A DIFFERENT ANGLE ON QUANTUM UNCERTAINTY

[ THE MEASURE ANGLE ]

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\[ |\psi\rangle, \{ |i\rangle \} : \text{How many } |i\rangle \text{ in } |\psi\rangle? \]
Anderson localization
many-body localization
effective counting
effective description of quantum states
effective supports
vacuum structure of gauge theories
topological insulators
quantum Hall Effect
holography
ETH & thermalization
Dirac eigenmodes
quantum information
quantum uncertainty
probability theory (effective choices)
measures of diversity
efficiency of quantum computation
measure theory (effective measure)
CANONICAL CONTEXT: spinless lattice Schrödinger particle wrt position basis

\[ |\psi\rangle \rightarrow \{ \psi_i \mid i = 1, \ldots, N \} \quad \text{How many positions (N) is particle simultaneously in?} \]

Quantum mechanics: \[ |\psi\rangle \rightarrow P = (p_1, p_2, \ldots p_N) \quad p_i = | \langle i | \psi \rangle |^2 = \psi_i^* \psi_i \]

What is \( N[|\psi\rangle, \{|i\}\}] = N[P] \) ???

Options:

(i) ill-posed question in QM

(ii) well-posed question in QM (how?)
Important convenience:

\[ P = (p_1, \ldots, p_N) \rightarrow W \equiv NP = (w_1, \ldots, w_N) \quad \text{[counting vector]} \]

\[ \mathcal{N} = \mathcal{N}[W] : W \in \mathcal{W} \equiv \{ (w_1, \ldots, w_N) \mid w_i \geq 0, \sum_{i=1}^{N} w_i = N, N \in \mathbb{N} \} \]

Participation Number: \qquad \text{Bell & Dean, 1970}

\[ \frac{1}{\mathcal{N}_p[W]} = \frac{1}{N^2} \sum_{i=1}^{N} w_i^2 \]

used profusely in localization studies to this day

\[ \mathcal{N}_p \notin \mathcal{N} \]

participation number doesn’t count
OLD KEY INGREDIENT : MONOTONICITY

enhancing the cumulation of probability cannot increase the effective number

\[
\mathcal{N}(\ldots w_i - \epsilon \ldots w_j + \epsilon \ldots) \leq \mathcal{N}(\ldots w_i \ldots w_j \ldots), \quad w_i \leq w_j
\]

monotonicity wrt cumulation
NEW KEY INGREDIENT: ADDITIVITY

\[ N_1, W_1 \quad N_2, W_2 \]

\[ \mu(S_1 \cup S_2) = \mu(S_1) + \mu(S_2) \quad S_1 \cap S_2 = \emptyset \]

\[ N_{12} = N_1 + N_2 \]

\[ \mathcal{N}[W_{12}] = \mathcal{N}[W_1] + \mathcal{N}[W_2] \]

Note: \[ W_1 \in \mathcal{W}_{N_1}, W_2 \in \mathcal{W}_{N_2} \Rightarrow W_1 \boxdot W_2 \in \mathcal{W}_{N_1+N_2} \]

\[ (a_1, \ldots, a_N) \boxdot (b_1, \ldots, b_M) \equiv (a_1, \ldots, a_N, b_1, \ldots, b_M) \]

\[ \mathcal{N}[W_1 \boxdot W_2, N_1 + N_2] = \mathcal{N}[W_1, N_1] + \mathcal{N}[W_2, N_2], \quad \forall W_1, W_2, N_1, N_2 \]
EFFECTIVE NUMBERS

(A) \( \mathcal{N}[W_1 \boxplus W_2] = \mathcal{N}[W_1] + \mathcal{N}[W_2] \)

(S) \( \mathcal{N}(\ldots w_i \ldots w_j \ldots) = \mathcal{N}(\ldots w_j \ldots w_i \ldots) \)

(B1) \( \mathcal{N}(1, 1, \ldots, 1) = \mathcal{N} \)

(B2) \( \mathcal{N}(N, 0, \ldots, 0) = 1 \)

(B) \( 1 \leq \mathcal{N}[W] \leq N \)

(C) \( \mathcal{N}[W] \) is continuous on \( \mathcal{W} \)

(M-) \( \mathcal{N}(\ldots w_i - \epsilon \ldots w_j + \epsilon \ldots) \leq \mathcal{N}(\ldots w_i \ldots w_j \ldots) \), \( w_i \leq w_j \)

monotonicity wrt cumulation

\( \mathcal{N} \) : set of functions satisfying (A), (S), (B2), (C), (M-)

[ (B1) and (B) follow ]

\( \mathcal{N}_p \notin \mathcal{N} \) (not additive)
THE CONSISTENCY GAME

number of objects
natural numbers

number of objects with weights
effective numbers

\[
\begin{array}{c}
\begin{array}{c}
[\ldots]\quad \longrightarrow \quad \mathbb{N}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\bullet \quad \longrightarrow \quad 1 \\
\bullet \bullet \quad \longrightarrow \quad 2 \\
\bullet \bullet \bullet \quad \longrightarrow \quad 3 \\
\bullet \bullet \bullet \ldots \quad \longrightarrow \quad \mathbb{N}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
[\bullet] \quad \sqcup \quad [\bullet \bullet] \quad \longrightarrow \quad [\bullet \bullet \bullet]
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
2 \quad + \\
3
\end{array}
\end{array}
\]

\[
5
\]

\[
\begin{array}{c}
\begin{array}{c}
\mathbb{N} \quad [W] \quad + \\
\mathbb{N} \quad [V]
\end{array}
\end{array}
\]

\[
\mathbb{N} \quad [W \sqcup V]
\]
Theorem

There are infinitely many elements in $\mathbb{N}$ and there exists $N_* \in \mathbb{N}$ such that

(a) $N_*[W] \leq N[W] \leq N_+[W] \quad \forall N \in \mathbb{N}, \quad \forall W \in \mathcal{W}$

(b) $N_*[W] < N_+[W] \implies \{ N[W] \mid N \in \mathbb{N} \} \supseteq [\alpha, \beta)$

where $W$ in (b) is arbitrary but fixed and $\alpha = N_*[W], \beta = N_+[W]$.

$N_*[W] = \sum_{i=1}^{N} n_*(w_i) \quad n_*(w) \equiv \min\{ w, 1 \}$

$N_+[W] = \sum_{i=1}^{N} n_+(w_i) \quad n_+(w) \equiv \begin{cases} 0, & w = 0 \\ 1, & w > 0 \end{cases} \quad N_+ \notin \mathbb{N}$

- THERE IS A “MINIMAL AMOUNT” OF OBJECTS WITH PROBABILITY WEIGHTS [ least element ]
- CONSISTENT EFF. AMOUNTS UP TO THE NUMBER OF NON-ZERO WEIGHTS [ no structure at the top: $\mathcal{N}_*$ represents the actual content of the concept ]
- CONTINUUM OF EFFECTIVE COUNTING SCHEMES PERFECTLY NATURAL
WE NOW KNOW HOW TO COUNT WITH PROBABILITIES!

How did you say it works?

Example: Buying one hat, 6 choices, assign preferences (probabilities)

\[
P = (0.01, 0.02, 0.10, 0.15, 0.30, 0.42)
\]

\[
W = (0.06, 0.12, 0.60, 0.90, 1.80, 2.52) \quad \text{[counting weights]}
\]

\[
N_{\star}[W] = 0.06 + 0.12 + 0.60 + 0.90 + 1.00 + 1.00 = 3.68
\]

Question: How many hats are you effectively choosing from?

Answer: About 2.5

Reply: You are a liar!

This application a basis for the notion of effective choices in probability theory.
BACK TO QUANTUM STATES:

\[ |\psi\rangle, \{ |i\rangle \} : \text{ How many } |i\rangle \text{ in } |\psi\rangle? \]

(i) ill-posed question in QM

(ii) well-posed question in QM (how?) ✓

Answer:

\[
N_\star[|\psi\rangle, \{|i\rangle\}] = N_\star[W] \quad W = (w_1, \ldots, w_N) \quad w_i = N |\langle i | \psi \rangle|^2
\]

\[
N_\star[W] = \sum_{i=1}^{N} n_\star(w_i) \quad n_\star(w) \equiv \min\{w, 1\}
\]
Quantum Uncertainty

canonical experiment: \[ \hat{\mathcal{O}} \leftrightarrow \{ (|i\rangle, O_i) \mid i = 1, 2, \ldots, N \} \quad \text{non-degenerate} \]

\[ |\psi\rangle \quad \overset{\text{measure} \hat{\mathcal{O}}}{\longrightarrow} \quad \{ (|i_\ell\rangle, O_{i_\ell}) \mid \ell = 1, 2, \ldots \} \]

uncertainty of \( |\psi\rangle \) wrt \( \hat{\mathcal{O}} \) = indeterminacy encoded by \( \{ (|i_\ell\rangle, O_{i_\ell}) \} \)

STANDARD
spread of outcomes

distance on the spectrum

metric uncertainty

\( \rho \)-uncertainty

\[ \Delta = \Delta[|\psi\rangle, \hat{\mathcal{O}}] \]

[e.g. standard deviation]

HERE

abundance of distinct outcomes

measure uncertainty

\( \mu \)-uncertainty

\[ \mathcal{N} = \mathcal{N}[|\psi\rangle, \{ |i\rangle \}] \]

[complete theory in arXiv:1807.03995]
MEASURE UNCERTAINTY PRINCIPLE

Set $\mathcal{N}$ of effective number functions $N$ exhausts all quantum $\mu$-uncertainties

$$\mathcal{N}[|\psi\rangle, \{|i\rangle\}] = \mathcal{N}[W], \quad W = (w_1, \ldots, w_N), \quad w_i = N |\langle i|\psi\rangle|^2$$

$[U_0]$ The $\mu$-uncertainty of $|\psi\rangle$ with respect to $\{|i\rangle\}$ is at least $N_\ast[W]$ states.

ECONOMICAL EXPRESSION OF FUNDAMENTAL DIFFERENCE BETWEEN QUANTUM AND CLASSICAL
When regularizations removed/continuous spectra: measure uncertainties are effective volumes

Example: Schrödinger particle in bounded region $A$ of $\mathbb{R}^D$ described by wave-function $\psi$

$$V_*[\psi] = \int_A \nu_*(x) \, d^D x \quad \nu_*(x) = \min \{ V \psi^*(x) \psi(x), 1 \}$$

QUANTUM UNCERTAINTY EXPRESSED AS A GENERALIZATION OF THE JORDAN CONTENT!

Formulation entirely general in terms of the setting (system, Hilbert space)
TAKE-AWAYS

- **IDENTITY-COUNTING PROBLEMS ARE WELL-DEFINED AND SOLVED IN QUANTUM MECHANICS**

- **BY VIRTUE OF EXTENDING THE CLASSICAL NOTION OF MEASURE VIA PROBABILITY**
  
  counting $\rightarrow$ effective counting , counting measure $\rightarrow$ diversity measure,

  Jordan content $\rightarrow$ effective Jordan content

- **CONSEQUENTLY MUCH MORE BASIC THAN “QUANTUM” SETTING**
  
  Effective counting arises classically virtually everywhere!

- **FRUITFUL EXTENSION OF QUANTUM UNCERTAINTY INTO MEASURE UNCERTAINTY**

  COMPLETELY UNDER CONTROL UNLIKE METRIC UNCERTAINTY

- **QUALITATIVELY NEW TYPE OF UNCERTAINTY PRINCIPLE**
Anderson localization

quantum uncertainty

quantum information

dirac eigenmodes

ETH & thermalization

topological insulators

effective description of quantum states

many-body localization

effective counting

effective supports

vacuum structure of gauge theories

quantum Hall Effect

holoography

quantum chaos

measure theory (effective measure)

probability theory (effective choices)

measures of diversity