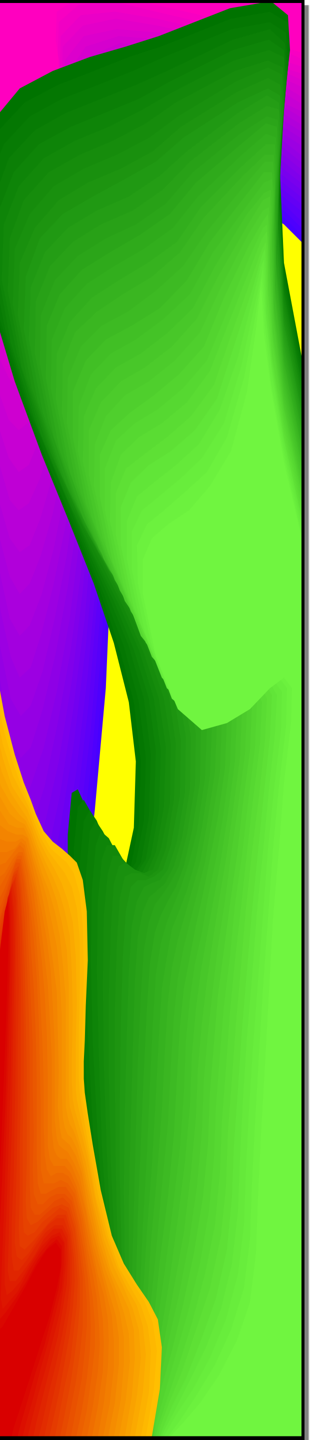
At the formal level, a generic 2-parameter solution can be found: **transseries** solution.

Garoufalidis, Its, Kapaev, Mariño – 2010
Aniceto, Schiappa, Vonk - 2011

- Transseries: multiple series expansion in different **transmonomials**, e.g. x, $e^{-A/x}$.
- These transmonomials have an **ordering**, e.g. $e^{-A/x} << x$. (Painlevé I: $x = g_s = z^{-5/4}$.)

Q1: “Sum” transseries into a function?

Q2: What is the underlying physics and mathematics?



Transseries solution

In the pole-free regions, Painlevé I solutions behave asymptotically as $u \sim \sqrt{z}$.

Perturbative **asymptotic** expansion:

$$u_{\text{pert}}(z) \simeq \sqrt{z}\left(1 - \frac{1}{48}\, z^{-\frac{5}{2}} - \frac{49}{4608}\, z^{-5} - \frac{1225}{55296}\, z^{-\frac{15}{2}} - \dots\right)$$

Coefficients grow as *(2g)!*

Physical interpretation: $z^{-5/4}$ is the **string coupling** g_s.

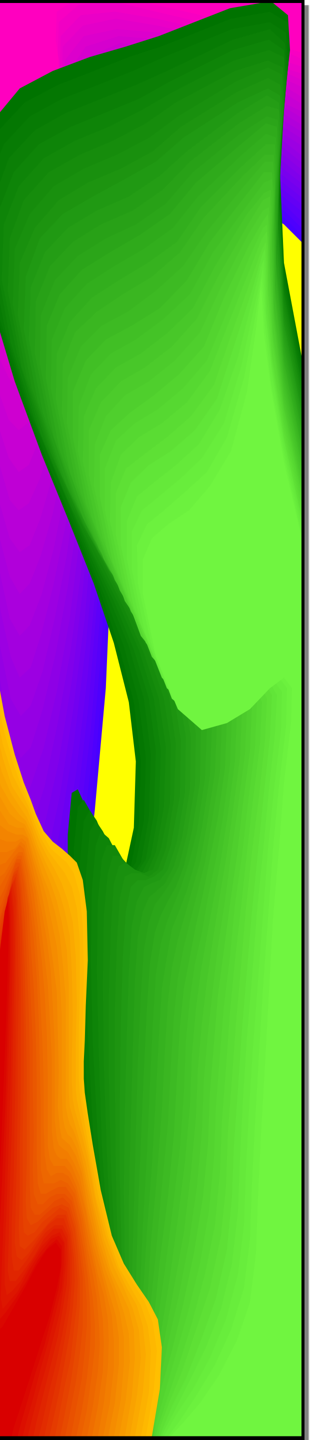
Transseries solution

To find the integration parameters, we must extend the perturbative series to a **resurgent transseries**.

Naïve way (use $x = z^{-5/4}$):

$$u(x;\sigma) = x^{-\frac{2}{5}} \sum_{n=0}^{+\infty} \sigma^n \, \mathrm{e}^{-\frac{nA}{x}} \, x^{n\beta} \sum_{g=0}^{+\infty} u_g^{(n)} \, x^g$$

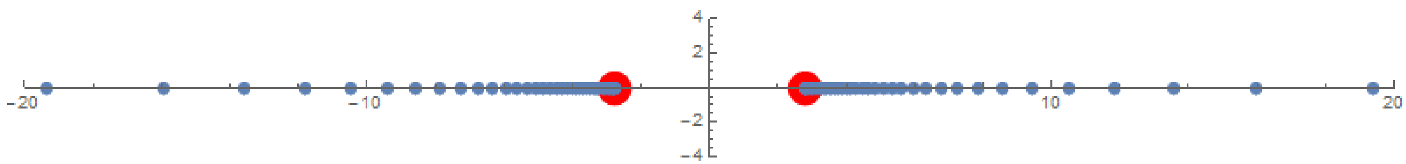
This does provide a **1-parameter family** of formal solutions, but not all!



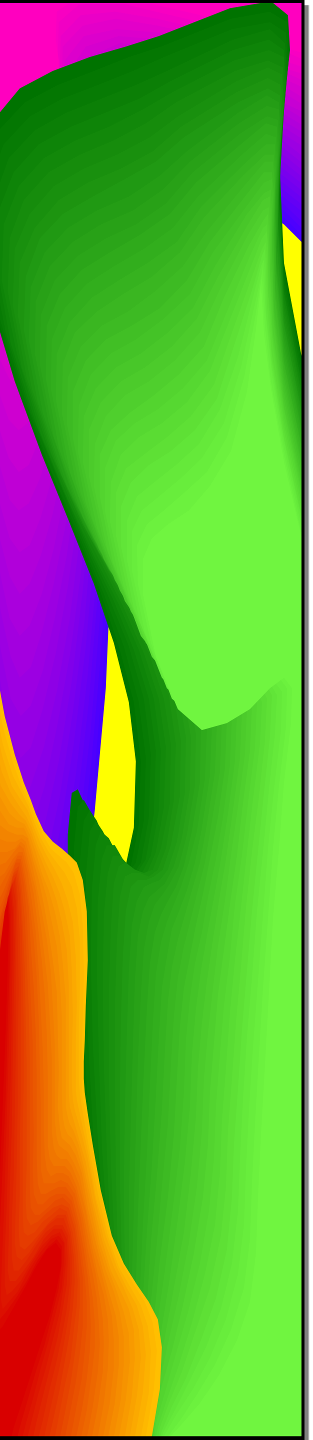
Transseries solution

Indications that there should be more:

- Painlevé I is a **2nd order** ODE, so we expect two constants of integration
- Instanton action can be **$A=\pm 8\sqrt{3}/5$**
- Borel plane has positive and negative branch points at these values



So at least formally, we expect to have a 2-parameter transseries solution.



Transseries solution

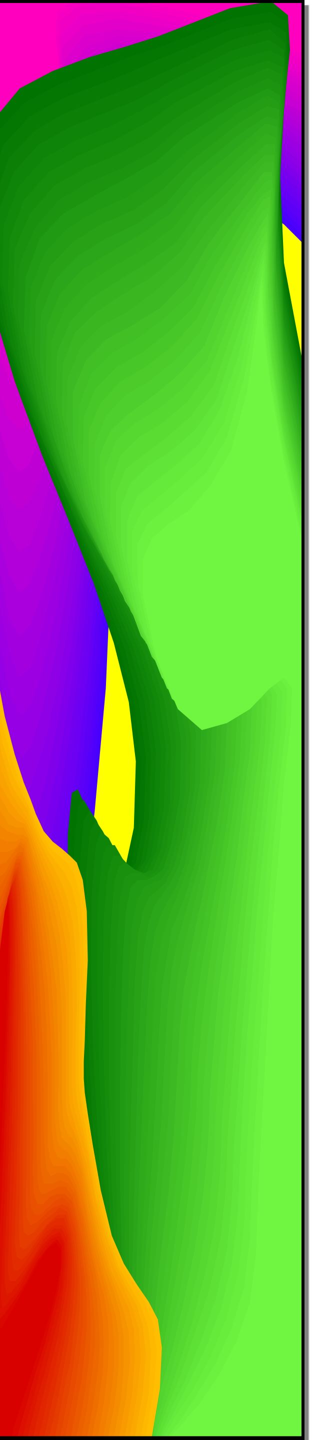
Indeed, such a solution can be found:

$$u(x;\sigma_1,\sigma_2) = x^{-\frac{2}{5}} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} L_{nm}(x;\sigma_1,\sigma_2)\,\sigma_1^n\sigma_2^m\, e^{\frac{(m-n)A}{x}}\, x^{\beta_{nm}}\;\Phi^{(n|m)}(x)$$

$$\Phi^{(n|m)}(x) \simeq \sum_{g=0}^{\infty} u_g^{(n|m)} x^g$$

$$L_{nm}(x;\sigma_1,\sigma_2) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{2}{\sqrt{3}}(m-n)\sigma_1\sigma_2 \log x\right)^k$$

- Four **transmonomials**:
  $e^{-A/x} \ll x \ll \log(x) \ll e^{+A/x}$.
- Two **parameters** σ_1 and σ_2.



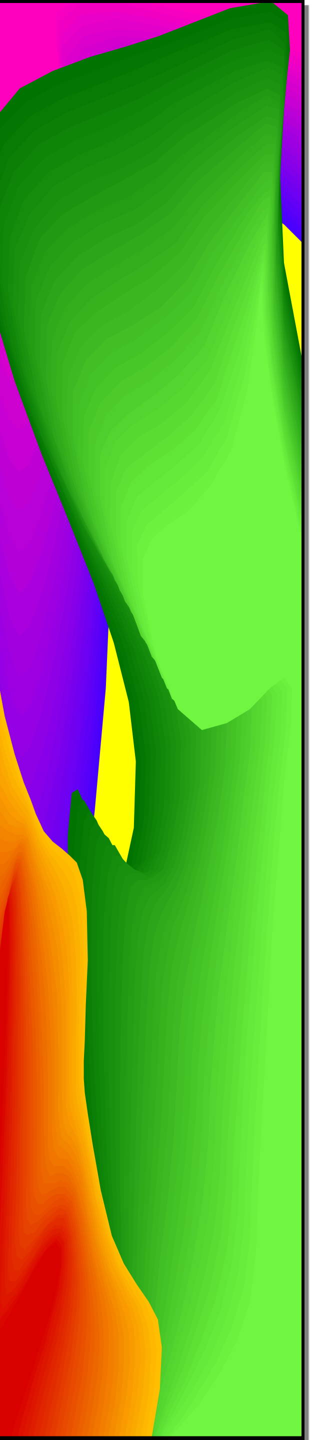
Transseries solution

$$u(x;\sigma_1,\sigma_2) = x^{-\frac{2}{5}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} L_{nm}(x;\sigma_1,\sigma_2)\, \sigma_1^n \sigma_2^m \, e^{\frac{(m-n)A}{x}} \, x^{\beta_{nm}} \Phi^{(n|m)}(x)$$

$$\Phi^{(n|m)}(x) \simeq \sum_{g=0}^{\infty} u_g^{(n|m)} x^g$$

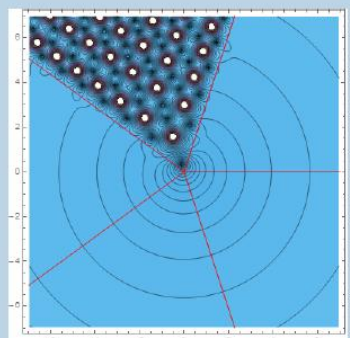
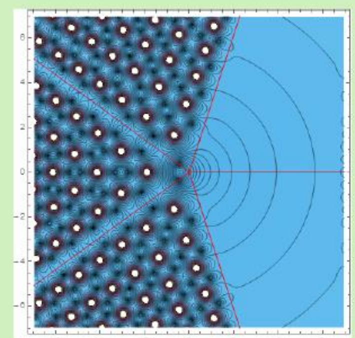
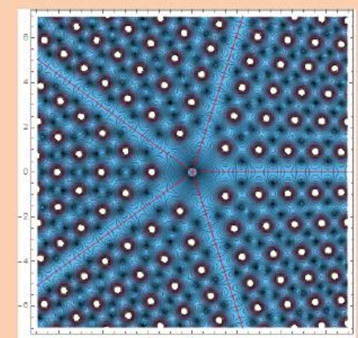
$$L_{nm}(x;\sigma_1,\sigma_2) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{2}{\sqrt{3}} (m-n)\sigma_1\sigma_2 \log x \right)^k$$

- Coefficients of **log terms** are multiples of those of non-log terms. They can always trivially be included; ignore them for now.
- Note the appearance of a different **starting order** β_{nm} for each sector.

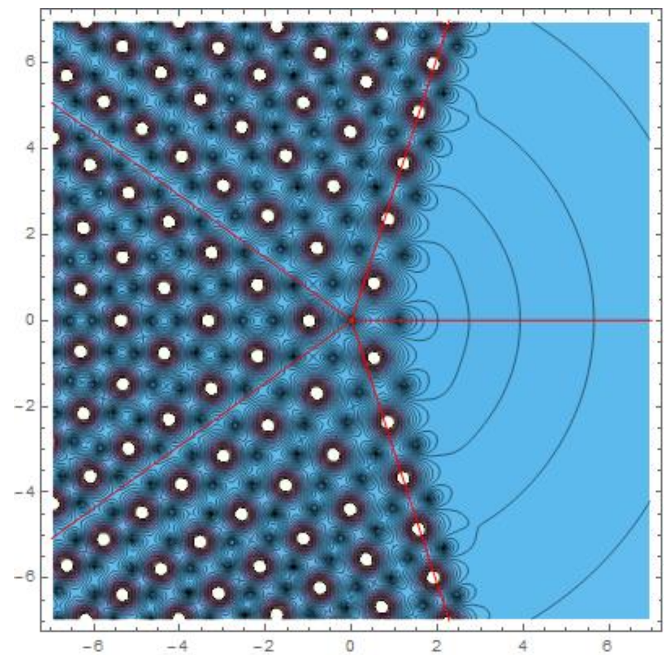


Transseries solution

Using clever numerics, one can resum the transseries for small parameters σ_1, σ_2:

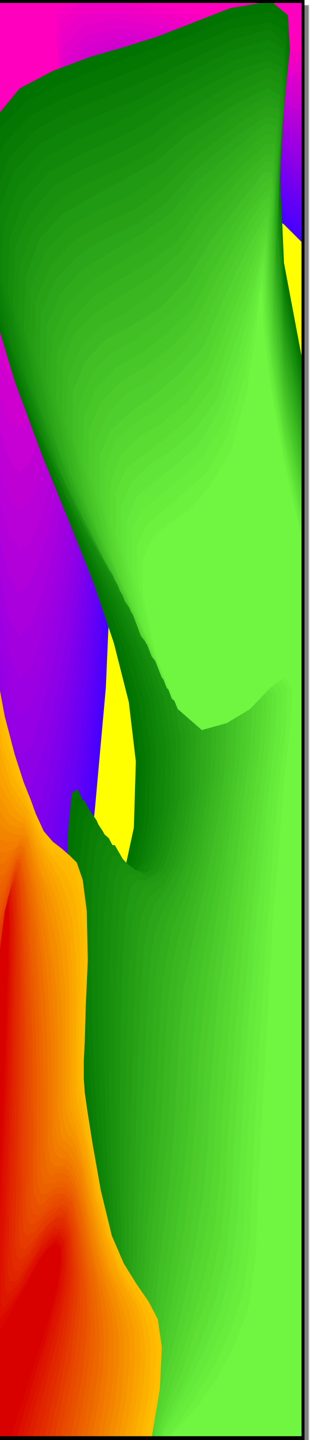
	0 parameter	1 parameter	2 parameter
Painlevé I	Tritronquée 	Tronquée 	General 

Side note: the same structure appears in the full matrix model.



4. Analytic transseries summation

linear case



Borel-Padé-Écalle summation

How do we turn a formal transseries into a **function**? Given values for *x*, σ_1 and σ_2, how do we compute a value for $u(x;\sigma_1,\sigma_2)$?

$$u(x;\sigma_1,\sigma_2) = x^{-\frac{2}{5}} \sum_{n,m=0}^{\infty} \sigma_1^n \sigma_2^m e^{\frac{(m-n)A}{x}} x^{\beta_{nm}} \sum_{g=0}^{\infty} u_g^{(n|m)} x^g$$

Borel-Padé-Écalle:

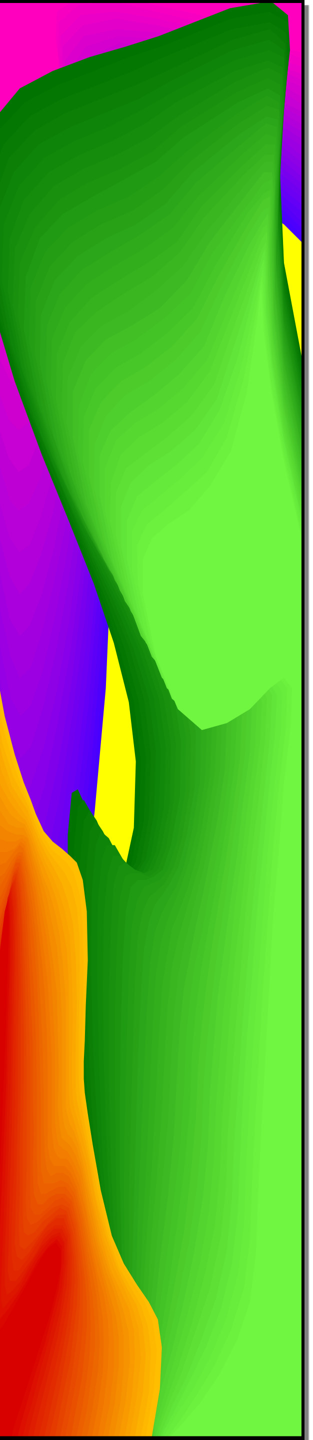
1. Use **Borel summation** for the asymptotic series using Padé approximants.
2. Do the other sums **order by order**.

Borel-Padé-Écalle summation

Remarks:

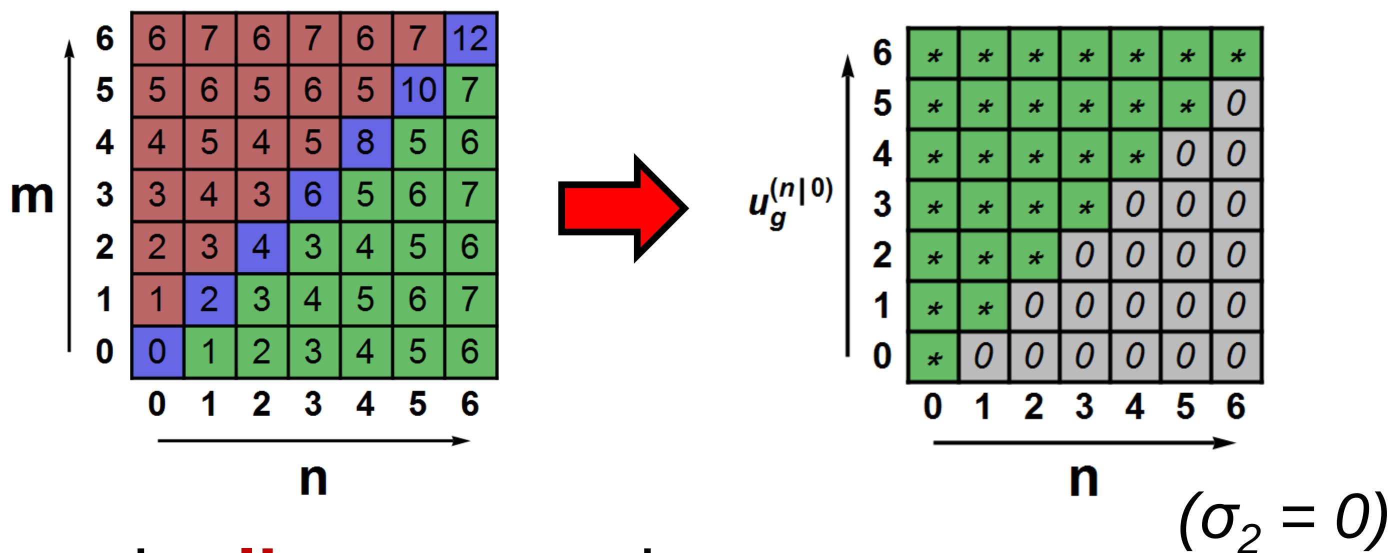
- Note that $\sigma_2\, e^{+A/x}$ can be made **small**, so makes sense numerically.
- On the other hand: restricted to regimes where σ_1 and σ_2 are small.
- Does this make sense when $e^{-A/x}$ becomes of order 1?
- Mathematically: **anti-Stokes line**.
- Physically: **phase transition**!

Can we do better?



Linear summation

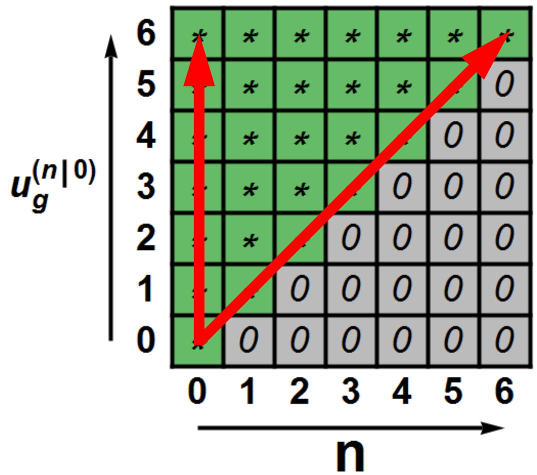
Let us look at the starting orders β_{nm} for the ***u*-transseries**:



Note the **linear** growth.

For simplicity, let us set $\boldsymbol{\sigma_2 = 0}$ and focus on the $m=0$ sectors.

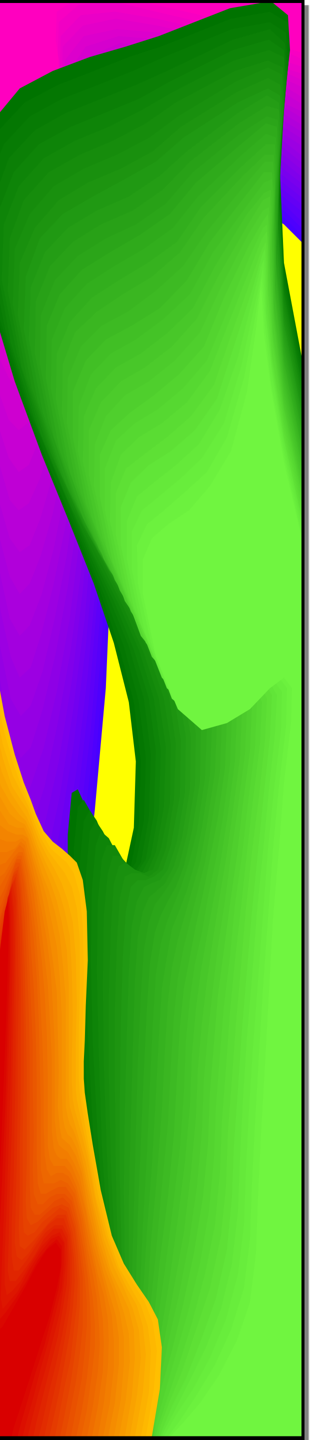
Linear summation



Borel-Padé-Écalle: sum “vertically”.

Would it be possible to sum the leading terms for all of the *n*-sectors?

Amazingly: yes, with an **exact** answer!



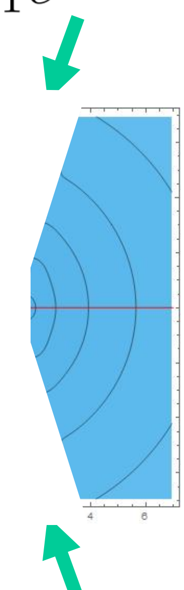
Linear summation

This procedure (*transasymptotic summation*) was first carried out by Costin and collaborators.

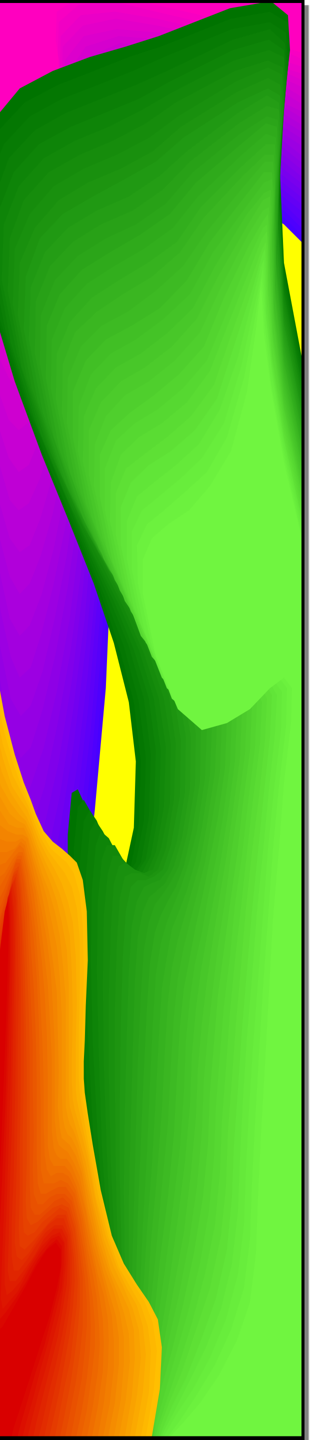
Costin – 1995/1998
Costin, Costin – 2001
Costin, Costin, Huang – 2013

$$u_0(x;\sigma_1) = \frac{1 + 10\tau + \tau^2}{(1-\tau)^2}$$

$$\tau = \frac{\sqrt{x}}{12}\sigma_1 e^{-A/x}$$



τ = *1*: array of **poles**!
This allows to "go inside the filled sectors".

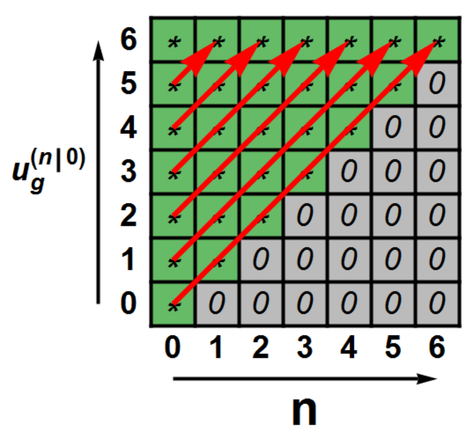


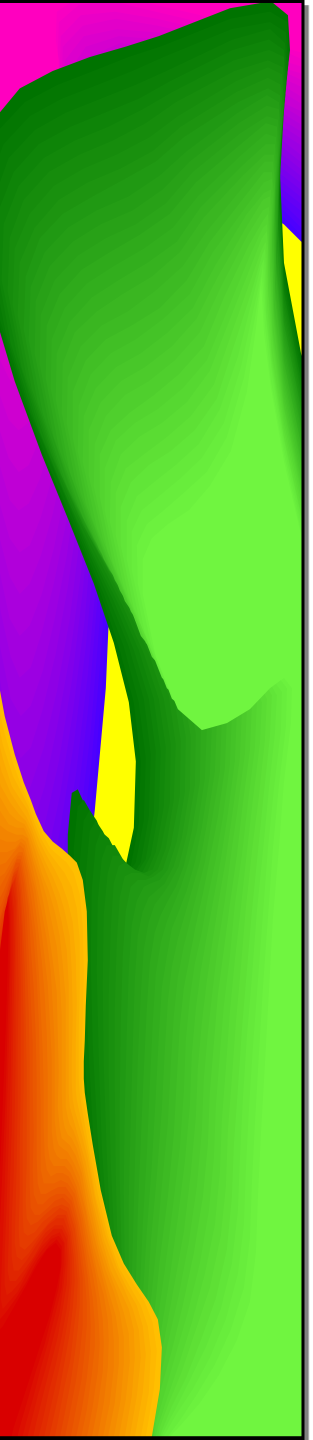
Linear summation

$$u_0(x;\sigma_1) = \frac{1+10\tau+\tau^2}{(1-\tau)^2} \qquad \tau = \frac{\sqrt{x}}{12}\sigma_1 e^{-A/x}$$

Remark: we have exchanged our transmonomials $x \ll e^{-A/x}$ for $x \ll \tau$.

Question: can we continue this process and sum subleading terms?





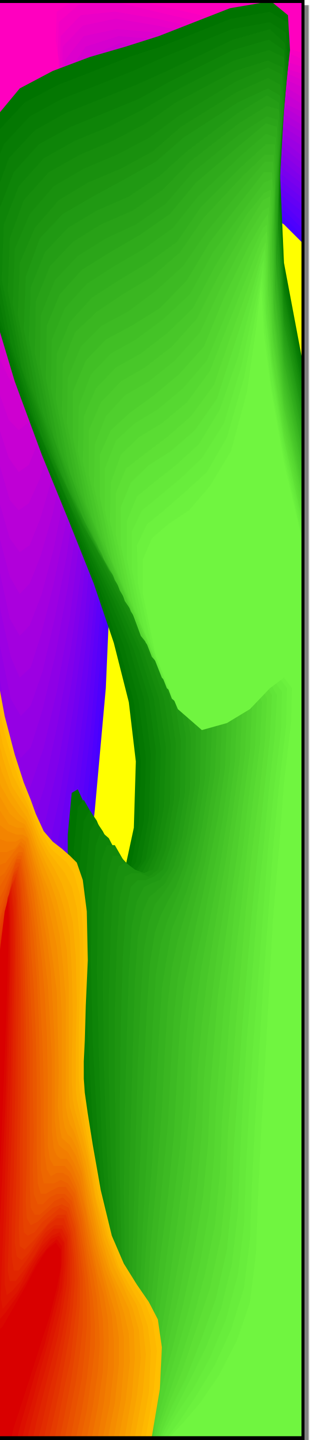
Linear summation

Costin et al.: **yes**, and this gives *O(x)* corrections (g_s-corrections) to the locations of the poles.

However, there is an even better way to look at this: study the **partition function** instead!

m \ n	0	1	2	3	4	5	6
6	36	25	16	9	4	1	–12
5	25	16	9	4	1	–10	1
4	16	9	4	1	–8	1	4
3	9	4	1	–6	1	4	9
2	4	1	–4	1	4	9	16
1	1	–2	1	4	9	16	25
0	0	1	4	9	16	25	36

Again, set $\sigma_2 = 0$ for simplicity.

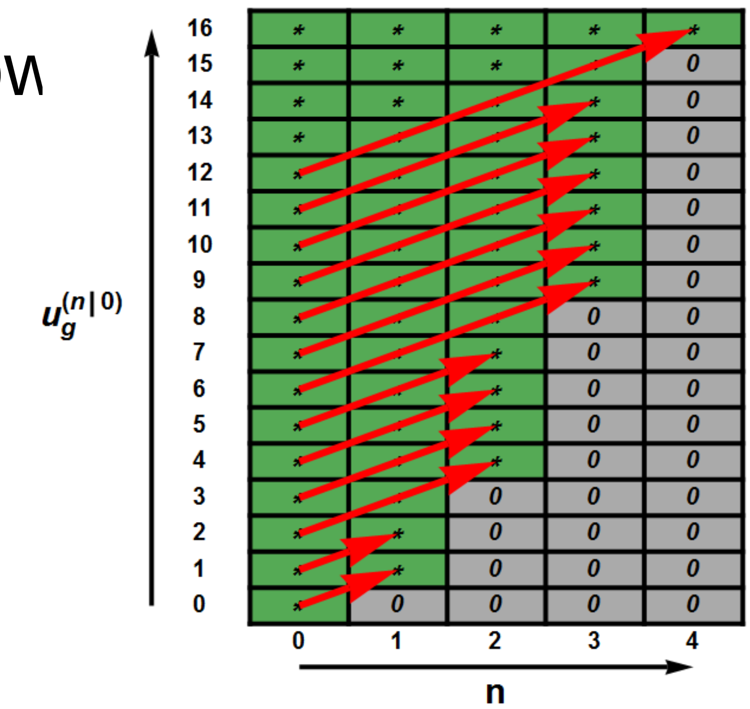


Linear summation

The diagonal sums now become **finite sums**!

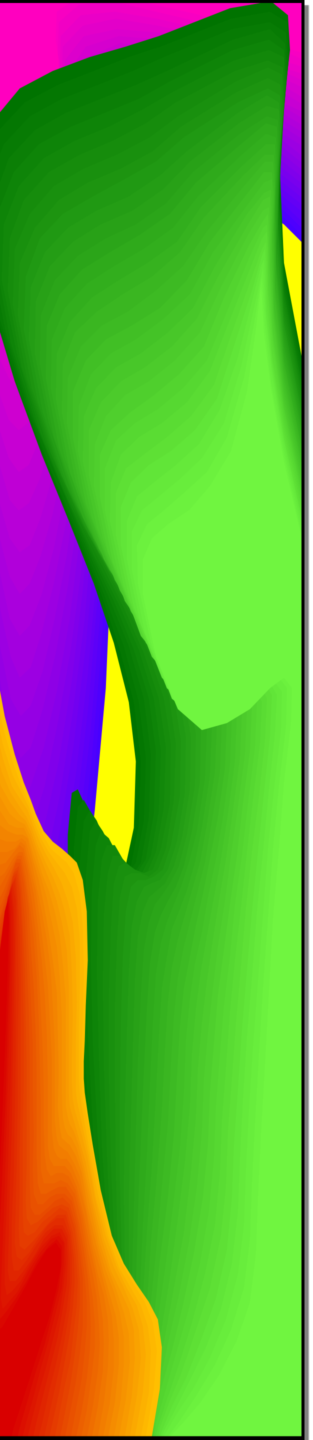
In particular, the leading order gives

$$Z_0(x;\tau) = 1 - \tau$$



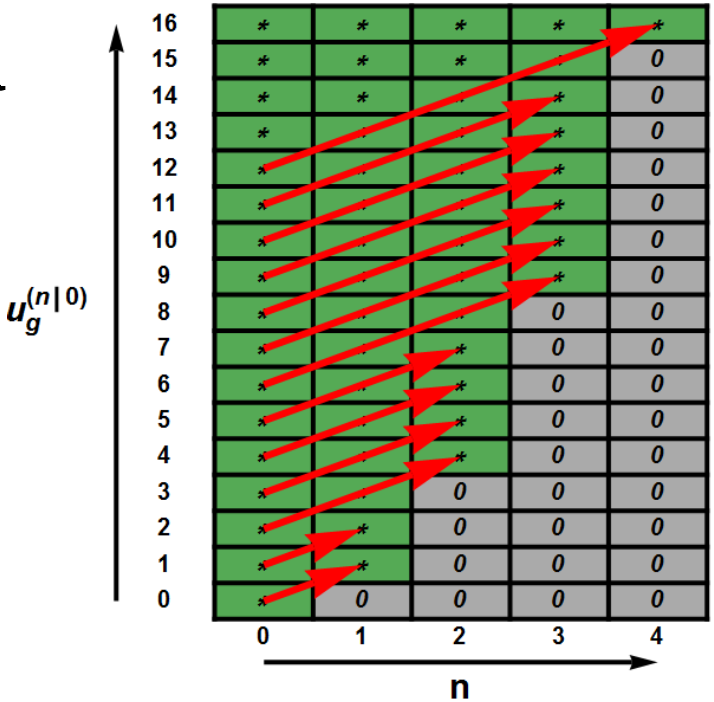
As expected, we find **zeroes** for *Z* at $\tau = 1$, where we found **poles** for *u*.

What about *x*- (or g_s-) corrections?



Linear summation

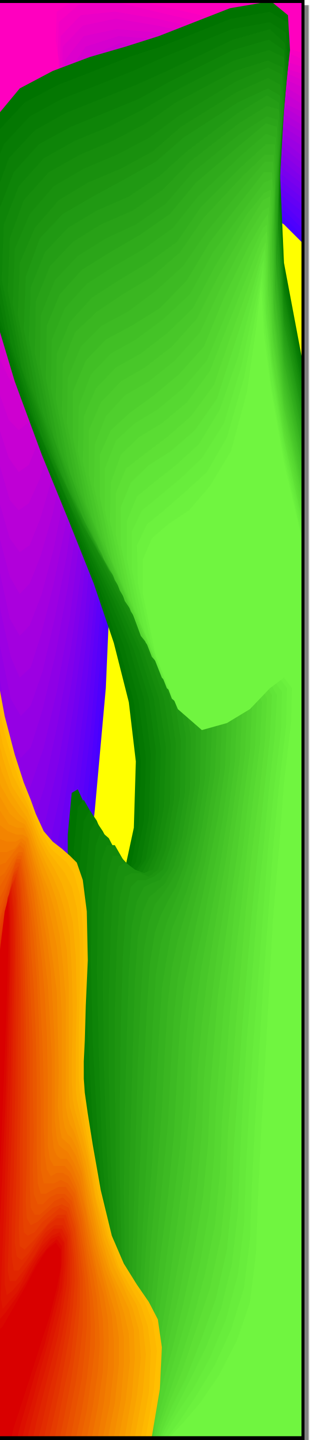
At third order, we find a **quadratic polynomial** in τ.



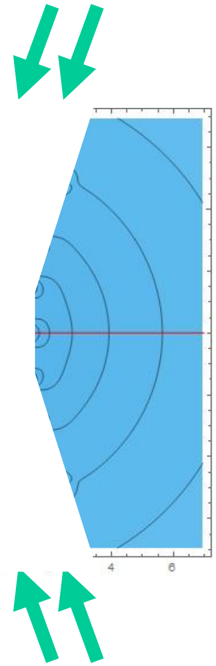
Therefore, we find two zeroes!

One is the g_s-corrected version of the zero at τ =1.

Other one is **new**; location scales as $1/g_s$.



Linear summation



Indeed, this gives us the correct **second line** of poles/zeroes. Continuing, we can go as deep into the pole region as we wish.

Linear summation

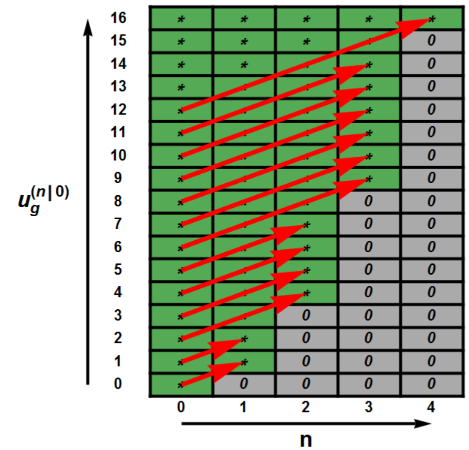
So **linear summation** provides:

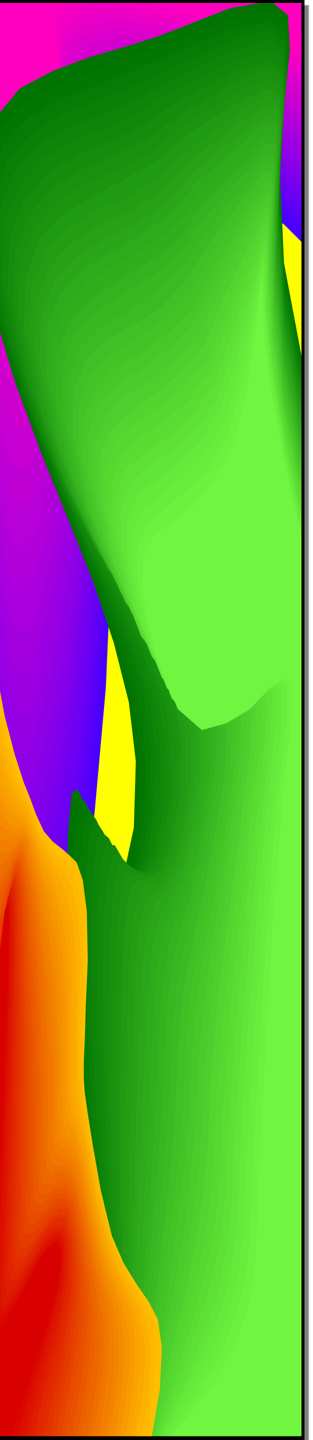
$$\tau_{\text{arr. }1} = 1 + \#\,x + \#\,x^2 + \#\,x^3 + \ldots$$

$$\tau_{\text{arr. }2} = \#\,x^{-1} + \# + \#\,x + \#\,x^2 + \ldots$$

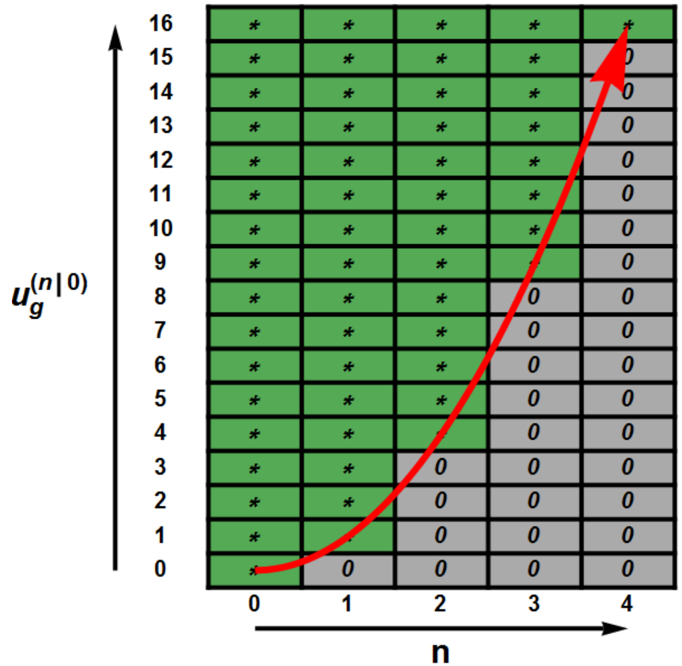
$$\tau_{\text{arr. }3} = \#\,x^{-2} + \#\,x^{-1} + \# + \#\,x + \ldots$$

$$\tau_{\text{arr. }4} = \#\,x^{-3} + \#\,x^{-2} + \#\,x^{-1} + $$



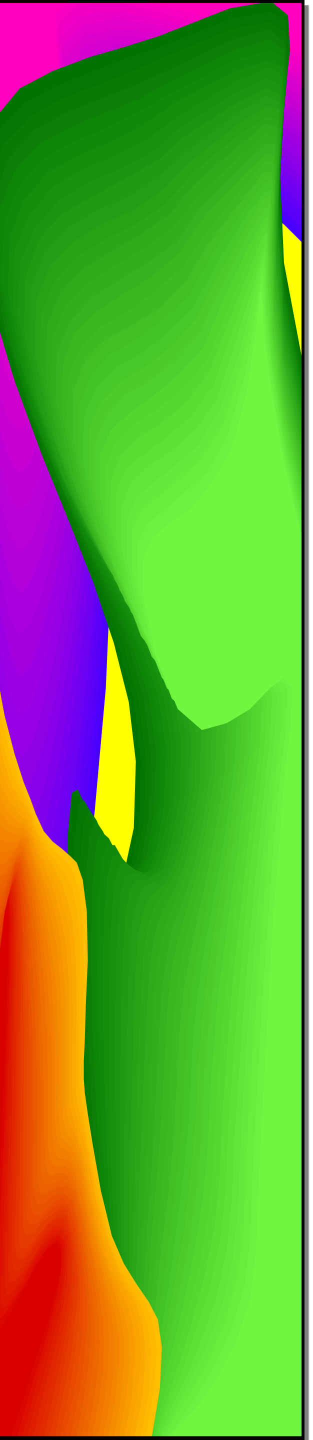


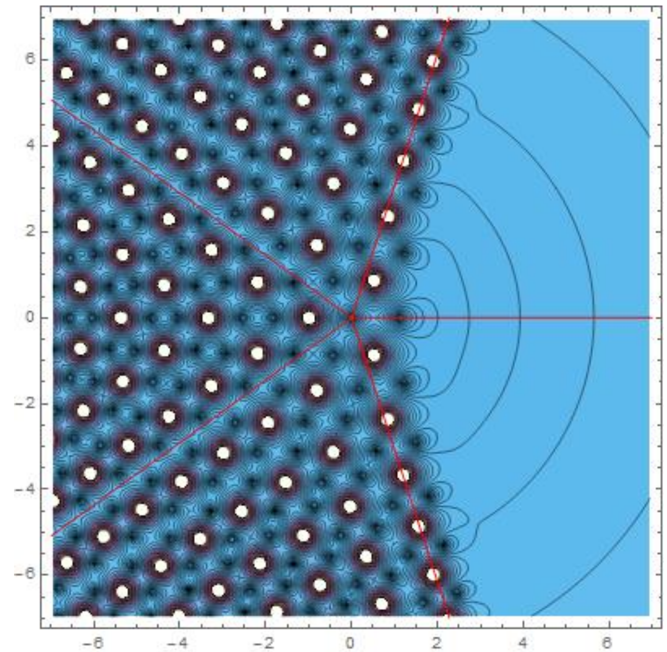
Linear summation



But... shouldn't we “listen” to what the problem is telling us? Why sum linearly?

Shouldn't we sum quadratically?



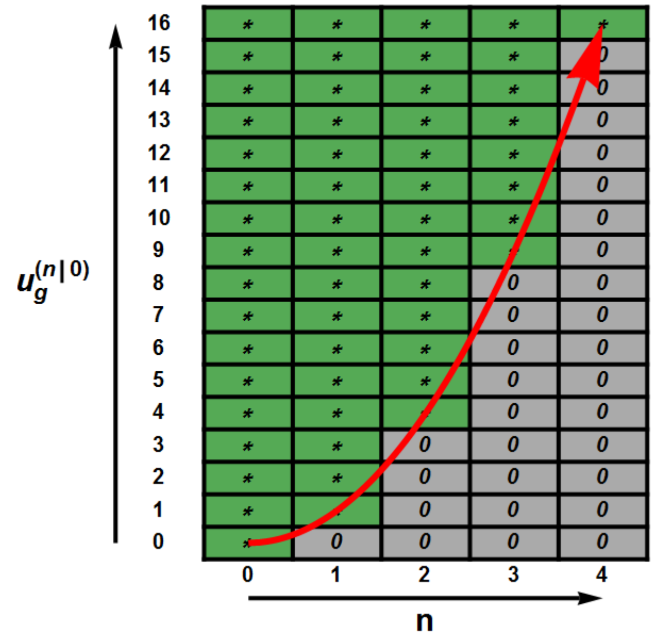


5. Analytic transseries summation

quadratic case

Quadratic summation

Again, one can find a closed form for the leading coefficients. This gives the sum



$$Z_0(\zeta, q) = \sum_{n=0}^{\infty} G_2(n+1)\zeta^n q^{n^2}$$

Here, G_2 is the Barnes function ("superfactorial") and

$$\zeta \equiv i\frac{2^{\frac{1}{2}}}{3^{\frac{1}{4}}}\sigma_1 e^{-A/x}, \qquad q \equiv i\frac{1}{2^{\frac{5}{2}}3^{\frac{3}{4}}}\sqrt{x}$$

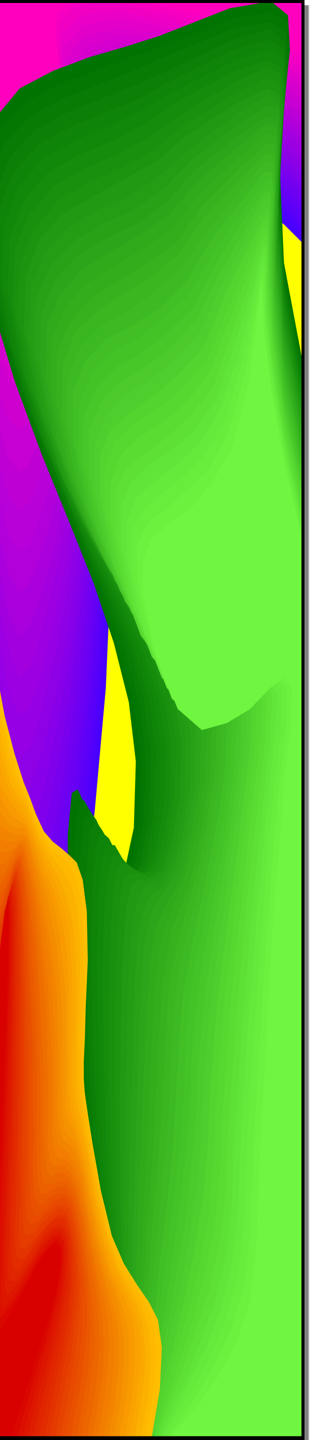
Quadratic summation

For the **1-parameter case**, this and similar q^2-expansions (and their g_s-corrections) have appeared in the literature before.

Bonnet, David, Eynard – 2000
Mariño, Schiappa, Weiss – 2008
Eynard, Mariño – 2008

In fact, close relations to **modularity**; more about this in the conclusions.

However, as we will see later, using the correct strategy it now becomes easy to include the **second parameter**!



Quadratic summation

Including *x*- (or g_s-) corrections then gives an expression of the form

$$Z(x;\zeta,q) = Z_0(\zeta,q) + xZ_1(\zeta,q) + x^2 Z_2(\zeta,q) + \dots$$

Note the philosophy:

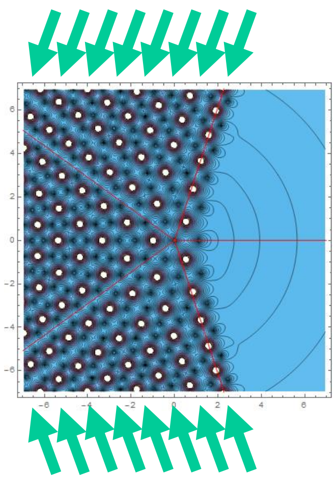
1. Introduce **additional** transmonomial $q \sim x^{1/2}$,
2. The ordering is now $x < q << \zeta$,
3. Judiciously **re-express** terms in q, ζ and x,
4. Sum (q,ζ)-expressions **exactly**.

Analytic transseries summation

Quadratic summation

Now, we find (to first order in x) the locations of **all** zeroes just from the first order analytic transseries summation!

$$Z_0(\zeta, q) = \sum_{n=0}^{\infty} G_2(n+1)\zeta^n q^{n^2}$$



Small print: this only works in the two adjoining sectors. To get into the fifth sector, we need a Stokes transition.

Quadratic summation

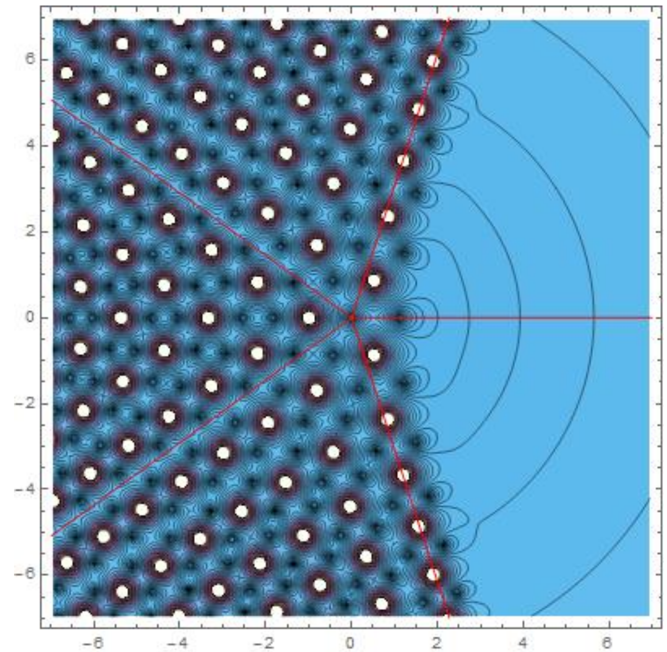
Thus, in the **quadratic** case we have this:

$$\tau_{\text{arr. 1}} = 1 + \# x + \# x^2 + \# x^3 + \ldots$$

$$\tau_{\text{arr. 2}} = \# x^{-1} + \# + \# x + \# x^2 + \ldots$$

$$\tau_{\text{arr. 3}} = \# x^{-2} + \# x^{-1} + \# + \# x + \ldots$$

$$\tau_{\text{arr. 4}} = \# x^{-3} + \# x^{-2} + \# x^{-1} + \# + \ldots$$



6. The second parameter

The first parameter revisited

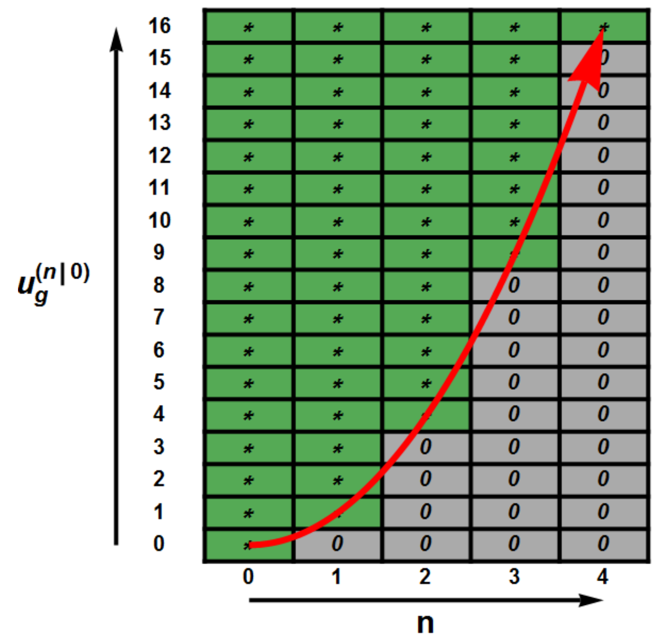
Summing higher g_s (or x-) corrections:

$$\mathcal{O}(g_s^0): \qquad Z_0^{(0)} = \sum_{n=0}^{\infty} G_2(n+1) q^{n^2} \zeta^n$$

$$\mathcal{O}(g_s^1): \qquad Z_1^{(0)} = \sum_{n=0}^{\infty} G_2(n+1) q^{n^2} \zeta^n \; p_1(n)$$

$$\mathcal{O}(g_s^1): \qquad Z_2^{(0)} = \sum_{n=0}^{\infty} G_2(n+1) q^{n^2} \zeta^n \; p_2(n)$$

$$\vdots$$



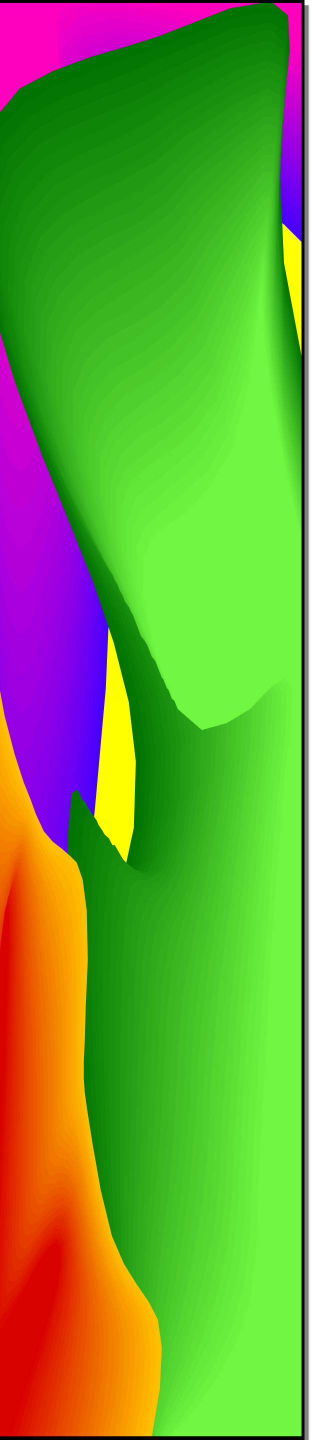
with

$$p_1(n) = -\frac{1}{192\sqrt{3}} \left(94n^3 + 17n\right)$$

$$p_2(n) = -\frac{1}{1105920} \left(44180n^6 + 170320n^4 + 74985n^2 + 1344\right)$$

$$\vdots$$

Mariño, Schiappa, Weiss – 2008



The second parameter

Summing leading σ_2-corrections:

$$\mathcal{O}(\sigma_2\, g_s^0): \qquad Z_0^{(1)} = \sum_{n=0}^{\infty} G_2(n+1) q^{n^2} \zeta^n \quad \phi_1(n)$$

$$\mathcal{O}(\sigma_2\, g_s^1): \qquad Z_1^{(1)} = \sum_{n=0}^{\infty} G_2(n+1) q^{n^2} \zeta^n \left(\phi_1(n) p_1(n) + p_1'(n)\right)$$

$$\mathcal{O}(\sigma_2\, g_s^2): \qquad Z_2^{(1)} = \sum_{n=0}^{\infty} G_2(n+1) q^{n^2} \zeta^n \left(\phi_1(n) p_2(n) + p_2'(n)\right)$$

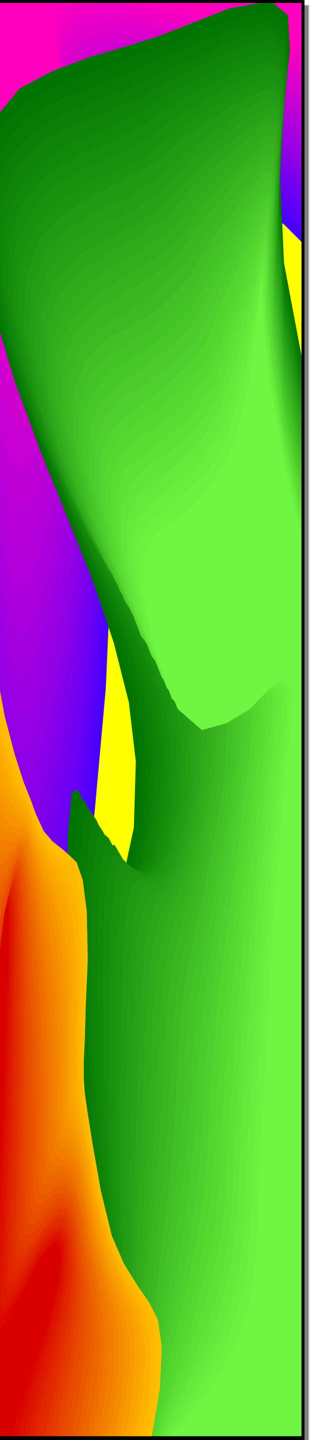
$$\vdots$$

m \ n	0	1	2	3	4	5	6
6	36	25	16	9	4	1	−12
5	25	16	9	4	1	−10	1
4	16	9	4	1	−8	1	4
3	9	4	1	−6	1	4	9
2	4	1	−4	1	4	9	16
1	1	−2	1	4	9	16	25
0	0	1	4	9	16	25	36

with the **same** polynomials $p_i(n)$ and

$$\phi_1(n) = \frac{2}{\sqrt{3}} \sum_{k=1}^{n-1} \frac{k}{n-k}$$

$$= \frac{2}{\sqrt{3}} n \left(\psi^{(0)}(n+1) - \psi^{(0)}(1) - 1\right)$$



The second parameter

Summing subleading σ_2-corrections:

$$\mathcal{O}(\sigma_2^2\, g_s^0): \qquad Z_0^{(2)} = \sum_{n=0}^{\infty} G_2(n+1) q^{n^2} \zeta^n \quad \phi_2(n)$$

$$\mathcal{O}(\sigma_2^2\, g_s^0): \qquad Z_1^{(2)} = \sum_{n=0}^{\infty} G_2(n+1) q^{n^2} \zeta^n \left(\phi_2(n) p_1(n) + \phi_1(n) p_1'(n) + \frac{1}{2} p_1''(n)\right)$$

$$\mathcal{O}(\sigma_2^2\, g_s^0): \qquad Z_2^{(2)} = \sum_{n=0}^{\infty} G_2(n+1) q^{n^2} \zeta^n \left(\phi_2(n) p_2(n) + \phi_1(n) p_2'(n) + \frac{1}{2} p_2''(n)\right)$$

$$\vdots$$

where now

$$\phi_2(n) = \frac{2}{3}\left(n\left(\psi^{(1)}(n+1) - \psi^{(1)}(1)\right) + \left(\psi^{(0)}(n+1) - \psi^{(0)}(1)\right) + \phi_1(n)^2\right)$$

We get a closed form for **any** order in σ_2!

The second parameter

It appears (as for adding g_s-corrections) we can add a single "enhanced instanton" ($e^{+nA/x}$) sector by acting with a **derivation**:

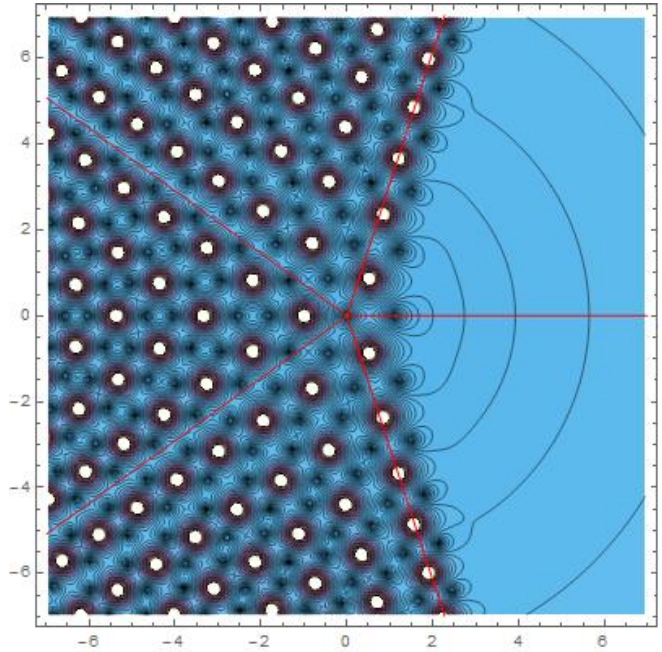
$$Z^{(n)} = \frac{1}{n!}\delta^n \left[Z^{(0)} \right]$$

As a result, we can write the **full** partition function in cartoon form as:

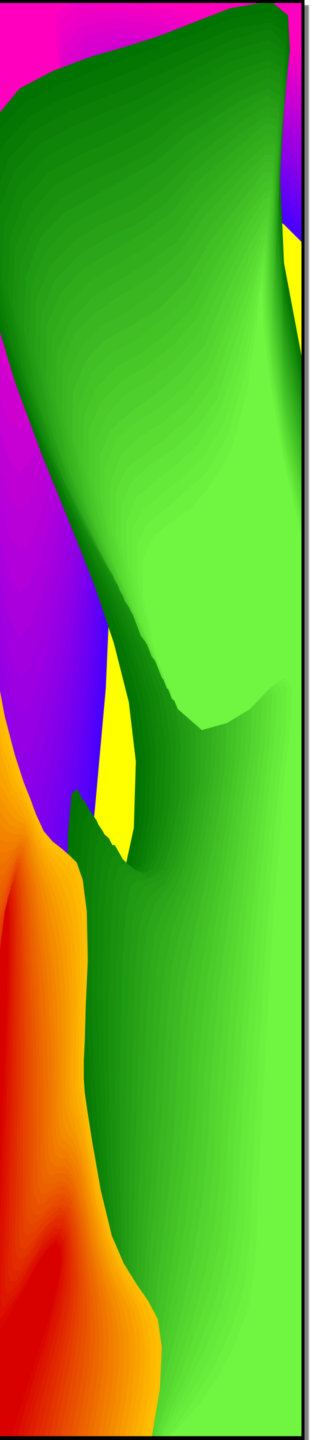
$$Z(x;\sigma_1,\sigma_2) = Z_{\text{diag}}(x;\sigma_1\sigma_2) \times e^{\sigma_2\delta}\left[Z^{(0)}(x;\sigma_1)\right] \times e^{\sigma_1\hat{\delta}}\left[\hat{Z}^{(0)}(x;\sigma_2)\right]$$

Here, hatted quantities refer to the upper diagonal sectors.

m \ n	0	1	2	3	4	5	6
6	36	25	16	9	4	1	−12
5	25	16	9	4	1	−10	1
4	16	9	4	1	−8	1	4
3	9	4	1	−6	1	4	9
2	4	1	−4	1	4	9	16
1	1	−2	1	4	9	16	25
0	0	1	4	9	16	25	36



7. Conclusion and outlook



Conclusion

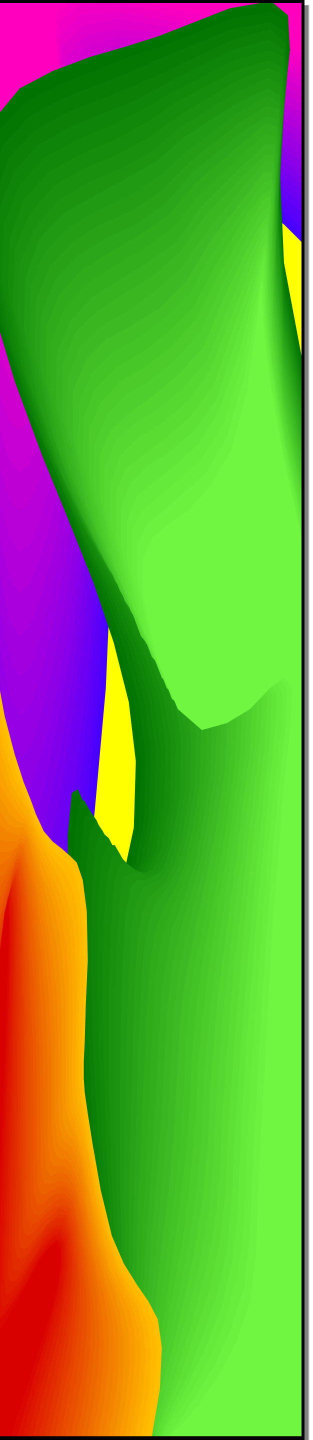
- Problems become tractable when the correct (here: quadratic) **analytic transseries summation** is used.
- Can sum the **full** transseries this way.
- In particular, this immediately gives us all poles for Painlevé I. In physics terms, we can study **phase transitions** where nonperturbative effects start competing with perturbative ones.

Topological strings

The **topological string** would be an interesting system to apply these techniques to.

HAE: Couso-Santamaría, Edelstein, Schiappa, Vonk – 2013, 2014
Couso-Santamaría – 2015
SC: Couso-Santamaría, Mariño, Schiappa – 2016
Codesido, Mariño, Schiappa – to appear

Here: no nonperturbative definition, but instead a family of solutions. Start from transseries; turn it into a function for yet undetermined σ_i. “**Semiclassics recoded**”.



Remarks on modularity

$$Z_0(\zeta, q) = \sum_{n=0}^{\infty} G_2(n+1)\zeta^n q^{n^2}$$

The q^2-expansions have a modularity flavor. In “filled cuts” matrix model, theta functions appear. Here, a related object appears to be the Weierstrass σ-function.

Very divergent coefficients, but

- Sum for **small q** can be done,
- Looks like a closed form for **zeroes** can be found.

Further open questions

- Extension to **matrix models**: in progress.
- Can the **diagonal sector** be written as $e^{\delta\ \sigma 1\ \sigma 2}\ [\ Z_{pert}\ ]$?
- Can we **classify** problems according to linear, quadratic, (cubic, …?) analytic transseries summation?

Thank you!