A DIFFERENT ANGLE ON QUANTUM UNCERTAINTY

[THE MEASURE ANGLE]

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$$|\psi\rangle\ ,\ \{\,|i\rangle\}\quad :\quad \text{How many } |i\rangle \text{ in } |\psi\rangle\,?$$

2/3 I.H. & R. Mendris, arXiv:1807.03995

1/3 not published

quantum chaos

many-body localization

effective supports

effective description of quantum states

vacuum structure of gauge theories

topological insulators

effective counting

quantum Hall Effect

holography

ETH & thermalization

Anderson localization

efficiency of quantum computation

Dirac eigenmodes

quantum uncertainty

probability theory
(effective choices)

measure theory
(effective measure)

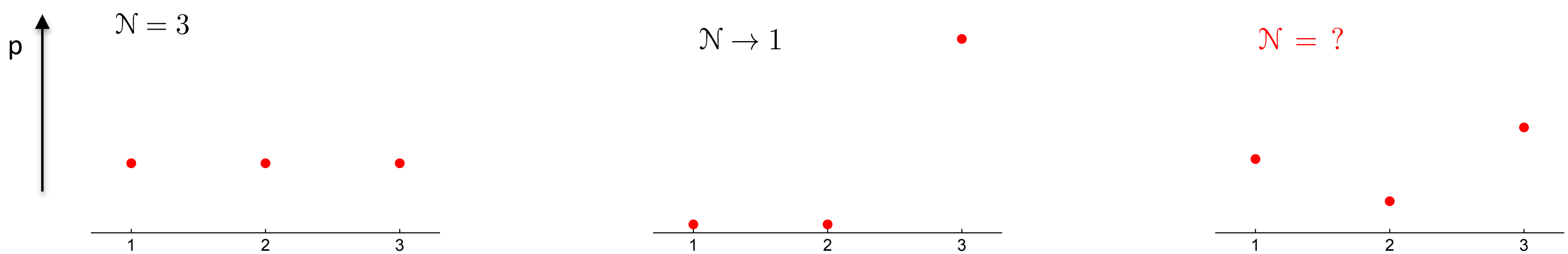
quantum information

measures of diversity

CANONICAL CONTEXT : spinless lattice Schrödinger particle wrt position basis

$|\,\psi\,\rangle \;\rightarrow\; \{\, \psi_i \;|\; i = 1, \ldots, N \,\}$ How many positions ($\mathcal{N}$) is particle simultaneously in?

Quantum mechanics: $|\psi\rangle \;\rightarrow\; P = (p_1, p_2, \ldots p_N)$ $\qquad p_i = |\,\langle i \mid \psi\rangle\,|^2 = \psi_i^\star \psi_i$



What is $\mathcal{N}[\,|\psi\rangle, \{|i\rangle\}\,] = \mathcal{N}[P]$????

Options:

(i) ill-posed question in QM

(ii) well-posed question in QM (how?)

?

STRATEGY :

(1) Axiomatically define the set $\mathfrak{N}$ of all $\mathcal{N} = \mathcal{N}[P]$ assigning effective number of states

(2) Study the content and structure of $\mathfrak{N}$

Important convenience :

$$P = (p_1, \ldots, p_N) \longrightarrow W \equiv NP = (w_1, \ldots, w_N)$$

[counting vector]

$$\mathcal{N} = \mathcal{N}[W] \quad : \quad W \in \mathcal{W} \equiv \left\{ (w_1, \ldots, w_N) \mid w_i \geq 0 \,, \sum_{i=1}^{N} w_i = N \,,\; N \in \mathbb{N} \right\}$$

Participation Number :

Bell & Dean, 1970

$$\frac{1}{\mathcal{N}_p[W]} = \frac{1}{N^2} \sum_{i=1}^{N} w_i^2$$

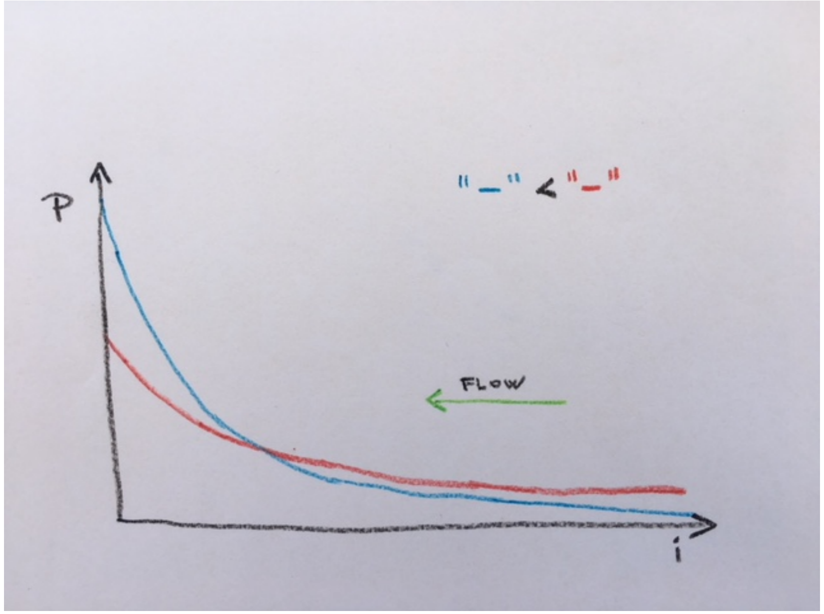
used profusely in localization studies to this day

$$\mathcal{N}_p \notin \mathfrak{N}$$

participation number doesn't count

OLD KEY INGREDIENT : MONOTONICITY

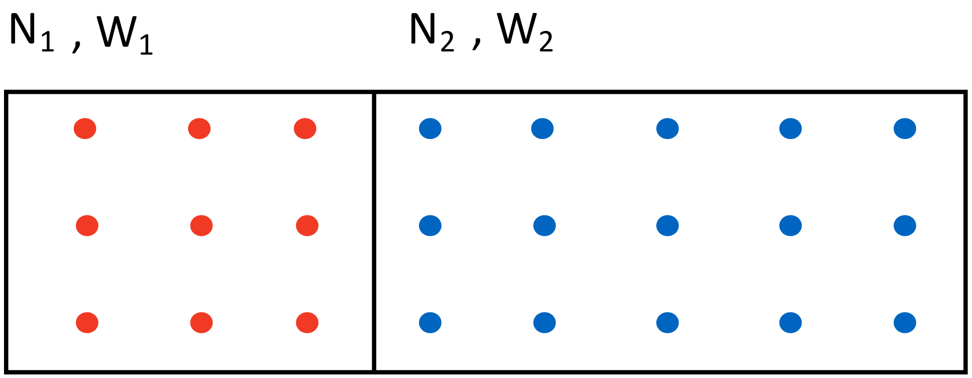
enhancing the cumulation of probability cannot increase the effective number



$$\text{(M-)} \qquad \mathcal{N}(\ldots w_i - \epsilon \ldots w_j + \epsilon \ldots) \leq \mathcal{N}(\ldots w_i \ldots w_j \ldots) \quad , \quad w_i \leq w_j$$

monotonicity wrt cumulation

NEW KEY INGREDIENT : ADDITIVITY



$$N \;\to\; \mathcal{N}[W]$$

Effective number of states has to be measure-like!

$$\mu(S_1 \cup S_2) = \mu(S_1) + \mu(S_2) \qquad S_1 \cap S_2 = \emptyset$$

$$N_{12} \;=\; N_1 + N_2$$

$$\mathcal{N}[W_{12}] \;=\; \mathcal{N}[W_1] + \mathcal{N}[W_2]$$

Note: $W_1 \in \mathcal{W}_{N_1} \;\; , \;\; W_2 \in \mathcal{W}_{N_2} \;\Rightarrow\; W_1 \boxplus W_2 \in \mathcal{W}_{N_1+N_2}$

$$(a_1, \ldots, a_N) \boxplus (b_1, \ldots, b_M) \equiv (a_1, \ldots, a_N, b_1, \ldots, b_M)$$

$$\mathcal{N}[W_1 \boxplus W_2, N_1 + N_2] \;=\; \mathcal{N}[W_1, N_1] + \mathcal{N}[W_2, N_2] \quad , \quad \forall\, W_1, W_2, N_1, N_2$$

EFFECTIVE NUMBERS

(A) $$\mathcal{N}[W_1 \boxplus W_2] \;=\; \mathcal{N}[W_1] + \mathcal{N}[W_2]$$

(S) $$\mathcal{N}(\ldots w_i \ldots w_j \ldots) \;=\; \mathcal{N}(\ldots w_j \ldots w_i \ldots)$$

(B1) $$\mathcal{N}(1, 1, \ldots, 1) \;=\; N$$

(B2) $$\mathcal{N}(N, 0, \ldots, 0) \;=\; 1$$

(B) $$1 \;\leq\; \mathcal{N}[W] \;\leq\; N$$

(C) $\mathcal{N}[W]$ is continuous on $\mathcal{W}$

(M-) $$\mathcal{N}(\ldots w_i - \epsilon \ldots w_j + \epsilon \ldots) \leq \mathcal{N}(\ldots w_i \ldots w_j \ldots) \quad , \quad w_i \leq w_j$$

monotonicity wrt cumulation

$\mathfrak{N}$: set of functions satisfying (A) , (S) , (B2) , (C) , (M-)

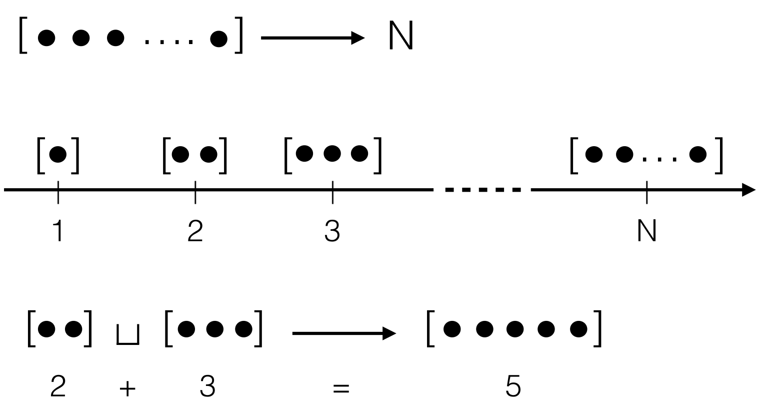
[(B1) and (B) follow]

$\mathcal{N}_p \notin \mathfrak{N}$ (not additive)

THE CONSISTENCY GAME

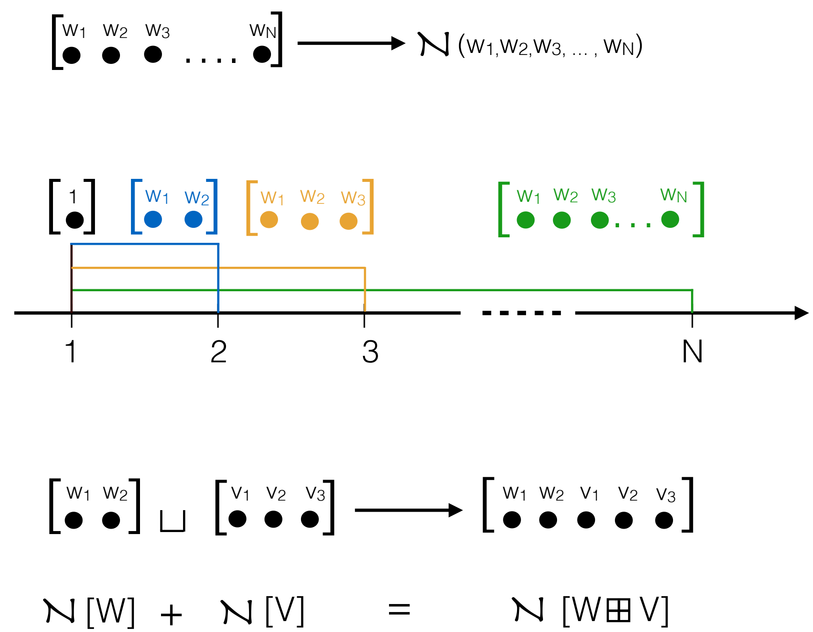
<u>number of objects</u>

natural numbers



<u>number of objects with weights</u>

effective numbers



Theorem

There are infinitely many elements in $\mathfrak{N}$ *and there exists* $\mathcal{N}_\star \in \mathfrak{N}$ *such that*

$$(a) \qquad \mathcal{N}_\star[W] \le \mathcal{N}[W] \le \mathcal{N}_+[W] \qquad \forall\, \mathcal{N} \in \mathfrak{N} \;,\; \forall\, W \in \mathcal{W}$$

$$(b) \qquad \mathcal{N}_\star[W] < \mathcal{N}_+[W] \quad \Longrightarrow \quad \{\, \mathcal{N}[W] \mid \mathcal{N} \in \mathfrak{N} \,\} \supseteq [\alpha, \beta\,)$$

where W *in* (b) *is arbitrary but fixed and* $\alpha = \mathcal{N}_\star[W]\;,\; \beta = \mathcal{N}_+[W]$.

$$\mathcal{N}_\star[W] = \sum_{i=1}^{N} \mathfrak{n}_\star(w_i) \qquad \mathfrak{n}_\star(w) \equiv \min\{\, w, 1 \,\}$$

$$\mathcal{N}_+[W] = \sum_{i=1}^{N} \mathfrak{n}_+(w_i) \qquad \mathfrak{n}_+(w) \equiv \begin{cases} 0\,, & w = 0 \\ 1\,, & w > 0 \end{cases} \qquad \mathcal{N}_+ \notin \mathfrak{N}$$

- THERE IS A “MINIMAL AMOUNT” OF OBJECTS WITH PROBABILITY WEIGHTS
  [least element]
- CONSISTENT EFF. AMOUNTS UP TO THE NUMBER OF NON-ZERO WEIGHTS
  [no structure at the top: $\mathcal{N}_\star$ represents the actual content of the concept]
- CONTINUUM OF EFFECTIVE COUNTING SCHEMES PERFECTLY NATURAL

WE NOW KNOW HOW TO COUNT WITH PROBABILITIES!

How did you say it works?

Example: Buying one hat, 6 choices, assign preferences (probabilities)

$$P = (0.01, 0.02, 0.10, 0.15, 0.30, 0.42)$$

$$W = (0.06, 0.12, 0.60, 0.90, 1.80, 2.52)$$ [counting weights]

$$\mathcal{N}_\star[W] = 0.06 + 0.12 + 0.60 + 0.90 + 1.00 + 1.00 = 3.68$$

Question: How many hats are you effectively choosing from?

Answer: About 2.5

Reply: You are a liar!

This application a basis for the notion of effective choices in probability theory.

BACK TO QUANTUM STATES :

$$|\,\psi\,\rangle\ ,\ \{\,|\,i\,\rangle\}\qquad :\qquad \text{How many}\ \ |\,i\,\rangle\ \text{in}\ \ |\,\psi\,\rangle\ ?$$

(i) ill-posed question in QM

(ii) well-posed question in QM (how?) ✓

Answer:

$$\mathcal{N}_\star[\,|\psi\rangle, \{|i\rangle\}\,] = \mathcal{N}_\star[W] \qquad W = (w_1, \ldots, w_N) \qquad w_i = N\,|\langle i\,|\,\psi\rangle|^2$$

$$\mathcal{N}_\star[W] = \sum_{i=1}^{N} \mathfrak{n}_\star(w_i) \qquad \mathfrak{n}_\star(w) \equiv \min\{\,w, 1\,\}$$

Quantum Uncertainty

canonical experiment: $\hat{O} \longleftrightarrow \{\, (\,|\, i\,\rangle, O_i\,) \mid i = 1,2,\ldots,N \,\}$ non-degenerate

$$|\,\psi\,\rangle \xrightarrow{\text{measure } \hat{O}} \{\, (\,|\, i_\ell\,\rangle, O_{i_\ell}\,) \mid \ell = 1,2,\ldots \}$$

uncertainty of $|\,\psi\,\rangle$ wrt $\hat{O}$ = indeterminacy encoded by $\{\, (\,|\, i_\ell\,\rangle, O_{i_\ell}\,) \}$

STANDARD	HERE
spread of outcomes	
distance on the spectrum	abundance of distinct outcomes
metric uncertainty	measure uncertainty
ρ-uncertainty	μ-uncertainty
$\Delta = \Delta[\,|\,\psi\,\rangle, \hat{O}\,]$	$\mathcal{N} = \mathcal{N}[\,|\,\psi\,\rangle, \{\,|\, i\,\rangle\,\}\,]$
[e.g. standard deviation]	[complete theory in arXiv:1807.03995]

MEASURE UNCERTAINTY PRINCIPLE

Set $\mathfrak{N}$ of effective number functions $\mathcal{N}$ exhausts all quantum μ-uncertainties

$$\mathcal{N}[\,|\,\psi\,\rangle, \{\,|\,i\,\rangle\,\}\,] \;=\; \mathcal{N}[W] \quad , \quad W = (w_1, \ldots, w_N) \quad , \quad w_i \;=\; N\;|\langle\, i\,|\,\psi\,\rangle|^2$$

$[\mathrm{U}_0]$ *The μ-uncertainty of $|\,\psi\,\rangle$ with respect to $\{\,|\,i\,\rangle\}$ is at least $\mathcal{N}_\star[W]$ states.*

classical state

quantum state

$$S \xrightarrow{\text{measurement}} S \qquad\qquad |\,\psi\,\rangle \xrightarrow{\text{measurement}} |\,i\,\rangle$$

μ-uncertainty $= 1$

minimal μ-uncertainty $= \mathcal{N}_\star$

ECONOMICAL EXPRESSION OF FUNDAMENTAL DIFFERENCE BETWEEN QUANTUM AND CLASSICAL

μ – UNCERTAINTY OF SCHRÖDINGER PARTICLE IN $\mathbb{R}^D$

When regularizations removed/continuous spectra: measure uncertainties are effective volumes

Example: Schrödinger particle in bounded region A of $\mathbb{R}^D$ described by wave-function ψ

$$\mathcal{V}_\star[\psi] \,=\, \int_A \nu_\star(x)\, d^D x \qquad\qquad \nu_\star(x) = \min\{\, V\psi^\star(x)\psi(x)\,,1\}$$

QUANTUM UNCERTAINTY EXPRESSED AS A GENERALIZATION OF THE <u>JORDAN CONTENT</u>!

Formulation entirely general in terms of the setting (system, Hilbert space)

TAKE-AWAYS

- IDENTITY-COUNTING PROBLEMS ARE WELL-DEFINED AND SOLVED IN QUANTUM MECHANICS [I.H. & R.M. arXiv:1807.03995]

- BY VIRTUE 0F EXTENDING THE CLASSICAL NOTION OF MEASURE VIA PROBABILITY

  counting $\rightarrow$ effective counting , counting measure $\rightarrow$ diversity measure ,

  Jordan content $\rightarrow$ effective Jordan content

- CONSEQUENTLY MUCH MORE BASIC THAN “QUANTUM” SETTING

  Effective counting arises classically virtually everywhere!

- FRUITFUL EXTENSION OF QUANTUM UNCERTAINTY INTO <u>MEASURE UNCERTAINTY</u>

  COMPLETELY UNDER CONTROL UNLIKE METRIC UNCERTAINTY

- QUALITATIVELY NEW TYPE OF UNCERTAINTY PRINCIPLE quantum chaos

many-body localization

effective supports

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