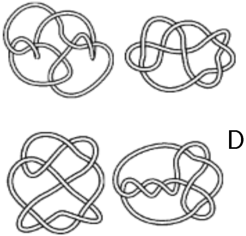
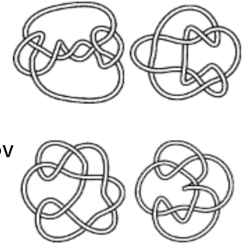


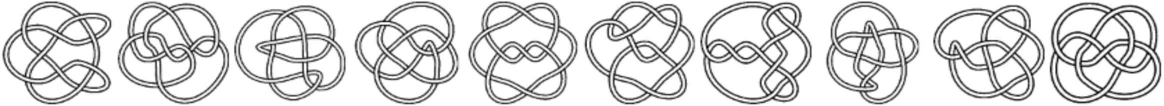


# Topological entanglement and knots

D. Melnikov, A. Mironov, S. Mironov, A. Morozov, An. Morozov

INR RAS, ITEP

Crete, OAC, July 12, 2018



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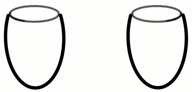
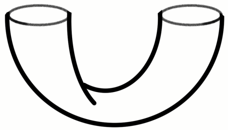
[Aravind], [Kauffman]

Let us show how topological entanglement works in easiest situations

# code space





The notion of separability of the state is quite clear in this case: the left state is clearly separable, while the right state is not. One can check the entanglement of two states computing their von Neumann entropy, for example, via the replica trick.

The above states are pictorial representations of the wavefunction which can explicitly be written as a functional integral. Hence those states are pure. The corresponding density matrices are obtained by a direct products of the wavefunction and its conjugate, that is a disjoint union of the three manifolds of two copies of the manifolds with oposite orientation of the boundaries.

The pictorial representations of (unnormalized) density matrices:

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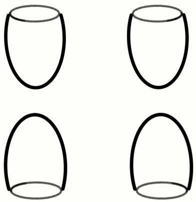

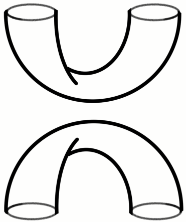

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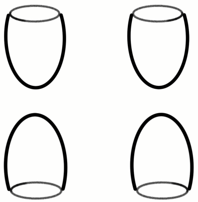
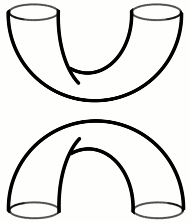
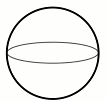
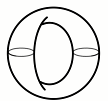




Let us denote the left density matrix as $\hat{\rho}_1$ and the right one as $\hat{\rho}_2$.

The normalized versions are

$\rho_1 = \Big[$  $\Big]^{-2} \hat{\rho}_1 ,\qquad \rho_2 = \Big[$  $\Big]^{-1} \hat{\rho}_2 \qquad (1)$

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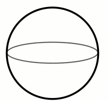
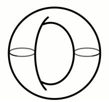
$\rho_1 = \Bigg[$  $\Bigg]^{-2} \hat{\rho}_1\,, \qquad \rho_2 = \Bigg[$  $\Bigg]^{-1} \hat{\rho}_2 \qquad (1)$

The normalized matrices satisfy the condition $\operatorname{tr}\rho = 1$, where the trace is understood as gluing together the oppositely oriented parts of the pictorial density matrices. In general, the result of such an operation would be a closed manifold, consequently a topological invariant. For example, the closed-manifold diagrams can be understood as

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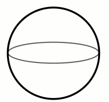
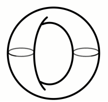
$\rho_1 = \Big[$  $\Big]^{-2} \hat{\rho}_1 ,$ $\quad \rho_2 = \Big[$  $\Big]^{-1} \hat{\rho}_2$ (1)

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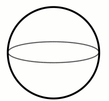
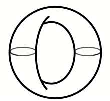
 $= Z(S^3)\,,$  $= Z(S^2 \times S^1)\,.$ (2)

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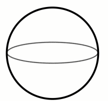
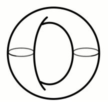
$\rho_1 = \Big[$  $\Big]^{-2} \hat{\rho}_1\,, \qquad \rho_2 = \Big[$  $\Big]^{-1} \hat{\rho}_2 \qquad (1)$

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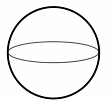
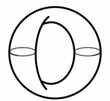
 $= Z(S^3)\,,$  $= Z(S^2 \times S^1)\,. \qquad (2)$

However, to keep the discussion more general, the diagrams above may stand for partition functions of closed three-manifolds with Wilson loops (links).

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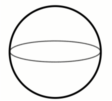
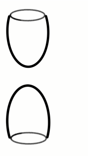
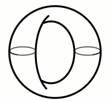
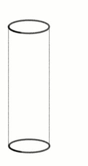
$\rho_1^A \equiv \mathrm{tr}_B \rho_1 = \Big[$  $\Big]^{-1} \times$  $\quad \rho_2^A \equiv \mathrm{tr}_B \rho_2 = \Big[$  $\Big]^{-1} \times$  (3)

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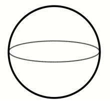
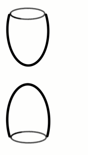
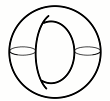
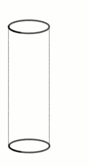
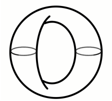
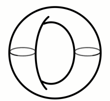
$\rho_1^A \equiv \mathrm{tr}_B \rho_1 = \Bigg[$  $\Bigg]^{-1} \times$  $\qquad \rho_2^A \equiv \mathrm{tr}_B \rho_2 = \Bigg[$  $\Bigg]^{-1} \times$  (3)

Consequently one computes

$\mathrm{tr}\,\left(\rho_1^A\right)^n = 1\,, \qquad \mathrm{tr}\,\left(\rho_2^A\right)^n = \Bigg[$  $\Bigg]^{1-n}$ (4)

and the entanglement entropy

$$S_{\mathrm{E}}(\rho_1) = 0\,, \qquad S_{\mathrm{E}}(\rho_2) = \log \Bigg[$$

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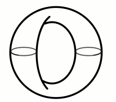

In the first case the entanglement entropy vanishes identically, while in the second case the result depends on the details of the Hilbert space (Wilson line insertions). In the most simple case the invariant in $S_{\mathrm{E}}(\rho_2)$ is $Z(S^2 \times S^1) = 1$, so the second entropy also vanishes. Below we will investigate the cases in which the entropy is non-trivial.

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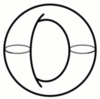
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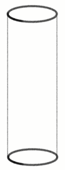
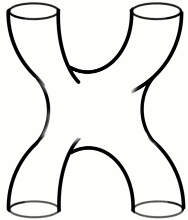
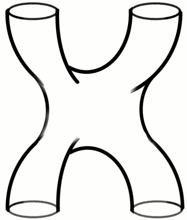
In particular, for a collection of Wilson lines closing around the non-contractible $S^1$ cycle of $S^2 \times S^1$ the answer will be

$$S_{\mathrm{E}} \;=\; \log\left(\dim \mathcal{H}_{S^2}\right)\,. \tag{6}$$

A pictorial representation of the density matrix of a mixed state (left).



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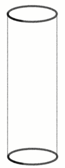
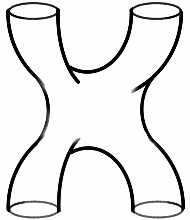
The same diagram stands for a local operator acting on a state.

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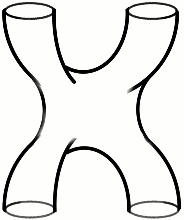

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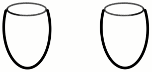
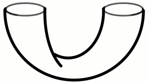


to a non-separable one



(7)

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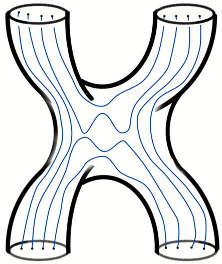

(8)

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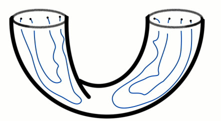

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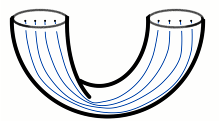

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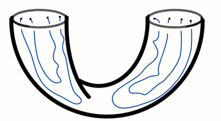

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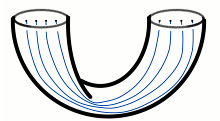
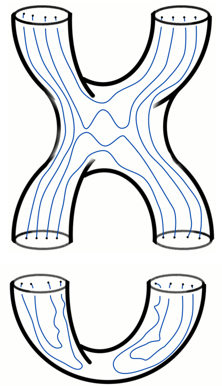


(9)



(10)

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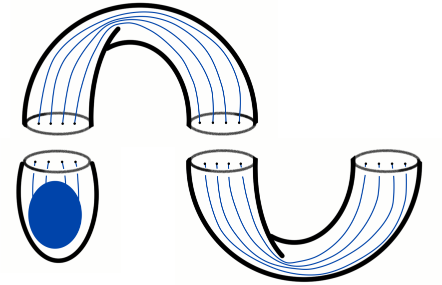

$$\langle M| \quad |\psi\rangle \quad |\Omega\rangle \tag{11}$$

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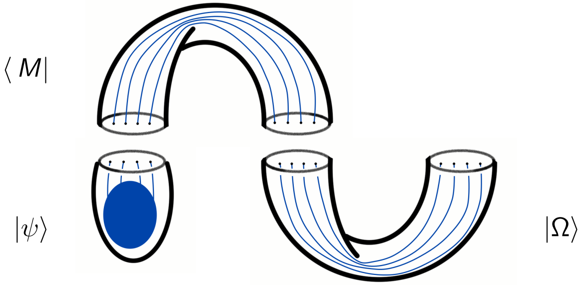


(11)

Teleportation looks somewhat trivial in the case of topological theories, since after all, those theories are independent from the space (and time) distance.

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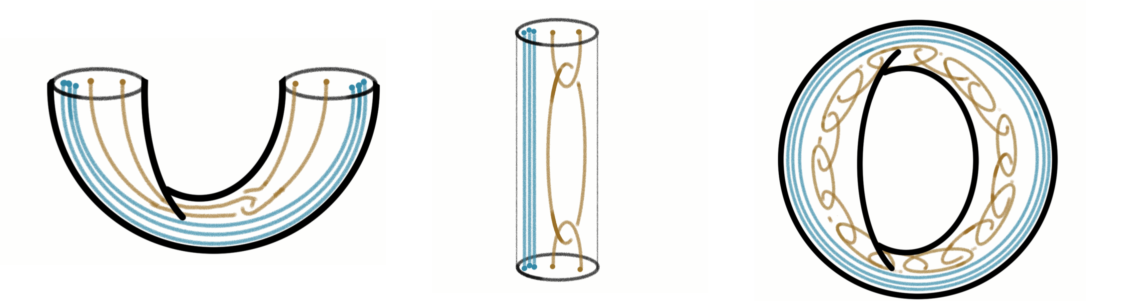

Figure: A state with two "chained" Wilson lines for arbitrary number of punctures (left). Reduced density matrix of the corresponding state (center). $S^2 \times S^1$ of a chain link that computes $\mathrm{tr}\,(\rho^A)^n$ (right).

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If we, finally, allow chaining (knotting) of the strings we get knots and links in 3-dimensional manifolds

Considering $S^3$ only leads to the whole theory of knot invariants

The most common and promising way to look at it is [Witten]:

Punctures on $S^2$ (the boundary) correspond to points of 2-dimensional conformal block

While strings in $S^3$ are Wilson loops in 3-dimensional Chern-Simons theory.

This provides one with the strong machinery for computations of knot invariants.

Namely, arbitrary representations of general SU(N) group provides one with colored HOMFLY polynomials.

## HOMFLY-PT

$$H_R^{\mathcal{K}}(q|A)\Big|_{A=q^N} = \left\langle \mathrm{Tr}_R P \exp \oint_{\mathcal{K}} \mathcal{A} \right\rangle_{SU(N)}$$

$3d$ Chern-Simons theory with the gauge group $SU(N)$ and the action

$$S = \frac{\kappa}{4\pi} \int_{d^3x} \mathrm{Tr}\left(\mathcal{A}d\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right) \qquad\qquad q = \exp\frac{2\pi i}{\kappa + N}$$

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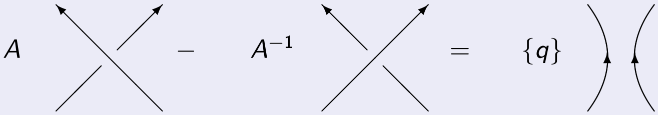
## Skein relations for $H_{\square}^{\mathcal{K}}(q|A)$:

$H_R^{\mathcal{K}}(q|A)$ is proportional to a Laurent polynomial in $q$ and $A$




$$\{x\} \equiv x - \frac{1}{x}$$

$$[n] \equiv \{q^n\}/\{q\}$$

## Examples

$$H^{2,3}_{[1]} = \left((q^2+q^{-2})A - A^{-1}\right)\frac{\{A\}}{\{q\}}$$

$$H^{2,3}_{[2]} = \left((q^{12}+q^6+q^4+1)A^2 - q^8 - q^6 - q^2 - 1 + q^2A^{-2}\right) q^4\frac{\{A\}\{Aq\}}{\{q\}\{q^2\}}$$

$$H^{2,3}_{[1,1]} = \left((q^{-12}+q^{-6}+q^{-4}+1)A^2 - q^{-8} - q^{-6} - q^{-2} - 1 + q^{-2}A^{-2}\right)\frac{\{A\}\{A/q\}}{q^4\{q\}\{q^2\}}$$

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## R-matrix

$$\mathcal{R}_{12}\mathcal{R}_{23}\mathcal{R}_{12} = \mathcal{R}_{23}\mathcal{R}_{12}\mathcal{R}_{23}$$

$$\mathcal{R}\Delta(g)\mathcal{R}^{-1} = \Delta(g)$$

Hence $\mathcal{R}$ is diagonal on the irreducible representation with eigenvalue

$$r_R \sim \pm q^{\frac{1}{2}\varphi_{[2]}(R)}$$

$\mathcal{R}$-matrix provides **entanglement**

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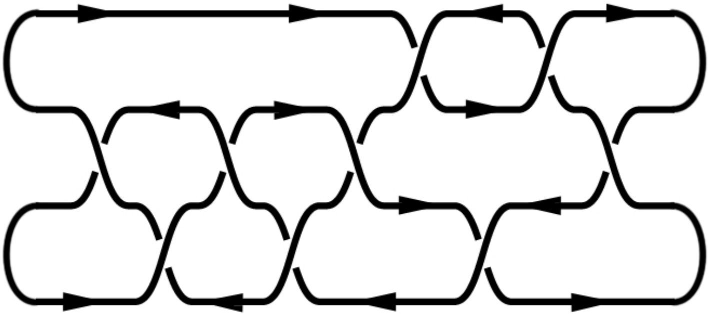

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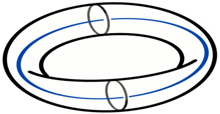
A distinct basis in such a Hilbert space can be introduced by considering an unknot (circle) along the non-contractible cycle in the bulk of the solid torus, colored with an integrable representation:

$R_i$  $\equiv \quad |\text{unknot}; R_i\rangle \quad \longrightarrow \quad e_i. \qquad (12)$

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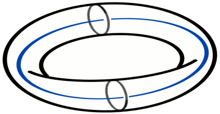
$R_i$ 

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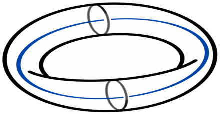
$R_i$  $\equiv \quad |\text{unknot}; R_i\rangle \quad \longrightarrow \quad e_i\,. \qquad (12)$

The scalar product of two basis vectors in the Hilbert space corresponds to gluing together two solid tori along the common $T^2$ boundary (with opposite orientation).

Consequently, the result of this operation is a link invariant on $S^2 \times S^1$:

$$\langle R_i | R_j \rangle \; = \; Z(S^2 \times S^1; \text{unlink}; R_i, R_j) \; = \; \delta_{R_i \bar{R}_j}\,, \tag{13}$$

*i.e.* it is indeed an orthonormal basis.

The vector spaces associated with tori come with a discrete set of operations, namely the mapping class group. In the case of the $SU(2)$ Chern-Simons, the $S$ and $T$ generators have a well-known unitary representation:

$$S_{mn} \;=\; \sqrt{\frac{2}{k+2}}\sin\left(\frac{\pi(m+1)(n+1)}{k+2}\right), \qquad T_{mn} \;=\; q^{2C_2}\delta_{mn}\,. \tag{14}$$

Matrices $S$ and $T$ are defined here in the basis of integrable representations, labeled by integer $m, n = 0, \ldots, k+1$.

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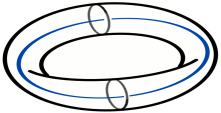
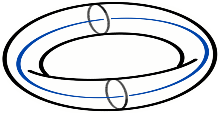
The $S$ transformation swaps the two fundamental cycles of $T^2$. Diagramatically we can illustrate this as follows:

$\Big\langle R_i$  $\Big|\; S \;\Big| R_j$  $\Big\rangle \;=\; Z(S^3; \mathrm{Hopf}; R_i, R_j)$ (15)

In general, the scalar products of states in the Hilbert space are path integrals of the TQFT on different manifolds. Different ways of gluing tori produce a set of closed three manifolds known as Lens spaces (Seifert spaces).

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Using this picture of cobordisms of $T^2$ one can compute entanglement entropy directly associated with links and thus quantify the relation of topological and quantum entanglements.

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Using this picture of cobordisms of $T^2$ one can compute entanglement entropy directly associated with links and thus quantify the relation of topological and quantum entanglements.

In particular, the state, which corresponds to a Hopf link, appears to be maximally entangled, while unlinked circles have zero entanglement entropy.

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$+$: topological stability

$-$: more complicated structure than qbit$\oplus$qbit$\oplus$qbit$\oplus$qbit..

THANK YOU FOR YOUR ATTENTION!

$$S_A = -\mathrm{tr}\,_A \rho_A \log \rho_A$$

$$S_A = \lim_{n \to 1} \frac{\mathrm{tr}\,_A \rho_A^n - 1}{1-n} = -\frac{\partial}{\partial n} \log \mathrm{tr}\,_A \rho_A^n \bigg|_{n=1}$$