Relativistic electron vortices
beyond the paraxial approximation

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A paraxial approximation in optics:

\[ \sigma_{\perp} \gg \lambda, \]
\[ k_{\perp} \ll k_z. \]

Naively, for massive wave-packets:

\[ \sigma_{\perp} \gg \lambda_{dB} = \frac{2\pi}{p}, \]
\[ p_{\perp} \ll p_z. \]

An example:
The Laguerre-Gaussian beams of electrons [see e.g., K. Y. Bliokh, et al., Physics Reports 690, 1 (2017)]

The problem: the lack of Lorentz invariance!
A Lorentz invariant condition of paraxiality:

\[ \sigma_\perp \gg \lambda_c = \frac{\hbar}{mc}, \]
\[ \sigma \sim \frac{1}{\sigma_\perp} \ll m, \]

That is, \( \lambda_{dB}(\neq \text{inv}) \rightarrow \lambda_c = \frac{\hbar}{mc}(= \text{inv}) \)

The vacuum is stable!

A non-paraxial Gaussian scalar packet is


\[
\psi(p) = \frac{2^{3/2} \pi}{\sigma} \frac{e^{-m^2/\sigma^2}}{\sqrt{K_1(2m^2/\sigma^2)}} \exp \left\{ \frac{(p - \bar{p})^2}{2\sigma^2} \right\}
\]

\[ \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\varepsilon} |\psi(p)|^2 = 1, \]

And when \( \sigma \ll m \), one has:

\[ \frac{(p - \bar{p})^2}{2\sigma^2} = -\frac{1}{2\sigma^2} (\delta_{ij} - \bar{u}_i \bar{u}_j) (p - \bar{p})_i (p - \bar{p})_j \]
Is the non-paraxial description really necessary?

\[ \sim \frac{\sigma^2}{m^2} \sim \frac{\chi^2_c}{\sigma^2_\perp} \ll 1 \]

For an LHC beam, it is less than \(10^{-22}\)

For modern electron accelerators: \(10^{-14}\) (and some 2-3 orders larger for ILC and CLIC)

For electron microscopes: \(10^{-6}\) (!)

However, these estimates are valid only for Gaussian packets!
The condition of paraxiality is different for vortex electrons with orbital angular momentum $\ell$:

$$\frac{\sigma^2}{m^2} \ll 1 \rightarrow |\ell|\frac{\sigma^2}{m^2} \ll 1,$$

or, alternatively, $\lambda_c \rightarrow \sqrt{|\ell|} \lambda_c \gg \lambda_c$ when $\ell \gg 1$

$$|\ell|\frac{\sigma^2}{m^2} = |\ell|\frac{\lambda_c^2}{\sigma^2(0)} \approx \ell^2 \frac{\lambda_c^2}{\langle \rho \rangle^2} \quad \text{Grows with the OAM!}$$

The current record is


$$\ell \sim 10^3$$

As a result,

$$|\ell|\frac{\sigma^2}{m^2} \text{ can reach } \sim 10^{-3}(!)$$
Vortex particles with orbital angular momentum (OAM)


A Bessel state of a free scalar particle:

$$\psi(r) = N J_\ell (\kappa \rho) e^{-i\omega t + ip_\parallel z + i\ell \phi_z}$$

Twisted photons: Allen, et al. 1992
Vortex particles with orbital angular momentum (OAM)

They form a complete and orthogonal set:

\[
\langle p'_{\parallel}, \kappa', \ell' | p_{\parallel}, \kappa, \ell \rangle = (2\pi)^2 2 \varepsilon(p) \delta(p - p') \frac{\delta(\kappa - \kappa')}{\kappa} \delta_{\ell \ell'}
\]

\[
\hat{\psi}(x) = \sum_{\ell} \int \frac{dp_{\parallel}\kappa d\kappa}{(2\pi)^2 2 \varepsilon} \langle x | p_{\parallel}, \kappa, \ell \rangle \hat{a}_{\{p_{\parallel}, \kappa, \ell\}} + \text{H.c.}
\]

\[
= \sum_{\ell} \int \frac{dp_{\parallel}\kappa d\kappa}{(2\pi)^2 \sqrt{2 \varepsilon}} (J_\ell(\kappa \rho) e^{-i \varepsilon t + ip_{\parallel} z + i \ell \phi_r} \hat{a}_{\{p_{\parallel}, \kappa, \ell\}} + \text{H.c.})
\]

\[
\left[ \hat{a}_{\{p_{\parallel}, \kappa, \ell\}}, \hat{a}^\dagger_{\{p'_{\parallel}, \kappa', \ell'\}} \right] = (2\pi)^2 \delta(p_{\parallel} - p'_{\parallel}) \frac{\delta(\kappa - \kappa')}{\kappa} \delta_{\ell \ell'}
\]

\[
[\hat{\psi}(x), \hat{\psi}^\dagger(x')] = \sum_{\ell} \int \frac{dp_{\parallel}\kappa d\kappa}{(2\pi)^2 2 \varepsilon} J_\ell(\kappa \rho) J_\ell(\kappa' \rho') (e^{-i \varepsilon (t - t') + ip_{\parallel} (z - z') + i \ell (\phi_r - \phi_{r'})} - \text{c.c.}).
\]

D.K., PRA 91 (2015) 013847
Why do the Bessel beams *not* describe the non-paraxial effects for large OAM?

1. The packet’s width, length, and energy-momentum uncertainties are ignored (unlocalized orthogonal states)

2. The mean transverse momentum *is independent* of the OAM

Why do the Laguerre-Gaussian beams *not* describe the non-paraxial effects for large OAM?

1. They are intrinsically paraxial

Non-paraxial relativistic wave packets with OAM are needed!
There are two recent papers where an attempt to achieve this was made:

Relativistic Electron Wave Packets Carrying Angular Momentum

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There are important differences between the nonrelativistic and relativistic description of electron beams. In the relativistic case the orbital angular momentum quantum number cannot be used to specify the wave functions and the structure of vortex lines in these two descriptions is completely different. We introduce analytic solutions of the Dirac equation in the form of exponential wave packets and we argue that they properly describe relativistic electron beams carrying angular momentum.

DOI: 10.1103/PhysRevLett.118.114801

Relativistic Electron Vortices

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The desire to push recent experiments on electron vortices to higher energies leads to some theoretical difficulties. In particular the simple and very successful picture of phase vortices of vortex charge $\ell^2\hbar$ associated with $\ell\hbar$ units of orbital angular momentum per electron is challenged by the facts that (i) the spin and orbital angular momentum are not separately conserved for a Dirac electron, which suggests that the existence of a spin-orbit coupling will complicate matters, and (ii) that the velocity of a Dirac electron is not simply the gradient of a phase as it is in the Schrödinger theory suggesting that, perhaps, electron vortices might not exist at a fundamental level. We resolve these difficulties by showing that electron vortices do indeed exist in the relativistic theory and show that the charge of such a vortex is simply related to a conserved orbital part of the total angular momentum, closely related to the familiar situation for the orbital angular momentum of a photon.

DOI: 10.1103/PhysRevLett.118.114802
Massive scalar packets with the OAM:

\[
\psi_{\ell}(p) = \frac{2^{3/2} \pi}{\sigma^{|\ell|+1} \sqrt{|\ell|!}} p^{\ell} \frac{e^{-m^2/\sigma^2}}{\sqrt{K_{|\ell|+1}(2m^2/\sigma^2)}} \exp \left\{ \frac{(p - \overline{p})^2}{2\sigma^2} + i\ell \phi_p \right\}
\]

They are orthogonal in OAM:

\[
\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\varepsilon} [\psi_{\ell'}(p)]^* \psi_{\ell}(p) = \delta_{\ell,\ell'}
\]

An exact solution to the Klein-Gordon equation:

\[
\psi_{\ell}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\varepsilon} \psi_{\ell}(p) e^{-ipx} = \frac{(i\rho)^{|\ell|}}{\sqrt{2|\ell|! \pi}} \frac{\sigma^{|\ell|+1}}{\sqrt{K_{|\ell|+1}(2m^2/\sigma^2)}} \frac{K_{|\ell|+1}(\zeta m^2/\sigma^2)}{\zeta^{|\ell|+1}} e^{i\ell \phi_r}
\]

\[
\zeta = \frac{1}{m} \sqrt{(\overline{p}_\mu + ix_\mu \sigma^2)^2} = \text{inv}, \quad \text{Re} \zeta > 0
\]

And analogously for a fermion with \( \langle \hat{j}_z \rangle = \ell + \lambda \)
The mean 4-momentum is

\[
\langle p^\mu_\ell \rangle = \{\langle \varepsilon_\ell \rangle, \langle p_\ell \rangle \} = \{\bar{\varepsilon}, \bar{p}\} \frac{K|\ell|+2}{K|\ell|+1} \frac{(2m^2/\sigma^2)}{(2m^2/\sigma^2)} \approx \{\bar{\varepsilon}, \bar{p}\} \left(1 + \left(\frac{3}{4} + \frac{|\ell|}{2}\right) \frac{\sigma^2}{m^2}\right)
\]

An invariant mass of this packet:

\[
m^2_\ell = \langle p_\ell \rangle^2 \simeq m^2 \left(1 + \left(\frac{3}{2} + \frac{|\ell|}{2}\right) \frac{\sigma^2}{m^2}\right)
\]

An enhancement due to the OAM!

What can already be achieved:

\[
\frac{\delta m_\ell}{m_{\text{inv}}} \approx \frac{\delta m_\ell}{m} \lesssim 10^{-3} \quad |\ell| \sim 10^3 \quad \sigma_\perp \gtrsim 0.1 \text{ nm}
\]
For the vortex electron’s magnetic moment:

\[ \mu_f = \frac{1}{2} \int d^3r \, \mathbf{r} \times \overline{\psi}_f(x) \gamma \psi_f(x) \simeq \frac{1}{2\xi} (\zeta + \hat{z} \ell) \left( 1 + \mathcal{O}(|\ell|\sigma^2/m^2) \right) \]

A large magnetic moment due to the OAM


(described by the Bessel beam)

An enhancement of the spin-orbit interaction due to the OAM!

(not described by the Bessel beam)
\[ \mu_f = \mu_s + \mu_b. \]

Bosonic (OAM) contribution:

\[ \mu_b = \frac{1}{2} \left< u \times \frac{\partial \varphi_\ell(p)}{\partial p} \right> = \hat{\varphi} \ell \left< \frac{1}{2 \varepsilon} \right> \approx \hat{\varphi} \ell \frac{1}{2 \varepsilon} \left( 1 - \frac{\sigma^2}{2m^2} \left( |\ell| + \frac{1}{2} + \frac{m^2}{\varepsilon^2} \right) \right). \]

Spin contribution:

\[ \mu_s = \left< \frac{1}{(2\varepsilon)^2} \left( \zeta(\varepsilon + m) + \frac{P(P\zeta)}{\varepsilon + m} \right) \right> \approx \zeta \frac{1}{2\varepsilon} \left( 1 - \frac{\sigma^2}{2m^2} \left[ \frac{1}{2} + \frac{3m}{2\varepsilon} + \frac{1}{2} \varepsilon^2 - \frac{3}{2} \varepsilon^3 \right] + \frac{m}{\varepsilon + m} \left( \frac{3}{2} - 2\frac{m^2}{\varepsilon^2} - \frac{3}{2} \frac{m^3}{\varepsilon^3} \right) \right. \]
\[ \left. + |\ell| \left( 1 + \frac{m}{\varepsilon} - \frac{m}{\varepsilon + m} \right) \right) \]

The spin-orbit interaction is also \(|\ell|\) times enhanced compared to the Bessel beam!

The reason:

\[ \left< p_\perp \right> \approx \sigma \sqrt{|\ell|} \text{ when } |\ell| \gg 1. \]

The conical angle now grows with the OAM:

\[ |\ell| \frac{\sigma^2}{m^2} \approx \frac{\left< p_\perp \right>^2}{m^2} = \left( \frac{\varepsilon^2}{m^2} - 1 \right) \tan^2 \theta_0 \approx \ell^2 \frac{\lambda_c^2}{\left< \rho \right>^2}. \]
In the paraxial regime, these packets reduce to the invariant Laguerre-Gaussian beams with $n = 0$:

$$
\psi_{\ell}^{\text{par}}(x) = \frac{i^\ell}{\sqrt{|\ell|!}} \frac{1}{\sqrt{2m}} \left( \frac{\sigma}{\sqrt{\pi}} \right)^{3/2} \frac{(\rho/\sigma_{\perp}(t))^{|\ell|}}{(1 + (t/t_d)^2)^{3/4}} \exp \left\{ i\ell \phi_r - i\bar{p}^\mu x_\mu - 
- i (|\ell| + 3/2) \arctan \left( \frac{t}{t_d} \right) - \frac{1}{2\sigma_{\perp}^2(t)} \left( 1 - it/t_d \right) \left( \rho^2 + \frac{\bar{\varepsilon}^2}{m^2} (z - \bar{u}t)^2 \right) \right\},
$$

- $\sigma_{\perp}(t) = \sigma^{-1} \sqrt{1 + (t/t_d)^2}$
- $t/t_d$ (where $t_d = \bar{\varepsilon}/\sigma^2$ is a diffraction time)
- $\rho^2 + \bar{\varepsilon}^2(z - \bar{u}t)^2/m^2$

A Gouy phase $(|\ell| + 3/2) \arctan (t/t_d)$:

- Is also Lorentz invariant
- In contrast to twisted photons, it depends on $t$ rather than on $z$
- It has $3/2$ instead of $1$ ($= 2/2$) due to localization in a 3D space
The Laguerre-Gaussian beams with $n \neq 0$:

$$\psi_{\ell,n}^{\text{par}}(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\varepsilon} \psi_{\ell,n}^{\text{par}}(p) e^{-ipx} = \sqrt{\frac{n!}{(|\ell| + n)!}} \frac{i^{2n+\ell}}{\pi^{3/4} \sqrt{2m}} \frac{\rho/\sigma_\perp(t)|^\ell}{\sigma_\perp^{3/2}(t)} L_n^{\ell}(\rho^2/\sigma_\perp^2(t)) \times \exp \left\{ i\ell \phi_r - i \vec{P}_\mu x^\mu - i(2n + |\ell| + 3/2) \arctan(t/t_d) - \frac{1}{2\sigma_\perp^2(t)} \left(1 - i \frac{t}{t_d}\right) \left(\rho^2 + \varepsilon^2(z - \bar{u}t)^2/m^2\right) \right\},$$

$$\int d^3 r \ 2\varepsilon |\psi_{\ell,n}^{\text{par}}(x)|^2 = 1, \quad \text{Invariant normalization}$$

A mean radius of the beam:

$$\langle \rho \rangle = \sigma_\perp(t) \frac{\Gamma(n + |\ell| + 3/2)}{\Gamma(n + |\ell| + 1)} \approx \sigma_\perp(t) \sqrt{|\ell|}, \quad |\ell| \gg n$$

The beam width grows with the OAM and the corresponding effects can be described only beyond the paraxial approximation!
The differences of these packets from those of Barnett
[S. M. Barnett, PRL 118, 114802 (2017)]:

These packets

1.) Applicable beyond the paraxial regime
2.) Lorentz invariant (scalar)
3.) The Gouy phase depends on \( t \)
   (applicable also for non-relativistic particles!)
4.) Correct description of spin
   (no issues with the spin operators)

Those of Barnett

1.) Not applicable beyond the paraxial regime
2.) Non-invariant
3.) The Gouy phase depends on \( z \)
   \((only\ for\ ultra-relativistic\ particles\ with\ z = t)\)
4.) Incorrect description of spin
   (see K. Bliokh et al., Physics Reports 690, 1 (2017)

\[
u_{\ell,n} = \sqrt{\frac{2n!}{\pi (n + |\ell|)!}} \frac{(\rho \sqrt{2})^{|\ell|}}{w^{|\ell| + 1}(z)} \exp \left( - \frac{p_0 \rho^2}{2(z_R + iz)} \right) \\
\times L_n^{|\ell|} \left( \frac{2\rho^2}{w^2(z)} \right) e^{i\ell \phi} \exp \left[ -i(2n + |\ell| + 1)\tan^{-1}(z/z_R) \right],
\]

(12)
The differences of these packets from those of Bialynicki-Birula and Bialynicka-Birula [I. Bialynicki-Birula, Z. Bialynicka-Birula, PRL 118, 114801 (2017)]:

**These packets**

1.) Applicable beyond the paraxial regime

2.) Lorentz invariant (scalar)

3.) At large distances decay as

\[ \psi_\ell(x) \propto \exp \left\{ -\sqrt{-x^2_{\mu}/\lambda_c} \right\} \]

**Those of Bialynicki-Birula**

1.) Also applicable ...

2.) Not invariant

3.) At large distances decay as (see Eq.(15))

\[ \psi_\ell(x) \propto \exp \left\{ -\frac{\varepsilon}{m} \rho/\lambda_c \right\} \]

The troubles with this behavior:

1. The decay is *not* Lorentz invariant

2. Worse: in relativistic case, the beam width

\[ \text{is smaller than the Compton wavelength} \] – impossible!
Summary

1) We put forward the wave packets of vortex bosons and fermions that preserve Lorentz invariance and are applicable beyond the paraxial regime.

2) In the paraxial approximation, these packets reduce to the invariant Laguerre-Gaussian beams.

3) There are notable differences of the latter from the twisted photons, relevant especially for non-relativistic particles.

4) Compared to packets with the non-singular phases (Gaussian, Airy, etc.), the non-paraxial effects turn out to be drastically enhanced for large orbital angular momenta.
So what?

I. Unlike the Bessel states, the wave packets do interfere, which is crucial for potential applications with entangled beams (interaction-free measurements, probing of Coulomb/hadronic phases, etc.)

II. The non-paraxial corrections can be important for describing:

- The spin-orbit interaction in relativistic highly twisted beams
- Scattering of wave packets by atoms, including quantum superpositions thereof (Schroedinger cats of electrons)
- Beam-beam collisions of the highly energetic twisted beams (say, the Compton effect)

These corrections can already compete with a contribution of the 2nd loop in QED:

\[ \frac{d\sigma^{(1)}}{d\sigma_{pw}} \sim |\ell|\sigma^2/m^2 \gtrsim \alpha_{em}^2 = 1/137^2 \]